ON THE MAXIMAL NUMBER OF INDEPENDENT CIRCUITS IN A GRAPH

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§ 1. Introduction

In this paper we are going to prove the following theorem:

Let $n, k(\geqq 1)$ be integers such that $n \geqq 3k$ and suppose that the valency of every vertex of a graph $\mathcal{G}$ of n vertices is not less than $2k$. Then $\mathcal{G}$ contains k independent circuits.[1]

The special case $k=1$ of this theorem is a well-known and almost trivial assertion of graph theory. This generalization of it has been conjectured by P. ERDŐS. A few years ago G. DIRAC stated the following slightly weaker conjecture (written communication).

If a graph $\mathcal{G}$ of $n \geqq 3k$ vertices is $2k$-fold connected then it contains k independent circuits. The special case $k=2$ of our result was already known to them too. In their paper [1] G. DIRAC and P. ERDŐS prove the following theorem.

To every $k \geqq 1$ and to $c \geqq 0$ there is a smallest integer $n(k, c)$ such that if $n > n(k, c)$ then every graph $\mathcal{G}$ of n vertices every vertex of which has valency $\geqq 2k$ except possible c vertices contains k independent circuits. Though this theorem for large n is stronger than ours, it is of quite different character and the proof needs different arguments.

Using our theorem already mentioned in paper [1] G. DIRAC and P. ERDŐS prove a generalisation of this theorem too. As to the further possible generalizations and problems arising here we also refer to [1].

We would like to mention that in the first version of this paper dated November 9, 1961 the proof of our results was more complicated. L. PÓSA called our attention to the fact that the proof of the case $n=3k$ can be considerably simplified. We will point out in the text where his idea is used.

§ 2. Definitions. Notations

A graph $\mathcal{G}$ is considered to be an ordered pair $\langle G, \mathcal{G}^* \rangle$ where G is the set of vertices denoted by $P, Q, \ldots$ etc. and $\mathcal{G}^*$ is a set of non ordered pairs (P, Q), $P \neq Q$, $P, Q \in G$ called the edges of $\mathcal{G}$.

A graph $\mathcal{H}$ is said to be a subgraph of $\mathcal{G}$ if $H \subseteq G$ and $\mathcal{H}^* \subseteq \mathcal{G}^*$. If $\mathcal{H}$ is a subgraph of $\mathcal{G}$ we briefly say that $\mathcal{G}$ contains $\mathcal{H}$ and we write $\mathcal{H} \subseteq \mathcal{G}$.

If $\mathcal{G}$ is a graph, and $H \subseteq G$ then the subgraph of $\mathcal{G}$ spanned by the vertices belonging to H will be denoted by $\mathcal{G}(H) = \langle H, \mathcal{G}^*(H) \rangle$.

[1] We only consider finite graphs without loops and multiple edges. For the detailed explanation of the terminology used in this paper see § 2.

$|A|$ denotes the number of elements of A for an arbitrary set A.

If $\mathcal{G}=\langle G, \mathcal{G}^*\rangle$ is a graph $|G|$ will be denoted by $\nu(\mathcal{G})$.

If $P\in G$ then the number of edges of $\mathcal{G}$ incident to P will be called the valency of P in $\mathcal{G}$ and will be denoted by $v(P, \mathcal{G})$.

If $P\in G$ and $H\subseteq G$, then the number of edges (P, Q), $Q\in H$ of $\mathcal{G}$ is said to be the valency of P in $\mathcal{G}$ with respect to H. This will be denoted by $v(P, \mathcal{G}, H)$.

Let $P_1, \ldots, P_j$ be different elements. The graph $\mathcal{C}[P_1, \ldots, P_j]$ the vertices of which are $P_1, \ldots, P_j$ and the edges of which are $(P_1, P_2), (P_2, P_3), \ldots, (P_{j-1}, P_j)$, (P_j, P_1) will be said a circuit of length j, for $j\geqq 3$. The graph $\mathcal{P}[P_1, \ldots, P_j]$ the vertices of which are $P_1, \ldots, P_j$ $(j\geqq 1)$ and the edges of which are $(P_1, P_2), (P_2, P_3), \ldots, (P_{j-1}, P_j)$ is said to be a path of length j.

If $\mathcal{C}[P_1, \ldots, P_j]$ is a circuit or $\mathcal{P}[P_1, \ldots, P_j]$ is a path the set of vertices $\{P_1, \ldots, P_j\}$ will be briefly denoted by $\bar{\mathcal{C}}$ or $\bar{\mathcal{P}}$ respectively.

Two vertices are said to be independent (in a graph $\mathcal{G}$) it they are not connected by an edge.

Two edges or two circuits are said to be independent if they have no common vertex. A collection of vertices (edges, circuits) is said to be independent if any two of them is independent respectively.

Let k be an integer $k\geqq 1$. We say that a graph is an $\bigcirc^k$ graph if it is the sum of k independent circuits more precisely if $\mathcal{C}_1, \ldots, \mathcal{C}_k$ is a collection of independent circuits, and the set of vertices and the set of edges of $\mathcal{G}$ are the union of the set of vertices and the set of edges of the circuits $\mathcal{C}_1, \ldots, \mathcal{C}_k$ respectively. We will use the notation $\bigcirc^k = \mathcal{C}_1 + \ldots + \mathcal{C}_k$.

§ 3. Proof of the theorems

THEOREM 1. *Let $\mathcal{G}=\langle G, \mathcal{G}^*\rangle$ be a graph such that $\nu(\mathcal{G})=n$. Suppose $n\geqq 3k$, $k\geqq 1$ and $v(P, \mathcal{G})\geqq 2k$ for every vertex $P\in G$.*

Then $\mathcal{G}$ contains an $\bigcirc^k$ graph.

REMARK. Theorem 1 is best possible of its kind as is shown by the following example:

Let $\mathcal{G}=\langle G, \mathcal{G}^*\rangle$ be the graph defined by the following stipulations.

Let G_1, G_2 be two disjoint sets, $|G_1| = 2k-1$, $|G_2| = n-2k+1\geqq k+1$. Put $G = G_1\cup G_2$. The edge (P, Q), $P, Q\in G$ belongs to $\mathcal{G}^*$ if and only if either $P, Q\in G_1$ or $P\in G_1$ and $Q\in G_2$.

It is obvious that $\nu(\mathcal{G})=n\geqq 3k$ and the valency $v(P, \mathcal{G})$ of every vertex of $\mathcal{G}$ is not less than $2k-1$. On the other hand, $\mathcal{G}$ does not contain a graph $\bigcirc^k = \mathcal{C}_1 + \ldots + \mathcal{C}_k$ since for every circuit $\mathcal{C}\subseteq\mathcal{G}$, $\bar{\mathcal{C}}$ contains at least two elements of G_1.[2]

Instead of Theorem 1 we prove the following stronger

THEOREM 2. *Let $\mathcal{G}=\langle G, \mathcal{G}^*\rangle$ be a graph such that $\nu(\mathcal{G})=n$. Suppose $n\geqq 3k$, $k\geqq 1$ and suppose that $v(P, \mathcal{G})\geqq 2k$ for every vertex $P\in G$. Put $n = lk+t$ where $0\leqq t<k$ $(l\geqq 3)$. Then $\mathcal{G}$ contains a graph $\bigcirc^k$ satisfying the following conditions;*

$$\bigcirc^k = \mathcal{C}_1 + \ldots + \mathcal{C}_k.$$

[2] This graph $\mathcal{G}$ was constructed by P. ERDŐS and T. GALLAI. See [2].

The length of $\mathcal{C}_i$ is not greater than l for $1 \leqq i \leqq k-t$. The length of $\mathcal{C}_i$ is not greater than $l+1$ for $k-t < i \leqq k$.

REMARK. In case $n=3k$ Theorems 1 and 2 give the same. For $n>3k$ Theorem 2 is stronger than Theorem 1. Thus it is sufficient to prove Theorem 2.

The estimation given by Theorem 2 for the length of $\mathcal{C}_i$ is certainly not best possible for large n. Although one can prove that it is best possible if $n \leqq 4k$. Perhaps one can carry out simpler proofs for the case $n>3k$ of Theorem 2 which gives no estimation for the length of $\mathcal{C}_i$ but we do not succeeded in obtaining this. Before proving Theorem 2 we formulate a well-known general argument of graph theory which will be used in the sequel.

Let $\Phi(\mathcal{H})$, $\Psi(\mathcal{H})$ denote properties of graphs. The property $\Phi(\mathcal{H})$ is said to be *monotonic* if $\Phi(\mathcal{H}_1)$ implies $\Phi(\mathcal{H}_2)$ provided $\mathcal{H}_1 \subseteqq \mathcal{H}_2$, $H_1 = H_2$.

The graph $\mathcal{H}_1$ is said to be *saturated* with respect to the property $\Phi(\mathcal{H})$, if every proper extension $\mathcal{H}_2$ of $\mathcal{H}_1$ with $\mathcal{H}_1 \subseteqq \mathcal{H}_2$, $\mathcal{H}_1^* \neq \mathcal{H}_2^*$ $H_1 = H_2$ fails to possess property Φ.

The property $\Phi(\mathcal{H})$ is said to be a *finite graph property* if for every graph $\mathcal{H}_1$ which has property Φ there exists a maximal graph $\mathcal{H}_2$ with $H_2 = H_1$ which contains $\mathcal{H}_1$ and which has property Φ i. e. $\mathcal{H}_1 \subseteqq \mathcal{H}_2$, $H_1 = H_2$, $\Phi(\mathcal{H}_2)$ and $\mathcal{H}_2$ is saturated with respect to the property Φ.

An almost trivial „reductio ad absurdum" proof shows that the following statement is true

LEMMA 1. *Suppose that $\Phi(\mathcal{H})$ is monotonic, and the negation of $\Psi(\mathcal{H})$ is a finite graph property. Suppose further that every graph $\mathcal{H}$ which possesses property Φ, and which is saturated with respect to the negation of property Ψ, possesses property Ψ.*

Then every graph wich has property Φ, possesses property Ψ too.

The application of this argument was proposed by L. PÓSA.

Now we turn to the proof of Theorem 2. We need here several lemmas, the proof of the lemmas is to be found in §4.

We distinguish the cases A) $n=3k$, B) $n>3k$.

Proof of Theorem 2 *in case* A).

Let $\mathcal{G}$ be a graph satisfying the conditions of Theorem 2.

Let $\Phi(\mathcal{H})$ denote the property of a graph $\mathcal{H}$ that $v(P, \mathcal{H}) \geqq 2k$ for every $P \in H$.

Let $\Psi(\mathcal{H})$ denote the property of a graph $\mathcal{H}$ that $\mathcal{H}$ contains an $\bigcirc^k$ graph.

It is obvious that $\Phi(\mathcal{H})$ is monotonic and that the negation of $\Psi(\mathcal{H})$ is a finite graph property. Thus by Lemma 1 we may suppose that

(1) $\mathcal{G}$ is saturated with respect to the negation of $\Psi(\mathcal{H})$.

Now we assume that

(2) $\mathcal{G}$ does not contain an $\bigcirc^k$ graph and we finish the proof by obtaining a contradiction.

By (2) there exist $P, Q \in G$, $P \neq Q$ such that

(3) $$(P, Q) \notin \mathcal{G}^*.$$

Hence $\mathcal{G}_1 = \langle G, \mathcal{G}_1^* \rangle$ with $\mathcal{G}_1^* = \mathcal{G}^* \cup \{(P, Q)\}$ is a proper extension of $\mathcal{G}$ and considering (1) $\mathcal{G}_1$ contains an $\bigcirc^k$ graph. That means there exists a graph $\bigcirc^k =$

$= \mathcal{C}_1 + \ldots + \mathcal{C}_k$ satisfying the condition:

(4) $$\mathrm{O}^k \subseteq \mathcal{G}_1 \qquad (\mathrm{O}^k = \mathcal{C}_1 + \ldots + \mathcal{C}_k).$$

Considering that the length of a circuit is at least 3 and that $n = 3k$ we have

(5) the length of the circuits $\mathcal{C}_i$ is 3 for $1 \leqq i \leqq k$; the sets $\overline{\mathcal{C}}_i$ are disjoint and

$$G = \bigcup_{1 \leqq i \leqq k} \overline{\mathcal{C}}_i.$$

It is obvious that at most one of the circuits $\mathcal{C}_i$ contains the edge (P, Q), and by (2) one of them, let us say $\mathcal{C}_1$, contains (P, Q). Thus we can choose the notations so that the following conditions hold.

(6) $$\mathcal{P}_1 = \mathcal{P}_1[P_1^1, P_2^1, P_3^1], \quad \mathcal{C}_i = \mathcal{C}_i[P_1^i, P_2^i, P_3^i]$$
$$P = P_1^1, \quad Q = P_3^1; \quad \mathcal{C}_i \subseteq \mathcal{G} \quad \text{for} \quad 2 \leqq i \leqq k,$$
$$\mathcal{P}_1 \subseteq \mathcal{G} \quad \text{and} \quad (P_1^1, P_3^1) \notin \mathcal{G}^*.$$

The idea of our proof is to show that in the subgraph spanned by the path $\overline{\mathcal{P}}_1$ and some, say four triangles $\overline{\mathcal{C}}_i$ one can find five independent triangles. However the technical excecution of the proof is not so simple

(7) Put $$H_1 = \overline{\mathcal{C}}_2 \cup \ldots \cup \overline{\mathcal{C}}_k.$$

Put further $$r = \sum_{j=1}^{3} v(P_j^1, \mathcal{G}, H_1).$$

By the assumption and by (6) we have $r \geqq 6k - 4$. Considering that $6k - 4 > 6(k-1)$ it follows that there exists an i_0, $2 \leqq i_0 \leqq k$ such that $\sum_{j=1}^{3} v(P_j^1, \mathcal{G}, \overline{\mathcal{C}}_{i_0}) \geqq 7$. We may assume that $i_0 = 2$. Hence we have

(8) $$\sum_{j=1}^{3} v(P_j^1, \mathcal{G}, \overline{\mathcal{C}}_2) \geqq 7.$$

Now we need the following

LEMMA 2. *Let $\mathcal{H}$ be a graph such that $v(\mathcal{H}) = 6$. $H = \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$, $\mathcal{P}[P_1, P_2, P_3] \subseteq \mathcal{H}$, $\mathcal{C} = \mathcal{C}[Q_1, Q_2, Q_3] \subseteq \mathcal{H}$ and*

$$\sum_{j=1}^{3} v(P_j, \mathcal{H}, \overline{\mathcal{C}}) \geqq 7.$$

Then either $\mathcal{H}$ contains two independent triangles or $v(P_2, \mathcal{H}, \overline{\mathcal{C}}) = 3$, $v(P_1, \mathcal{H}, \overline{\mathcal{C}}) = v(P_3, \mathcal{H}, \overline{\mathcal{C}}) = 2$ and there is a j, $1 \leqq j \leqq 3$ for which $(P_1 Q_j) \notin \mathcal{H}^*$ and $(P_3 Q_j) \notin \mathcal{H}^*$.

If $\mathcal{G}(\overline{\mathcal{P}}_1 \cup \overline{\mathcal{C}}_2)$ contains two independent triangles then by (6) $\mathcal{G}$ contains an O^k which contradicts (2). Thus by (8) and by Lemma 2 we have

(9) $$(P_2^1, P_j^2) \in \mathcal{G}^* \quad \text{for} \quad j = 1, 2, 3.$$

We may assume that

$$(P_1^1, P_1^2) \notin \mathcal{G}^*, \quad (P_3^1, P_1^2) \notin \mathcal{G}^*$$

and

$$(P_1^1, P_j^2) \in \mathcal{G}^*, \quad (P_3^1, P_j^2) \in \mathcal{G}^* \quad \text{for} \quad j = 2, 3.$$

(10) Put
$$H_2 = \overline{\mathcal{C}}_3 \cup \dots \cup \overline{\mathcal{C}}_k.$$

Considering that by (6) and (9)

$$v(P_1^1, \mathcal{G}, \overline{\mathscr{P}}_1 \cup \overline{\mathcal{C}}_2) = v(P_3^1, \mathcal{G}, \overline{\mathscr{P}}_1 \cup \overline{\mathcal{C}}_2) \leqq 3$$

we have

$$v(P_1^1, \mathcal{G}, H_2) + v(P_3^1, \mathcal{G}, H_2) \geqq 4k - 6 > 4(k-2).$$

It results from (10) that there exists an i_0, $3 \leqq i_0 \leqq k$ such that

$$v(P_1^1, \mathcal{G}, \overline{\mathcal{C}}_{i_0}) + v(P_3^1, \mathcal{G}, \overline{\mathcal{C}}_{i_0}) \geqq 5.$$

We may assume $i_0 = 3$. Thus we have

(11)
$$v(P_1^1, \mathcal{G}, \overline{\mathcal{C}}_3) + v(P_3^1, \mathcal{G}, \overline{\mathcal{C}}_3) \geqq 5.$$

By symmetry we may assume that

(12)
$$v(P_1^1, \mathcal{G}, \overline{\mathcal{C}}_3) = 3 \quad \text{and} \quad v(P_3^1, \mathcal{G}, \overline{\mathcal{C}}_3) \geqq 2$$

and that

$$(P_3^1, P_2^3) \in \mathcal{G}^*, \quad (P_3^1, P_3^3) \in \mathcal{G}^*.$$

Put

$$C_3' = C_3'[P_1^1, P_1^3, P_2^3].$$

Then we have by (12)

(13)
$$C_3' \subseteqq \mathcal{G} \quad \text{and} \quad v(P_3^3, \mathcal{G}, \overline{\mathcal{C}}_3^1) = 3.$$

Put

(14)
$$D_1 = \{P_2^1, P_3^1, P_1^2, P_3^3\}, \quad D_2 = \{P_2^2, P_3^2\},$$

$$H_3 = \overline{\mathcal{C}}_3' \cup \overline{\mathcal{C}}_4 \cup \dots \cup \overline{\mathcal{C}}_k.$$

We need a lower estimation for $\sum_{Q \in D_1} v(Q, \mathcal{G}, H_3)$.

(15)
$$\sum_{Q \in D_1} v(Q, \mathcal{G}, H_3) > 8(k-2).$$

To see this, we recall first that by (9) $(P_3^1, P_1^2) \notin \mathcal{G}^*$. Taking into consideration (6) and (9) an easy discussion shows that $(P_2^1, P_3^3) \notin \mathcal{G}^*$ and that $v(P_3^3, \mathcal{G}, D_1 \cup D_2) \leqq 3$ for if not then $\mathcal{G}(D_1 \cup D_2)$ would contain two independent triangles, and these would form with $\mathcal{C}_3', \mathcal{C}_4, \dots, \mathcal{C}_k$ k independent circuits in $\mathcal{G}$ in contradiction with (2).

Thus we obtain (15) as follows

$$\sum_{Q\in D_1} v(Q, \mathcal{G}, H_3) = v(P_2^1, \mathcal{G}, H_3)+v(P_3^1, \mathcal{G}, H_3)+v(P_1^2, \mathcal{G}, H_3)+$$

$$+v(P_3^3, \mathcal{G}, H_3)\geqq(2k-4)+(2k-4)+(2k-4)+(2k-3) = 8k-15.$$

It follows from (15) that either (16) or (17) holds

$$\sum_{Q\in D_1} v(Q, \mathcal{G}, \overline{\mathcal{C}}_3')\geqq 9. \tag{16}$$

$$\sum_{Q\in D_1} v(Q, \mathcal{G}, \overline{\mathcal{C}}_{i_0})\geqq 9 \quad \text{for an} \quad i_0, 4\leqq i_0\leqq k. \tag{17}$$

We may suppose that $i_0=4$.

Now we need the following

LEMMA 3. *Let $\mathcal{H}$ be a graph such that $v(\mathcal{H})=7, H=\{P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3\}$. Suppose that $(P_1, P_2)\in\mathcal{H}^*$; $(P_3, P_4)\in\mathcal{H}^*$; $\mathcal{C}=\mathcal{C}[Q_1, Q_2, Q_3]\subseteqq\mathcal{H}$ and*

$$\sum_{i=1}^{4} v(P_i, \mathcal{H}, \overline{\mathcal{C}})\geqq 9.$$

Then there exist two independent triangles $\mathcal{C}', \mathcal{C}''\subseteqq\mathcal{H}$. If $v(P_{j_0}, \mathcal{H}, \overline{\mathcal{C}})=3$ for an arbitrary j_0, $1\leqq j_0\leqq 4$ then $\mathcal{C}', \mathcal{C}''$ can be choosen so that $P_{j_0}\in\overline{\mathcal{C}}'\cup\overline{\mathcal{C}}''$, $\overline{\mathcal{C}}\subseteqq\overline{\mathcal{C}}'\cup\overline{\mathcal{C}}''$.

Now we show that any of the statements (16), (17) leads to a contradiction.

Suppose that (16) holds. Considering that by (9) and (12) $(P_2^1, P_1^2)\in\mathcal{G}^*$ and $(P_3^1, P_3^3)\in\mathcal{G}^*$, and that by (13) $\mathcal{C}_3'\subseteqq\mathcal{G}$ and $v(P_3^3, \mathcal{G}, \overline{\mathcal{C}}_3')=3$ we may apply Lemma 3 to the graph $\mathcal{H}=\mathcal{G}\ (D_1\cup\overline{\mathcal{C}}_3')$ with

$$P_1=P_2^1,\quad P_2=P_1^2,\quad P_3=P_3^1,\quad P_4=P_3^3,$$

$$\mathcal{C}=\mathcal{C}_3',\quad Q_1=P_1^1,\quad Q_2=P_1^3,\quad Q_3=P_2^3,$$

where $v(P_4, \mathcal{H}, \overline{\mathcal{C}})=3$.

It follows that $\mathcal{G}(D_1\cup\overline{\mathcal{C}}_3')$ contains two independent circuits $\mathcal{C}'$, $\mathcal{C}''$ and $D_1\cup\overline{\mathcal{C}}_3' = \{P\}\cup\overline{\mathcal{C}}'\cup\overline{\mathcal{C}}''$ where P is one of the vertices P_2^1, P_1^2, P_3^1. It follows from (6) and (9) that the circuit $\mathcal{C}'''=\mathcal{C}'''[P, P_2^2, P_3^2]$ belongs to $\mathcal{G}$. By (6), (9), (13) and (14) $\mathcal{C}', \mathcal{C}'', \mathcal{C}''', \mathcal{C}_4, \ldots, \mathcal{C}_k$ are k independent circuits in $\mathcal{G}$. This contradicts (2).

Suppose that (17) holds then again by (9) and (12)

$$(P_2^1, P_1^2)\in\mathcal{G}^* \quad \text{and} \quad (P_3^1, P_3^3)\in\mathcal{G}^*.$$

Hence we may apply Lemma 3 to the graph $\mathcal{H}=\mathcal{G}(D_1\cup\overline{\mathcal{C}}_4)$ with

$$P_1=P_2^1,\quad P_2=P_1^2,\quad P_3=P_3^1,\quad P_4=P_3^3,$$

$$\mathcal{C}=\mathcal{C}_4,\quad Q_1=P_1^4,\quad Q_2=P_2^4,\quad Q_3=P_3^4.$$

It follows that $\mathcal{G}(D_1\cup\overline{\mathcal{C}}_4)$ contains two independent circuits $\mathcal{C}'$, $\mathcal{C}''$ and $D_1\cup\overline{\mathcal{C}}_4 = \{P\}\cup\overline{\mathcal{C}}'\cup\overline{\mathcal{C}}''$ where $P\in D_1$.

We distinguish two cases (i) $P \neq P_3^3$, (ii) $P = P_3^3$. If (i) holds then by (9) the circuit $\mathcal{C}''' = \mathcal{C}'''[P, P_2^2, P_3^2]$ belongs to $\mathcal{G}$, and by (6), (13) and (14) $\mathcal{C}', \mathcal{C}'', \mathcal{C}'''$, $\mathcal{C}_3', \mathcal{C}_5, \ldots, \mathcal{C}_k$ are k independent circuits in $\mathcal{G}$. This contradicts (2).

If (ii) holds then by (9) the circuit $\mathcal{C}''' = \mathcal{C}'''[P_1^1, P_2^2, P_3^2]$ belongs to $\mathcal{G}$ and considering that $P = P_3^3$ belongs to $\overline{\mathcal{C}}_3$, using again (6) and (14) we obtain that $\mathcal{C}'$, $\mathcal{C}''$, $\mathcal{C}'''$, $\mathcal{C}_3$, $\mathcal{C}_5, \ldots, \mathcal{C}_k$ are k independent circuits in $\mathcal{G}$.

This again contradicts (2). Thus the proof of the case A) is complete.

REMARK. It seems that using our method the proof of the case $n = 3k$ can not be simplified essentially. Namely, in case $k = 4$ it is easy to construct a graph $\mathcal{G}$ satisfying the requirements of our theorem such that it contains the triangles $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and the path $\mathcal{P}_1$ of length three, further $G = \overline{\mathcal{P}}_1 \cup \overline{\mathcal{C}}_2 \cup \overline{\mathcal{C}}_3 \cup \overline{\mathcal{C}}_4$, but $\mathcal{C}_i \nsubseteqq \mathrm{O}^4$ $(i = 2, 3, 4)$ for every O^4 contained in $\mathcal{G}$. Though this difficulty does not enter in the proof of the case $n > 3k$ this proof is also complicated.

Proof of Theorem 2 *in case* B).

The theorem is trivial for $k = 1$. We assume $k > 1$. First we state a Lemma.

LEMMA 4. *Let* $\mathcal{H}$ *be a graph,* $v(\mathcal{H}) = r + 1$. *Suppose that* $\mathcal{C} = \mathcal{C}[Q_1, \ldots, Q_r] \subseteqq \mathcal{H}$, $H = \{Q_1, \ldots, Q_r, P\}$. *Then* $\mathcal{H}$ *contains a circuit* $\mathcal{C}'$ *of length* $< r$ *provided one of the following conditions holds:*

a) $$r > 4 \quad \textit{and} \quad v(P, \mathcal{H}, \overline{\mathcal{C}}) \geqq 2,$$

b) $$r > 3 \quad \textit{and} \quad v(P, \mathcal{H}, \overline{\mathcal{C}}) \geqq 3.$$

We prove

(18) $\mathcal{G}$ contains a circuit of length $\leqq l$.

It is obvious that $\mathcal{G}$ contains a circuit. Let $\mathcal{C} = \mathcal{C}[Q_1, \ldots, Q_r]$ be a circuit contained in $\mathcal{G}$ of minimal length, i. e. $\mathcal{C} \subseteqq \mathcal{G}$ and $v(\mathcal{C}') \geqq r$ for every $\mathcal{C}' \subseteqq \mathcal{G}$. We have to prove $r \leqq l$. Suppose that this is false, i. e. $r > l$. Put $H_1 = G - \overline{\mathcal{C}}$. Then $|H_1| = lk + t - r$. Considering the minimality of r we have $v(Q_j, \mathcal{G}, \overline{\mathcal{C}}) = 2$ for $1 \leqq j \leqq r$.

It follows from the assumption that

$$V = \sum_{j=1}^{r} v(Q_j, \mathcal{G}, H_1) \geqq 2kr - 2r = 2(k-1)r.$$

Considering that $H_1 \cap \overline{\mathcal{C}} = 0$ we have $V = \sum_{P \in H_1} v(P, \mathcal{G}, \overline{\mathcal{C}})$. It results that there exists a $P_0 \in H_1$ for which $v(P_0, \mathcal{G}, \overline{\mathcal{C}}) \geqq 3$ for if not, then

$$2(k-1)r \leqq V \leqq 2(lk + t - r),$$

$$kr \leqq lk + t < (l+1)k \quad \text{hence} \quad r < l + 1$$

in contradiction with the assumption $r > l$.

But then we can apply Lemma 4 to the graph $\mathcal{G}(\overline{\mathcal{C}} \cup \{P_0\})$ Considering that $r > l \geqq 3$, we obtain that $\mathcal{G}$ contains a circuit $\mathcal{C}'$ of length $< r$. This contradicts the minimality of r. Thus (18) is proved.

It follows from (18) that

(19) there exists an integer $s(\geqq 1)$, for which there exists an $\bigcirc^s$ graph satisfying the following conditions:

a) $$\bigcirc^s = \mathcal{C}_1 + \ldots + \mathcal{C}_s \subseteqq \mathcal{G},$$

b) $$v(\mathcal{C}_i) \leqq l \quad \text{for} \quad 1 \leqq i \leqq k-t, \quad i \leqq s,$$

c) $$v(\mathcal{C}_i) \leqq l+1 \quad \text{for} \quad k-t < i \leqq k, \quad i \leqq s.$$

(If $i > s$ then conditions b) and c) hold vacuously).

(20) Let s_0 be the greatest integer $\geqq 1$ satisfying the conditions of (19).

We have to prove $s_0 \geqq k$. We assume

(21) $$s_0 < k,$$

and we finish the proof of case B) by obtaining a contradiction.

(22) Let v_0 be the least integer for which there exists a graph $\bigcirc^{s_0}$ satisfying the conditions of (19) with $s = s_0$, such that $v(\bigcirc^{s_0}) = v_0$. Put briefly $H = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_{s_0}$ for such an $\bigcirc^{s_0} = \overline{\mathcal{C}}_1 + \ldots + \overline{\mathcal{C}}_{s_0}$.

(23) Let u be the maximal number $1 \leqq u \leqq l+1$ for which there exists a graph $\bigcirc^{s_0}$ satisfying the conditions of (19) such that $v(\bigcirc^{s_0}) = v_0$ and such that there exists a path $\mathcal{P} = \mathcal{P}[P_1, \ldots, P_u]$ with $\overline{\mathcal{P}} \subseteqq G - H$.

We are going to prove that $u = l+1$. To prove this let

(24) $\bigcirc_0^{s_0} = \mathcal{C}_1^0 + \ldots + \mathcal{C}_{s_0}^0$ be a graph satisfying the conditions of (19), such that $v(\bigcirc_0^{s_0}) = v_0$, $H_0 = \overline{\mathcal{C}}_1^0 \cup \ldots \cup \overline{\mathcal{C}}_{s_0}^0$ and let $\mathcal{P} = \mathcal{P}[P_1, \ldots, P_u]$ be a path $\mathcal{P} \subseteqq \mathcal{G}$ such that $\overline{\mathcal{P}} \subseteqq G - H_0$.

Put

(25) $$D = G - H_0, \quad D_1 = D - \overline{\mathcal{P}}$$

and assume that

(26) $$u < l+1.$$

(The assertions (27)—(35) will all depend on the indirect assumption (26).) It follows from (26) that

(27) $\mathcal{G}(D)$ does not contain a circuit.

In fact if $\mathcal{C}' \subseteqq \mathcal{G}(D)$ then if $v(\mathcal{C}') \leqq l$ then by (24) $\bigcirc^{s_0+1} = \mathcal{C}_0^1 + \ldots + \mathcal{C}_{s_0}^0 + \mathcal{C}'$ would be an $\bigcirc^{s_0+1}$ graph satisfying the conditions of (19) with $s = s_0+1$ in contradiction with (20). Hence $\mathcal{C}' \subseteqq \mathcal{G}(D)$ implies $v(\mathcal{C}') \geqq l+1$. But then by (24) and (25) $\mathcal{G}(D)$ would contain a path of length $> u$ in contradiction with (23).

Considering (23), (24) and (25) we have $v(P_1, \mathcal{G}, D_1) = 0$ and $v(P_u, \mathcal{G}, D_1) = 0$ and it follows from (27) that $v(P_1, \mathcal{G}, \overline{\mathcal{P}}) = v(P_u, \mathcal{G}, \overline{\mathcal{P}}) \leqq 1$. Hence we have

(28) $$v(P_1, \mathcal{G}, D) \leqq 1, \quad v(P_u, \mathcal{G}, D) \leqq 1.$$

Now we show

(29) D_1 is non empty.

Considering (19) and (24) we have

$$|H_0| \leqq ls_0 \quad \text{if} \quad s_0 < k-t,$$
$$|H_0| \leqq l(k-t)+\big(s_0-(k-t)\big)(l+1) \quad \text{if} \quad s_0 \geqq k-t.$$

It results from (19), (24) and (25) that

$$|D_1| = |G|-u-|H_0|.$$

Thus by the assumption and by (26) we have

$$|D_1| \geqq lk+t-l-ls_0 \quad \text{if} \quad s_0 < k-t,$$
$$|D_1| \geqq lk+t-l-l(k-t)-\big(s_0-(k-t)\big)(l+1) \quad \text{if} \quad s_0 \geqq k-t.$$

Considering that $lk+t = l(k-t)+(l+1)t$ we get

$$|D_1| \geqq l(k-1-s_0)+t \quad \text{if} \quad s_0 < k-t.$$
$$|D_1| \geqq (l+1)(k-s_0)-l \quad \text{if} \quad s_0 \geqq k-t.$$

Considering (21) we obtain that $|D_1|>0$ except if $s_0 < k-t$, $k-1 = s_0$, $t=0$, $u=l$, $|H_0| = l(k-1)$ and as a consequence of these and of (19), (24)

$$v(\mathcal{C}_i^0)=l \quad \text{for} \quad 1 \leqq i \leqq k-1.$$

But in this case $|D_1|=0$ leads to a contradiction as follows. Considering that $t=0$ and that $n = lk+t > 3k$ (by B)) we have $l \geqq 4$. It follows from (28) that $v(P_1, \mathcal{G}, D) \leqq 1$ hence by the assumption $v(P_1, \mathcal{G}, H_0) \geqq 2k-1$. Considering again (19), (24) and (25) we obtain that there is an i_0, $1 \leqq i_0 \leqq j_0 = k-1$ such that $v(P_1, \mathcal{C}, \mathcal{C}_{i_0}^0) \geqq 3$. Using $v(\mathcal{C}_{i_0}^0)=l \geqq 4$ we can apply Lemma 4 for the graph $\mathcal{G}(\mathcal{C}_{i_0}^0 \cup \{P_1\})$. It results that it contains a circuit $\mathcal{C}'$ with $v(\mathcal{C}')<v(\mathcal{C}_i^0)$. But then $\mathcal{C}_1^0+\ldots+\mathcal{C}_{i_0-1}^0+\mathcal{C}'+\mathcal{C}_{i_0+1}^0+\ldots+\mathcal{C}_{s_0}^0$ is an $\mathcal{O}^{s_0}$ graph satisfying the conditions of (19) with $v(\mathcal{O}^{s_0})<v(\mathcal{O}_0^{s_0})=v_0$ in contradiction with (22). Thus (29) is proved.

If $v(P, \mathcal{G}, D_1) \geqq 2$ for every $P \in D_1$ then by (29) $\mathcal{G}(D_1)$ contains a circuit in contradiction with (27). Thus there exists a P_0 such that

(30) $$P_0 \in D_1 \quad \text{and} \quad v(P_0, \mathcal{G}, D_1) \leqq 1.$$

It follows again from (27) that $v(P_0, \mathcal{G}, \overline{\mathcal{P}}) \leqq 1$ thus by (25) and (30) we have

(31) $$P_0 \in D \quad \text{and} \quad v(P_0, \mathcal{G}, D) \leqq 2.$$

Considering the assumption, (28) and (31) we obtain thet

(32) $$v(P_0, \mathcal{G}, H_0)+v(P_1, \mathcal{G}, H_0)+v(P_u, \mathcal{G}, H_0) \geqq 6k-2-1-1 > 6(k-1)$$

if $P_1 \neq P_u$ i. e. if $u>1$,
and

$$v(P_0, \mathcal{G}, H_0)+v(P_1, \mathcal{G}, H_0) \geqq 4k-2-1 > 4(k-1)$$

if $u=1$.

Put

(33) $D_2=\{P_0, P_1, P_u\}$. Then $D_2 \subseteqq D$ and $|D_2|=3$ if $u>1$, $|D_2|=2$ if $u=1$.

Considering (21), (24) and (25) it results from (32) and (33) that there exists an i_0, $1 \leqq i_0 \leqq s_0 \leqq k-1$ such that

(34) $$\sum_{P \in D_2} v(P, \mathcal{G}, \overline{\mathcal{C}}^0_{i_0}) \geqq \begin{cases} 7 & \text{if} \quad u>1 \\ 5 & \text{if} \quad u=1. \end{cases}$$

It follows from (34) that $v(P', \mathcal{G}, \overline{\mathcal{C}}^0_{i_0}) \geqq 3$ for a $P' \in D_2$ and it follows that

(35) $$\nu(\mathcal{C}^0_{i_0})=3.$$

Since if $\nu(\mathcal{C}^0_{i_0})>3$ then by Lemma 4 $\mathcal{G}(\overline{\mathcal{C}}^0_{i_0} \cup \{P'\})$ contains a circuit $\mathcal{C}'$ with $\nu(\mathcal{C}')<\nu(\mathcal{C}^0_{i_0})$ and then $\bigcirc^{s_0} = \mathcal{C}^0_1+\dots\mathcal{C}^0_{i_0-1}+\mathcal{C}'+\mathcal{C}^0_{i_0+1}+\dots+\mathcal{C}'_{s_0}$ would **be** an $\bigcirc^{s_0}$ graph satisfying (19) with $\nu(\bigcirc^{s_0})<\nu_0$ in contradiction with (22).

Now we need the following

LEMMA 5. *Let $\mathcal{H}$ be a graph with $\nu(\mathcal{H}) = v+4$, $v \geqq 1$. $H=\{Q_1, Q_2, Q_3, P_0, P_1, \dots, P_v\}$. Suppose that $\mathcal{C}=\mathcal{C}[Q_1, Q_2, Q_3] \subseteqq \mathcal{H}$, $\mathcal{P}=\mathcal{P}[P_1, \dots, P_v] \subseteqq \mathcal{H}$. Put $B=\{P_0, P_1, P_v\}$ and suppose further that*

$$\sum_{P \in B} v(P, \mathcal{H}, \overline{\mathcal{C}}) \geqq \begin{cases} 7 & \text{if} \quad v>1 \\ 5 & \text{if} \quad v=1. \end{cases}$$

Then there exists a triangle $\mathcal{C}'$ and a path $\mathcal{P}'$ of length $v+1$ such that $\mathcal{C}' \subseteqq \mathcal{H}$, $\mathcal{P}' \subseteqq \mathcal{H}$ and $\overline{\mathcal{C}}' \cap \overline{\mathcal{P}}'=0$.

Now we may apply Lemma 5 for the graph $\mathcal{H}=\mathcal{G}(\overline{\mathcal{C}}^0_{i_0} \cup \mathcal{P} \cup \{P_0\})$ with $\mathcal{C}=\mathcal{C}^0_{i_0}$, $B=D_2$, by (33), (34) and (35). Thus there exists a triangle $\mathcal{C}$ and a path $\mathcal{P}'$ of length $u+1$ satisfying the conditions $\mathcal{C}' \subseteqq \mathcal{G}(\overline{\mathcal{C}}^0_{i_0} \cup \overline{\mathcal{P}} \cup \{P_0\})$, $\mathcal{C}' \subseteqq \mathcal{G}(\overline{\mathcal{C}}^0_{i_0} \cup \overline{\mathcal{P}} \cup \{P_0\})$ and $\overline{\mathcal{C}}' \cap \overline{\mathcal{P}}'=0$.

It follows that the graph $\bigcirc^{s_0} = \mathcal{C}^0_1+\dots+\mathcal{C}^0_{i_0-1}+\mathcal{C}'+\mathcal{C}^0_{i_0+1}+\dots+\mathcal{C}^0_{s_0}$ satisfies the conditions of (19) and $\nu(\bigcirc^{s_0})=\nu(\bigcirc^{s_0}_0)=\nu_0$ (by (35)). Hence the number $u+1$ satisfies the conditions of (23) with this graph $\bigcirc^{s_0}$ and with the path $\mathcal{P}'$. This contradicts (23) and it follows that the assumption (26) $u < l+1$ leads to a contradiction. Thus we have

(36) $$u = l+1.$$

We may assume that $\bigcirc^{s_0}_0$ and $\mathcal{P}$ satisfy the conditions of (24) with $u = l+1$. Considering that in (24) $\mathcal{P}$ can be, choosen for an arbitrary path of length $l+1$ contained in $\mathcal{G}(D)$ and that (19), (20) and (24) imply that $\mathcal{G}(D)$ does not contain a circuit of length $\leqq l$ we may assume that

(37) one of the following statements (i), (ii) holds

(i) $\mathcal{G}(D)$ does not contain a circuit of length $\leqq l+1$,

(ii) $\mathcal{C}= \mathcal{C}[P_1, \dots, P_{l+1}] \subseteqq \mathcal{G}$,

where $\mathcal{P}= \mathcal{P}[P_1, \dots, P_{l+1}]$ denotes the path satisfying the conditions of (24).

We need some further definitions and notations.

(38) Put $c_i = v(\mathcal{C}_i^0)$ for $1 \leqq i \leqq s_0$.
Then

$$v_0 = \sum_{i=1}^{s_0} c_i.$$

Put further

(39) $$V = \sum_{j=1}^{l+1} v(P_j, \mathcal{G}, H_0), \quad V_1 = \sum_{j=1}^{l+1} v(P_j, \mathcal{G}, \overline{\mathcal{P}}), \quad V_2 = \sum_{j=1}^{l+1} v(P_j, \mathcal{G}, D_1).$$

Considering the assumption and (25) we have

(40) $$V + V_1 + V_2 = \sum_{j=1}^{l+1} v(P_j, \mathcal{G}) \geqq 2k(l+1).$$

We need a lower estimation for V. First we prove some preliminaries.

(41) $$v(Q, \mathcal{G}, \overline{\mathcal{P}}) \leqq 2 \quad \text{for every} \quad Q \in D_1.$$

For if not then $(P_{j_1}, Q), (P_{j_2}, Q), (P_{j_3}, Q) \in \mathcal{G}^*$ for some $1 \leqq j_1 < j_2 < j_3 \leqq l+1$ and the circuits $\mathcal{C}' = \mathcal{C}'[Q, P_{j_1}, \ldots, P_{j_2}]$, $\mathcal{C}'' = \mathcal{C}''[Q, P_{j_2}, \ldots, P_{j_3}]$ belong to $\mathcal{G}$ and at least one of them is of length $\leqq l$.

Thus by (24) either $O_0^{s_0} + \mathcal{C}'$ or $O_0^{s_0} + \mathcal{C}''$ is an O^{s_0+1} graph satisfying the conditions of (19) in contradiction with (20).

(42) Let D_2 be the set of those $Q \in D_1$ for which $v(Q, \mathcal{G}, \overline{\mathcal{P}}) = 2$.

We prove:

(43) Suppose $l \geqq 4$ or $l = 3$, $k - t \leqq s_0$. Then

$$|D_2| \leqq 1 \text{ if (37) (i) holds;}$$
$$|D_2| = 0 \text{ if (37) (ii) holds.}$$

If (i) holds and $v(Q_1, \mathcal{G}, \overline{\mathcal{P}}) = y(Q_2, \mathcal{G}, \overline{\mathcal{P}}) = 2$ for $Q_1 \neq Q_2 \in D_1$ then (Q_1, P_1); (Q_1, P_{l+1}); (Q_2, P_1); $(Q_2, P_{l+1}) \in \mathcal{G}^*$ for if not then $\mathcal{G}(D)$ contains a circuit of length $\leqq l+1$. But then $\mathcal{C}' = \mathcal{C}'[Q_1, P_1, Q_2, P_{l+1}]$ is a circuit of length $4 \leqq l+1$ contained in $\mathcal{G}$ and this is a contradiction.

Suppose now that (ii) holds. Then $s_0 < k - t$ for if not then by (24) $O_0^{s_0} + \mathcal{C}[P_1, \ldots, P_{l+1}]$ is an O^{s_0+1} graph satisfying (19) in contradiction with (20). Hence we may assume $l \geqq 4$. That means $v(\mathcal{C}[P_1, \ldots, P_{l+1}]) > 4$, and then $|D_2| > 0$ leads to a contradiction for if $Q \in D_2$ then $v(Q, \mathcal{G}, \overline{\mathcal{C}}) = 2$ and then by Lemma 4 $\mathcal{G}(\overline{\mathcal{C}} \cup \{Q\})$ would contain a circuit $\mathcal{C}'$ of length $\leqq l < l+1$. But then by (24) $O_0^{s_0} + \mathcal{C}'$ would be an O^{s_0+1} graph satisfying (19) in contradiction with (20). Thus (43) is proved.

Considering again that, by the maximality (20) of s_0, $\mathcal{G}(D)$ does not contain a circuit $\mathcal{C}'$ with $v(\mathcal{C}') \leqq l$ if $s_0 < k - t$ and with $v(\mathcal{C}') \leqq l+1$ if $s_0 \geqq k - t$ we have

(44) $$V_1 = 2(l+1) - 2 \quad \text{if (37) (i) holds}$$
$$V_1 = 2(l+1) \quad \text{if (37) (ii) holds.}$$

Considering that by (25) and (39) $V_2 = \sum_{Q \in D_1} v(Q, \mathcal{G}, \bar{\mathcal{P}})$ it follows from (41) that

(45) $$V_2 \leqq |D_1| + |D_2|.$$

Now we are going to prove

(46) $$V > (l+2)s_0 + v_0 \quad \text{if} \quad s_0 < k-t \quad \text{and}$$
$$V > (l+1)s_0 + v_0 \quad \text{if} \quad s_0 \geqq k-t.$$

By (40) we have

$$V \geqq 2k(l+1) - V_1 - V_2.$$

Hence we have

$$V > 2(k-1)(l+1) - V_2 \quad \text{if} \quad (37)\text{ (i) holds,}$$
$$V \geqq 2(k-1)(l+1) - V_2 \quad \text{if} \quad (37)\text{ (ii) holds.}$$

On the other hand by the assumption and by (25) we have

$$|D_1| = lk + t - (l+1) - v_0 = (l+1)(k-1) - (k-t) - v_0.$$

Using (45) we obtain

(O) $$V > (l+1)(k-1) + k - t + v_0 - |D_2| \quad \text{if} \quad (37)\text{ (i) holds,}$$
$$V \geqq (l+1)(k-1) + k - t + v_0 - |D_2| \quad \text{if} \quad (37)\text{ (ii) holds.}$$

Now we distinguish the cases

(α) $$l \geqq 4 \quad \text{or} \quad l = 3 \quad \text{and} \quad s_0 \geqq k-t,$$

($\alpha\alpha$) $$l = 3 \quad \text{and} \quad s_0 < k-t.$$

Ad (α): It follows from (O) (37) and (43) that $V \geqq (l+1)(k-1) + k - t + v_0$ holds.

Suppose first $s_0 \geqq k-t$. Considering that by (21) $s_0 \leqq k-1$ and that $k-t>0$ by the assumption we get

$$(l+1)(k-1-s_0) + k - t > 0$$

hence

$$V > (l+1)s_0 + v_0.$$

Suppose now $s_0 < k-t$. Then considering that $0 \leqq t \leqq k-1$

$$V \geqq (l+1)(k-1) + k - t + v_0 = (l+2)(k-t-1) + (l+1)t + 1 + v_0 >$$
$$> (l+2)(k-t-1) + v_0 \geqq (l+2)s_0 + v_0.$$

Hence (46) is proved if (α) holds.

Ad ($\alpha\alpha$): $l=3$, $s_0 < k-t$.

Considering that by (42) we have $|D_2| \leqq |D_1|$ we obtain from (O) that

$$V \geqq 2(k-t) + 2v_0.$$

Considering that $(\alpha\alpha)$ holds it follows from (19) and (24) that $v_0 = 3s_0$, hence

$$V \geqq 2(k-t)+3s_0+v_0 > 5s_0+v_0 = (l+2)s_0+v_0$$

holds in this case too and (46) is proved.

We prove:

(47) There exists an i_0, $1 \leqq i_0 \leqq s_0$ such that

$$\sum_{j=1}^{l+1} v(P_j, \mathcal{G}, \overline{\mathcal{C}}_{i_0}^0) \geqq l+3+c_{i_0} \quad \text{if} \quad s_0 < k-t,$$

$$\sum_{j=1}^{l+1} v(P_j, \mathcal{G}, \overline{\mathcal{C}}_{i_0}^0) \geqq l+2+c_{i_0} \quad \text{if} \quad s_0 \geqq k-t.$$

Suppose that (47) is false. Then by (25) and (38)

$$V = \sum_{i=1}^{s_0} \sum_{j=1}^{l+1} v(P_j, \mathcal{G}, \overline{\mathcal{C}}_i^0) \leqq \sum_{i=1}^{s_0} (l+2+c_i) \left(\text{or } \sum_{i=1}^{s_0} (l+1+c_i) \right) \leqq$$

$$\leqq (l+2)s_0+v_0 \quad (\text{or } (l+1)s_0+v_0),$$

respectively, which contradicts (46).

If follows from (47) that there exists a $P_{j_0} \in \overline{\mathcal{P}}$ such that $v(P_{j_0}, \mathcal{G}, \overline{\mathcal{C}}_{i_0}^0) \geqq 2$ for if not then $\sum_{j=1}^{l+1} v(P_j, \mathcal{G}, \overline{\mathcal{C}}_{i_0}^0) \leqq l+1$.

If follows that

(48) $$v(\overline{\mathcal{C}}_{i_0}^0) \leqq 4.$$

For if not then by Lemma 4 $\mathcal{G}(\overline{\mathcal{C}}_{i_0}^0 \cup \{P_{j_0}\})$ contains a circuit $\mathcal{C}'$ with $v(\mathcal{C}') < v(\overline{\mathcal{C}}_{i_0}^0)$ and thus by (24) $\mathcal{O}^{s_0} = \mathcal{C}_1^0 + \ldots + \mathcal{C}_{i_0-1}^0 + \mathcal{C}' + \mathcal{C}_{i_0+1}^0 + \ldots + \mathcal{C}_{s_0}^0$ would be an $\mathcal{O}^{s_0}$ graph satisfying (19) with $v(\mathcal{O}^{s_0}) < v_0$ in contradiction with (20).

Now we need the following lemmas

LEMMA 6. *Let $\mathcal{H}$ be a graph, $\mathcal{C}$ a circuit and $\mathcal{P}$ a path, $\mathcal{C} \subseteqq \mathcal{H}$, $\mathcal{P} \subseteqq \mathcal{H}$, $\overline{\mathcal{C}} \cap \overline{\mathcal{P}} = 0$, $\overline{\mathcal{C}} \cup \overline{\mathcal{P}} = H$. Suppose that $l \geqq 3$, $v(\mathcal{C}) \leqq \min(l, 4)$ and $v(\mathcal{P}) = l+1$. Suppose further that*

$$\sum_{P \in \overline{\mathcal{P}}} v(P, \mathcal{H}, \overline{\mathcal{C}}) \geqq l+3+v(\mathcal{C}).$$

Then $\mathcal{H}$ contains two independent circuits $\mathcal{C}'$, $\mathcal{C}''$ such that $v(\mathcal{C}') \leqq l$, $v(\mathcal{C}'') \leqq l$.

LEMMA 7. *Let $\mathcal{H}$ be a graph, $\mathcal{C}$ a circuit and $\mathcal{P}$ a path, $\mathcal{C} \subseteqq \mathcal{H}$, $\mathcal{P} \subseteqq \mathcal{H}$, $\overline{\mathcal{C}} \cap \overline{\mathcal{P}} = 0$, $\overline{\mathcal{C}} \cup \overline{\mathcal{P}} = H$. Suppose that $l \geqq 3$ $v(\mathcal{C}) \leqq \min(l+1, 4) = 4$ and $v(\mathcal{P}) = l+1$. Suppose further that*

$$\sum_{P \in \overline{\mathcal{P}}} v(P, \mathcal{H}, \overline{\mathcal{C}}) \geqq l+2+v(\mathcal{C}).$$

Then $\mathcal{H}$ contains two independent circuits $\mathcal{C}', \mathcal{C}''$ such that $v(\mathcal{C}') \leqq l+1$, $v(\mathcal{C}'') \leqq l+1$. As a corollary of this if in addition $v(\mathcal{C}) \leqq l$ holds then $\min\left(v(\mathcal{C}'), v(\mathcal{C}'')\right) \leqq l$ holds too.

Consider now the graph $\mathcal{H} = \mathcal{G}(\overline{\mathcal{C}}_{i_0}^0 \cup \overline{\mathcal{P}})$ and distinguish the cases (β) $s_0 < k-t$, $(\beta\beta)$ $s_0 \geqq k-t$.

Considering that if (β) holds then $v(\mathcal{C}_{i_0}^0) \leqq l$ the graph $\mathcal{H}$, the circuit $\mathcal{C} = \mathcal{C}_{i_0}^0$, and the path $\mathcal{P}$ satisfy all the conditions of Lemma 6 and Lemma 7 if (β) or $(\beta\beta)$ holds respectively. Suppose now that (β) holds. Then by (24) (25) and Lemma 6, there exist circuits $\mathcal{C}', \mathcal{C}''$ such that $\mathcal{C}_1^0 + \ldots + \mathcal{C}_{i_0-1}^0 + \mathcal{C}' + \mathcal{C}_{i_0+1}^0 + \ldots + \mathcal{C}_{s_0}^0 + \mathcal{C}''$ is an $\bigcirc^{s_0+1}$ graph satisfying the requirements of (19) in contradiction with (20).

Suppose that $(\beta\beta)$ holds. Then by Lemma 7 there exist circuits $\mathcal{C}', \mathcal{C}''$ satisfying the requirements of Lemma 7. By (24) and (25) $\mathcal{C}_1^0 + \ldots + \mathcal{C}_{i_0-1}^0 + \mathcal{C}' + \mathcal{C}_{i_0+1}^0 + \ldots + \mathcal{C}_{s_0}^0 + \mathcal{C}''$ is an $\bigcirc^{s_0+1}$ graph satisfying the requirements of (19) since if $i_0 \leqq k-t$ then $v(\mathcal{C}_{i_0}^0) \leqq l$, hence then we may assume $v(\mathcal{C}') \leqq l$. This contradicts (20).

We obtained a contradiction in both cases hence the indirect assumption (21) is false and Theorem 2 is proved in case B) too.

§ 4. Proof of the Lemmas

PROOF OF LEMMA 2. Assume that that $\mathcal{H}$ does not contain two independent triangles.

First we show that

$$v(P_1, \mathcal{H}, \overline{\mathcal{C}}) \leqq 2; \quad v(P_3, \mathcal{H}, \overline{\mathcal{C}}) \leqq 2. \tag{49}$$

By symmetry it is sufficient to prove the first statement. Suppose that $v(P_1, \mathcal{H}, \mathcal{C}) = 3$. Considering that by the assumptions we have

$$\sum_{j=1}^{3} v(P_j, \mathcal{H}, \overline{\mathcal{C}}) = \sum_{i=1}^{3} v(Q_i, \mathcal{H}, \overline{\mathcal{P}}) \geqq 7$$

we obtain

$$\sum_{i=1}^{3} v\left(Q_i, \mathcal{H}, \{P_2, P_3\}\right) \geqq 4$$

and as a consequence of this $v\left(Q_{i_0}, \mathcal{H}, \{P_2, P_3\}\right) = 2$ for an i_0, $1 \leqq i_0 \leqq 3$. By symmetry we may assume that $i_0 = 1$. But then $\mathcal{C}' = \mathcal{C}'[P_1, Q_2, Q_3]$, $\mathcal{C}'' = \mathcal{C}''[Q_1, P_2, P_3]$ are two independent triangles contained in $\mathcal{H}$. This contradicts our assuption, hence (49) is proved.

As a corollary of the assumption $\sum_{j=1}^{3} v(P_j, \mathcal{H}, \overline{\mathcal{C}}) \geqq 7$ using (49), we obtain

$$v(P_1, \mathcal{H}, \overline{\mathcal{C}}) = v(P_3, \mathcal{H}, \overline{\mathcal{C}}) = 2, \quad v(P_2, \mathcal{H}, \overline{\mathcal{C}}) = 3. \tag{50}$$

By symmetry we may assume $(P_1, Q_1) \notin \mathcal{H}^*$ and we have to prove that then $(P_3, Q_1) \notin \mathcal{H}^*$. Suppose that $(P_3, Q_1) \in \mathcal{H}^*$. Then by (50) the triangles $\mathcal{C}' = \mathcal{C}'[P_1, Q_2, Q_3]$, $\mathcal{C}'' = \mathcal{C}''[Q_1, P_2, P_3]$ belong to $\mathcal{H}$ and we obtain a contradiction. This proves Lemma 2.

PROOF OF LEMMA 3. Put $V=\sum_{j=1}^{4} v(P_j, \mathcal{H}, \bar{\mathcal{C}})$. The assumption $V \geqq 9$ implies that $v(P_{j_0}, \mathcal{H}, \bar{\mathcal{C}})=3$ for a j_0, $1 \leqq j_0 \leqq 4$, for if not then $V \leqq 4 \cdot 2=8$, and by symmetry we may assume that

(51) $$V(P_1, \mathcal{H}, \bar{\mathcal{C}})=3.$$

We have to prove that there exist independent triangles $\mathcal{C}'$, $\mathcal{C}''$ satisfying the conditions

(52) $$\mathcal{C}' \subseteqq \mathcal{H}, \quad \mathcal{C}'' \subseteqq \mathcal{H}, \quad H=\bar{\mathcal{C}}' \cup \bar{\mathcal{C}}'' \cup \{P_j\} \quad \text{where} \quad 2 \leqq j \leqq 4.$$

Assume that (52) is false. Then $v(Q_i, \mathcal{H}, \{P_3, P_4\}) \leqq 1$ for $1 \leqq i \leqq 3$. For if not then by symmetry we may assume $v(Q_1, \mathcal{H}, \{P_3, P_4\})=2$ and considering that by the assumption $(P_3, P_4) \in \mathcal{H}^*$ the triangles $\mathcal{C}'=\mathcal{C}'[P_1, Q_2, Q_3]$, $\mathcal{C}''=$ $=\mathcal{C}''[Q_1, P_3, P_4]$ belong to $\mathcal{H}$ and satisfy (52) with $P_j=P_2$. If follows that

(53) $$\sum_{i=1}^{3} v(Q_i, \mathcal{H}, \{P_3, P_4\}) \leqq 3.$$

Taking into consideration that

$$V = v(P_1, \mathcal{H}, \bar{\mathcal{C}})+v(P_2, \mathcal{H}, \bar{\mathcal{C}})+\sum_{i=1}^{3} v(Q_i, \mathcal{H}, \{P_3, P_4\})$$

it follows from (51) and (53) that

(54) $$v(P_2, \mathcal{H}, \bar{\mathcal{C}})=3.$$

On the other hand $v(P_j, \mathcal{H}, \bar{\mathcal{C}}) \geqq 2$ either for $j=3$ or for $j=4$, for if not then $v \leqq 3+3+1+1 = 8$. By symmetry we may assume

(55) $$v(P_3, \mathcal{H}, \bar{\mathcal{C}}) \geqq 2 \quad \text{and} \quad (P_3, Q_2) \in \mathcal{H}^*; \quad (P_3, Q_3) \in \mathcal{H}^*.$$

Considering that by the assumption $(P_1, P_2) \in \mathcal{H}^*$ then the triangles

$$\mathcal{C}'=\mathcal{C}'[P_3, Q_2, Q_3], \quad \mathcal{C}''=\mathcal{C}''[P_1, P_2\, Q_1]$$

satisfy by (51), (54) and (55) the conditions of (52) with $P_j=P_4$. Thus we obtained a contradiction and Lemma 3 is proved.

PROOF OF LEMMA 4. Suppose first that a) holds. Then $r>4$, and $(Q_{j_1}, P) \in \mathcal{H}^*$, $(Q_{j_2}, P) \in \mathcal{H}^*$ for some $1 \leqq j_1 < j_2 \leqq r$. Then the circuits

$$\mathcal{C}'=\mathcal{C}'[P, Q_{j_1}, \dots Q_{j_2}], \quad \mathcal{C}''=\mathcal{C}''[P, Q_{j_2}, \dots Q_r, Q_1 \dots Q_{j_1}]$$

belong to $\mathcal{H}$ by the assumption $\mathcal{C} \subseteqq \mathcal{H}$. We have to prove that one of them is of length $\leqq r$.

In fact $v(\mathcal{C}')+v(\mathcal{C}'') = r+4<2r$ if $r>4$ hence either $v(\mathcal{C}')<r$ or $v(\mathcal{C}'')<r$.

Suppose that b) holds. If $r>4$ then the satement is true by the case a) of Lemma 4 already proved. Thus we may assume that $v(\mathcal{C})=4$. But then by the assumption $(Q_j, P) \notin \mathcal{H}^*$ holds for at most one j, by symmetry we may assume that if any then $(Q_1, P) \notin \mathcal{H}^*$. But then e. g. $\mathcal{C}'=\mathcal{C}'[P, Q_2, Q_3] \subseteqq \mathcal{H}$ and $3=v(\mathcal{C}')<v(\mathcal{C})=4$.

PROOF OF LEMMA 5. Put $V=\sum_{P\in B} v(P,\mathcal{H},\overline{\mathcal{C}})$. It follows from the assumptions $V\geqq 7$ (or $V\geqq 5$ respectively) that there exists a $P\in B$ such that

$$v(P,\mathcal{H},\overline{\mathcal{C}})=3. \tag{56}$$

Suppose first that P_0 satisfies (56). Then by the assumption $V\geqq 7$ or $V\geqq 5$, respectively, there is a Q_{i_0} for which $(Q_{i_0},P_1)\in\mathcal{H}^*$. We may assume that $i_0=1$. Then the triangle $\mathcal{C}'=\mathcal{C}'[P_0,Q_2,Q_3]$ and the path $\mathcal{P}'=\mathcal{P}'[Q_1,P_1,\dots,P_v]$ obviously satisfy the requirements of Lemma 5. Thus we may assume that (56) holds either for $P=P_1$ or for $P=P_v$. By symmetry we may assume

$$v(P_1,\mathcal{H},\overline{\mathcal{C}})=3. \tag{57}$$

Suppose now that $v=1$. Then by the assumption $V\geqq 5$ $(Q_{i_0},P_0)\in\mathcal{H}^*$ for an i_0 and we may assume $i_0=1$. Then the triangle $\mathcal{C}'=\mathcal{C}'[P_1,Q_2,Q_3]$ and the path $\mathcal{P}'=\mathcal{P}'[Q_1,P_0]$ of length $v+1=2$ satisfy the requirements of Lemma 5. Hence we may assume $v>1$.

Considering that by the assumption

$$7\leqq V=v(P_1,\mathcal{H},\overline{\mathcal{C}})+\sum_{i=1}^{3} v(Q_i,\mathcal{H},\{P_0,P_v\})$$

it follows that $\sum_{i=1}^{3} v(Q_i,\mathcal{H},\{P_0,P_v\})\geqq 4$. As a consequence of this $v(Q_{i_0},\mathcal{H},\{P_0,P_v\})=2$ for an i_0 and by symmetry we may assume $i_0=1$. Hence we have

$$V(Q_1,\mathcal{H},\{P_0,P_v\})=2. \tag{58}$$

Put $\mathcal{C}'=\mathcal{C}'[P_1,Q_2,Q_3]$, $\mathcal{P}'=\mathcal{P}'[P_2,\dots,P_v,Q_1,P_0]$. We have $\overline{\mathcal{C}}'\cap\overline{\mathcal{P}}'=0$. By the assumption $\mathcal{C}\subseteqq\mathcal{H}$ and by (57) we have $\mathcal{C}'\subseteqq\mathcal{H}$. By the assumption $\mathcal{P}\subseteqq\mathcal{H}$ and by (58) we obtain that $\mathcal{P}'\subseteqq\mathcal{H}$ and $v(\mathcal{P}')=v-1+2=v+1$. Hence $\mathcal{C}'$ and $\mathcal{P}'$ satisfy the requirements of Lemma 5.

PROOF OF THE LEMMAS 6 AND 7. Put $v(\mathcal{C})=r$, $V=\sum_{P\in\mathcal{P}} V(P,\mathcal{H},\overline{\mathcal{C}})$. Put further $\mathcal{C}=\mathcal{C}[Q_1,\dots,Q_r]$, $\mathcal{P}=\mathcal{P}[P_1,\dots,P_{l+1}]$. We use induction on l. One can verify by a discussion that Lemma 6 is true if $l=r$, $l=r+1$ for $r=3,4$ and that Lemma 7 is true if $l=r-1$, $l=r$ for $r=4$ and if $l=r=3$. For example the special case $r=3$, $l=3$ of Lemma 6 is a corollary of Lemma 3. We omit the proof of the other special cases which can be carried out using the same ideas.

Thus we assume that $l>r+1$ and $l>r$ in case of Lemma 6 and Lemma 7, respectively, and that both lemmas are true for $l-1$. We may suppose that $V=l+3+r$ or $V=l+2+r$ respectively. First we prove that there exists an i_0, $1\leqq i_0\leqq l+1$ such that $v(P_{i_0},\mathcal{H},\overline{\mathcal{C}})\leqq 1$.

For if not than $V\geqq 2l+2$ and $2l+2>l+3+r$ if $l>r+1$; $2l+2>l+2+r$ if $l>r$. Let $\mathcal{H}_1$ be the graph defined by the following stipulations. Put $H_1=H-\{P_{i_0}\}$.

$$\mathcal{H}_1=\langle H_1,\mathcal{H}^*(H_1)\cup\{(P_{i_0-1},P_{i_0+1})\}\rangle \quad\text{if}\quad 1<i_0<l+1$$

and

$$\mathcal{H}_1=\mathcal{H}(H_1)\quad\text{if}\quad i_0=1\ \text{or}\ i_0=l+1.$$

$\mathcal{H}_1$ contains the path $\mathcal{P}_1 = \mathcal{P}_1[P_1, \ldots, P_{i_0-1}, P_{i_0+1}, \ldots, P_{l+1}]$ of length l, and the circuit $\mathcal{C}$. Then

$$V_1 = \sum_{P \in \overline{\mathcal{P}}_1} v(P, \mathcal{H}_1, \overline{\mathcal{C}}) \geqq l-1+3+r \quad (\text{or } l-1+2+r),$$

respectively. Considering that $l-1 \geqq 3$, $l-1 \geqq r$, $\mathcal{H}_1$ satisfies the assumptions of Lemma 6 and Lemma 7 for $l-1$, respectively. Thus $\mathcal{H}_1$ contains two independent circuits $\mathcal{C}_1'$, $\mathcal{C}_1''$ of length $\leqq l-1$ or of length $\leqq l$, respectively. Considering that $\mathcal{C}_1'$, $\mathcal{C}_1''$ are independent, at most one ot them contains the edge (P_{i_0-1}, P_{i_0+1}). Replacing this if necessary by the path $\mathcal{P}' = \mathcal{P}'[P_{i_0-1}, P_{i_0}, P_{i_0+1}]$ we get two independent circuits of $\mathcal{C}'$, $\mathcal{C}''$ of length $\leqq l$ or of length $\leqq l+1$ contained in $\mathcal{H}$, respectively.

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