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LOCAL NON-SQUEEZING THEOREMS AND STABILITY

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Dedicated to Misha Gromov on the occasion of his 50th birthday.

1. Introduction

One of the most fundamental results in symplectic topology is the non-squeezing theorem which asserts that there is no symplectic embedding which takes a standard $2n+2$-ball of radius 1 into a cylinder $(M\times D^2(a), \omega\oplus\sigma)$ whose base $D^2(a)$ is a closed 2-disc of σ-area $a < \pi$. This was first proved by Gromov ([G]) for a range of manifolds including standard Euclidean space, and was generalized to all manifolds by Lalonde–McDuff ([LM1]). In this paper we consider "local" versions of this theorem. The word local can here be interpreted in two ways. Sometimes we localize in space and think of embedding not a whole set such a ball or ellipsoid but just its germ along a central 2-disc $0 \times D$. Sometimes we localize in time and look for embeddings which are close to a given inclusion.

Our problem can be formulated as follows. Let $(W, \Omega) = (M \times D, \omega \oplus \sigma)$ be a symplectic cylinder, where D is a closed 2-disc of σ-area π and (M, ω) is some symplectic manifold. Suppose that S is a compact subset of W, whose boundary is a smooth hypersurface. When can S be moved symplectically to lie strictly inside W? The main Theorem below gives an essentially complete answer to this question. As one might expect, the answer lies in the geometry of S near the points which meet the boundary ∂W of the cylinder. The interesting case is when S meets ∂W along some closed characteristic $x \times \partial D$, and we will see that our problem is closely connected to the properties of the linearization of the characteristic flow around this closed orbit. As a corollary, we prove the sufficiency of the condition for the stability of geodesics in Hofer's metric.

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its application in Proposition 2.1. We are also grateful to Leonid Polterovich for carefully reading the paper and making useful suggestions.

1.1 Local squeezing and characteristic flows. To state our results precisely, we will need the following definitions.

DEFINITION 1.1. *We say that S is* `squeezable by isotopy` *in W if there is a smooth 1-parameter family $\psi_{t \in [0,1]}$ of symplectic embeddings of S in W starting at the inclusion such that*

$$\psi_1(S) \subset \operatorname{Int} W \ ,$$

where $\operatorname{Int} W = M \times \operatorname{Int} D$. S is `locally squeezable by isotopy` *in W if in addition $\psi_t(S) \subset \operatorname{Int} W$ for all $t > 0$. Finally, S is* `locally squeezable` *in W if there is a sequence ψ_i, $i \geq 1$, of symplectic embeddings $S \hookrightarrow \operatorname{Int} W$ such that ψ_i converges C^1 to the inclusion as $i \to \infty$.*

Clearly, a set which is locally squeezable by isotopy is both locally squeezable and squeezable by isotopy. However, the exact relationship between the latter two concepts is somewhat complicated because an isotopy of S in W usually will not extend to an isotopy from W to W. This point is discussed further below.

From now on, we assume that S is a compact subset of W whose boundary is a smooth hypersurface. Observe that all characteristics on the boundary ∂W of W are flat circles $pt \times \partial D$. The following observation is a well-known fact:

LEMMA 1.2. *If $S \cap \partial W$ contains no closed characteristics $pt \times \partial D$ then S is locally squeezable by isotopy in W.*

Proof: To prove this, one constructs a Hamiltonian H on ∂W whose flow points into W at all points of $\partial S \cap \partial W = S \cap \partial W$. For this, we need $\partial H/\partial t < 0$ at all points of $S \cap \partial W$, which is possible if this set contains no closed characteristic (here $t \in [0,1]$ is the angle coordinate of the boundary of the 2-disc D, the base of the cylinder). To be complete, here is a more detailed proof.

Because S is compact, the set $P = S \cap \partial W$ is compact too. Thus $P^c = (M \times S^1) - P$ is an open subset of the manifold $N = M \times S^1$, which contains $(M - K) \times S^1$ for some compact subset $K \subset M$. Because there is no closed loop $\{pt\} \times S^1$ in P, P^c contains some non-empty open interval $\{p\} \times I$ for each $p \in M$. Now choose any smooth function $f : N \to (-\infty, 0]$ which is strictly negative on $K \times S^1$, and has compact support. Let $g : M \to \mathbf{R}$ be its integral over the S^1-factor, and $h : N \to [0, \infty)$ be a smooth positive function with compact support inside the open set P^c, whose integral on each S^1-factor $\{p\} \times S^1$ is equal to $-g(p)$. (The existence of h is obvious: simply take a finite open covering $U_i \subset M$ of the projection on M of supp$f$, and a partition of unity ϕ_i associated to it. Choose open subsets $O_i = U_i \times I_i \subset P^c$, refining the open covering $\{U_i\}$ if needed. Then define for each O_i a positive function h_i with compact support in O_i whose S^1-integral is $-\phi_i g$, and set $h = \sum_i h_i$.)

Hence $f + h$ is a smooth function with compact support in N whose integral over each $\{pt\} \times S^1$ vanishes, and which is negative on P. Fixing a point $t_0 \in S_1$, the integral $\int_{t_0}^{t_0+2\pi}(f + g)$ is a smooth function on N whose derivative with respect to $t \in S^1 = \mathbf{R}/\mathbf{Z}$ is negative everywhere in P, and which vanishes outside some compact set in N. Finally, let $H : M \times \mathbf{R}^2 \to \mathbf{R}$ be any extension of $f + g$, with compact support. Then the flow ϕ_t induced by H sends S strictly inside W for sufficiently small times $t > 0$. □

Therefore the interesting case is when ∂S contains a closed characteristic $x_0 \times \partial D$. Then, because S is a subset of W, ∂S must be tangent to ∂W at all points y of this circle. Since the characteristics of ∂S point along the null direction of $\Omega|_{T_y\partial S} = \Omega|_{T_y\partial W}$, this circle is a characteristic on ∂S as well. What turns out to be crucial is the linearization of the characteristic flow of ∂S around this circle. If we identify all the tangent spaces $T_yM \subset T_y\partial W$ with $\mathbf{R}^{2n} = T_{x_0M}$ in the obvious way, this linearization is a family of symplectic linear maps

$$A_t : \mathbf{R}^{2n} \to \mathbf{R}^{2n} , \quad t \in [0,1] .$$

We will say that A_t has a non-constant closed orbit at time T if there is some fixed point $v \in T_{x_0}M$ of A_T which is not fixed by all the A_t, $t \in [0,T]$. Thus v is in the kernel of $A_t - \mathbb{1}$ for $t = T$ but not for all $t < T$. Here is our main result.

THEOREM 1.3. *Let S be a compact subset of a symplectic cylinder (W, Ω) which intersects ∂W along at least one closed characteristic $x \times \partial D$.*

(i) If $S \cap \partial W$ contains only finitely many closed characteristics $x \times \partial D$, and if the linearized flow around each has a non-constant closed orbit in time < 1, then S is locally squeezable by isotopy in W.

(ii) If there is a closed characteristic in $S \cap \partial W$ along which the linearized flow has no non-constant closed orbit in time ≤ 1, then S is neither locally squeezable nor squeezable by isotopy in W.

Remark 1.4: (*i*) Recall that when $\det(A_1 - \mathbb{1}) \neq 0$, the index of the closed characteristic $x_0 \times \partial D$ is given by counting (with multiplicities) the number of times $T < 1$ at which $\det(A_T - \mathbb{1}) = 0$: see Ekeland [E2] for example. Therefore, the hypothesis in part (*ii*) is equivalent to assuming that the closed characteristic $x_0 \times \partial D$ has the same index on S as it does on W. This suggests that symplectic homology might be directly applicable to prove part (ii). Unfortunately, this does not seem to work, although it does work in the special case of ellipsoids: see § 2.

(ii) We will see that any small neighborhood S of the disc $0 \times D$ in the $(2n+2)$-ball of radius 1 in $\mathbf{R}^{2n} \times D$ is squeezable by isotopy but not locally squeezable. This is a rather exceptional example which occurs because the closed characteristic $0 \times \partial D$ is not isolated in the set of all closed characteristics on ∂S. In general one would expect that if S intersects ∂W along an isolated closed characteristic then S is squeezable by isotopy only if it is also locally squeezable. It is not obvious how to prove this in all cases, though Theorem 1.3 shows that this does hold generically. See also Lemma 2.6.

The above theorem is very closely related with the question of the stability of geodesics in Hofer's metric on the group of Hamiltonian symplectomorphisms which we studied in [LM2,3]. In fact, we will show in § 3 that part (i) is essentially just a restatement of Theorem 1.6 in [LM2] where we used the non-constant closed orbit to construct a local squeezing of S near each closed characteristic in $S \cap \partial W$.

In view of this, we now concentrate on explaining the proof of part (ii). Observe that this is again local in S, that is to say it depends only on the germ of S near the flat disc bounded by the closed characteristics in $S \cap \partial W$. Given such a closed characteristic $x_0 \times \partial D$, choose a Darboux chart near $x_0 \in M$ which takes x_0 to $0 \in \mathbf{R}^{2n}$ and write the ε-neighborhood S_ε of the corresponding disc $0 \times D$ in S as

$$S_\varepsilon = \{(x,c,t) : c \le \pi - H_t(x) \ , \ \|x\| < \varepsilon\} \subset \mathbf{R}^{2n} \times D$$

where $H_t(x) \ge 0$, $H_t(0) = 0$ for all t. Here we have used action-angle coordinates (c,t) on D where $t \in \mathbf{R}/\mathbf{Z}$ as before and $c = \pi r^2$. Thus the area form is $dc \wedge dt$. Therefore, the characteristics of S are given by the Hamiltonian flow of the function $\pi - H_t(x) - c$ and the linearized flow A_t is generated by the 2-jet of $-H_t$. Observe that the linearized flow has a non-constant closed trajectory at time t exactly when A_t has an eigenvalue 1, i.e. $\det(A_t - \mathbb{1}) = 0$. We will call a path $A_{t \in [0,t_0]} \in \mathrm{Sp}(2n, \mathbf{R})$ *short* if $\det(A_t - \mathbb{1}) \neq 0$ for $t > 0$, and *positive* if it is generated by $-Q_t$ where Q_t is a positive definite non-degenerate quadratic Hamiltonian.

Because the conditions considered in part (ii) are open and we may replace S by a slightly smaller set, it suffices to prove the result in the case when H_t is quadratic and non-degenerate for all t. Thus, we will suppose that S_ε is a quadratic slice of the form

$$S_{Q_t,\varepsilon} = \{(x,c,t) : c < \pi - Q_t(x) \ , \ \|x\| < \varepsilon\} \ ,$$

where each Q_t is a positive definite quadratic form on $\mathbf{R}^{2n}$. If $\mathcal{K} = Q_t$ is independent of t, the slice $S_{\mathcal{K},\varepsilon}$ is contained in the ellipsoid

$$E_{\mathcal{K}} = \{(x,c,t) : c + Q(x) \leq \pi\} .$$

In this case part (i) of the above theorem may be proved directly, and part (ii) follows by using the theory of symplectic homology: see §2.

Now suppose that Q_t depends on time, and let

$$x \mapsto A_t x , \quad t \in [0,1]$$

be the corresponding linear flow, where $A_t \in \mathrm{G} = \mathrm{Sp}(2n,\mathbf{R})$. By slightly perturbing the family $Q_{t\in[0,1]}$ we may suppose that A_1 is diagonalizable. Let $\mathcal{U}$ denote the set of diagonalizable matrices with all eigenvalues on the unit circle, and observe that the flow of a time-independent positive quadratic form lies in $\mathcal{U}$. The following proposition allows us to reduce to the time-independent case provided that the time 1-map A_1 of Q_t lies in $\mathcal{U}$.

PROPOSITION 1.5. *Suppose that Q_t, $t \in \mathbf{R}/\mathbf{Z}$ is a 1-periodic family of positive definite quadratic forms on $\mathbf{R}^{2n}$ which generates a path $A_t \in \mathrm{Sp}(2n,\mathbf{R})$ such that $A_1 \in \mathcal{U}$. Then there is a time-independent positive definite quadratic form $\mathcal{K}$ and $\varepsilon > 0$ such that there is a symplectic isotopy*

$$\Phi_s : S_{\mathcal{K},\varepsilon} \to W , \quad 0 \leq s \leq 1 ,$$

which fixes all points of the disc $D_0 = 0 \times D$, starts at the inclusion and ends at a map Φ_1 which takes the ellipsoidal slice $S_{\mathcal{K},\varepsilon}$ into $S_{Q_t,\varepsilon}$. Moreover, if $A_{t\in[0,1]}$ is short, so is the path generated by $\mathcal{K}$.

Roughly speaking, this proposition says that any slice S which satisfies the conditions of Theorem 1.3(ii) and whose monodromy is conjugate to a unitary matrix, contains an ellipsoidal slice which satisfies the same conditions. A few extra arguments are needed to deduce non-squeezing results for S from those for the ellipsoidal slice: see § 2.2. In the general case we prolong the path A_t to a path with endpoint in $\mathcal{U}$ without introducing any new closed trajectories: see Theorem 1.6(i) below.

The proof of Proposition 1.5 is based on the study of positive paths in G which is carried out in [LM4]. We will only consider paths in G which start at $\mathbb{1}$. We will write

$$\mathcal{S}_1 = \{A \in \mathrm{G} : \det(A - \mathbb{1}) = 0\} ,$$

and denote by $\mathcal{P}_{\mathrm{Aut}}$ the space of short positive paths in G which are generated by time-independent Hamiltonians $\mathcal{K}$. If $\mathcal{K}(x) = -\frac{1}{2}x^T P x$, for some positive definite symmetric matrix P then it is easy to check that the corresponding flow is

$$A_t = e^{JPt} ,$$

where J is multiplication by i in $\mathbf{R}^{2n} = \mathbf{C}^n$. In particular, all elements A_t lie in $\mathcal{U}$, and the corresponding slice $S_{\mathcal{K},\varepsilon}$ is part of an ellipsoid. Let

$$e : \mathcal{P}_{\mathrm{Aut}} \to \mathcal{U}$$

be the endpoint map. It is not hard to see that every element of $\mathcal{U}$ with no multiple eigenvalues is the endpoint of a unique element of $\mathcal{P}_{\mathrm{Aut}}$. However, elements with multiple eigenvalues have larger inverse image in $\mathcal{P}_{\mathrm{Aut}}$ and hence it is not possible to lift every path in $\mathcal{U}$ to a path in $\mathcal{P}_{\mathrm{Aut}}$.

The main result of [LM4] is:

THEOREM 1.6. *(i) Every element of* G *is the endpoint of a positive path. An element of* $\mathrm{G} - \mathcal{S}_1$ *is the endpoint of a short positive path if and only if it has an even number of real eigenvalues* λ *with* $\lambda > 1$.

(ii) Any short positive path may be extended to a short positive path with endpoint in $\mathcal{U}$.

(iii) If $A_{t\in[0,1]}$ *is a short positive path with endpoint in* $\mathcal{U}$, *it is homotopic through short positive paths with endpoint in* $\mathcal{U}$ *to a short autonomous path. Moreover, we may choose this homotopy so that the path formed by its endpoints lifts to* $\mathcal{P}_{\mathrm{Aut}}$.

In the language of Ekeland [E2], this implies that every stable positive linear periodic Hamiltonian system is homotopic through such systems to one generated by an autonomous Hamiltonian. Moreover the homotopy may be chosen so that the index of the system does not change.

1.2 Stability of geodesics in $\mathrm{Ham}^c(M)$. Let $\mathrm{Ham}^c(M)$ be the group of Hamiltonian symplectomorphisms of the symplectic manifold (M, ω), generated by compactly supported Hamiltonian $M \times [0,1] \to \mathbf{R}$, with the Hofer norm $\|\phi\|$ defined by:

$$\|\phi\| = \inf \mathcal{L}(\phi_{t\in[0,1]}) \ ,$$

where the infimum is taken over all paths in $\mathrm{Ham}^c(M)$ from the identity $\mathbb{1}$ to ϕ, and where the length $\mathcal{L}$ of the path generated by the Hamiltonian $H_{t\in[0,1]}$ is defined to be

$$\mathcal{L}(\phi_t) = \mathcal{L}(H_t) = \int_0^1 \big(\max_{x\in M} H_t(x) - \min_{x\in M} H_t(x)\big)dt \ .$$

A path $\gamma = \phi_{t\in[0,1]}$ is said to be a stable geodesic if it is a local minimum for $\mathcal{L}$ on the space of all paths from $\mathbb{1}$ to ϕ_1. (This path space is given the C^1-topology.) It was shown in [BP],[LM2],[U] that a stable geodesic γ is quasi-autonomous, that is it has at least one fixed maximum P (a point at which H_t assumes its maximum for all t) and one fixed minimum p. Moreover, if there are only finitely many such fixed extrema, there must be one such pair P, p at which the linearised flow has no non-trivial closed trajectory in time < 1. We will show below that this statement is essentially equivalent to part (i) of Theorem 1.3 above. Similarly, part (ii) is equivalent to the converse:

THEOREM 1.7. *Suppose that γ has a fixed maximum and minimum at which the linearised flow A_t has no non-trivial closed trajectory in time ≤ 1. Then γ is a stable geodesic.*

As shown by Ustilovsky in [U], this result is a fairly easy consequence of the second variation formula for $\mathcal{L}$ provided that the Hessian of H_t at the fixed extrema are non-degenerate at all times. We prove the general result in § 3.

2. Local Squeezing

2.1 Ellipsoidal slices. Let Q be a positive definite quadratic form on $\mathbf{R}^{2n}$, and consider the ellipsoidal slice

$$S_{Q,\varepsilon} = \left\{(x, c, t) \in \mathbf{R}^{2n} \times \mathbf{R}^2 \mid c \leq \pi - Q(x) \text{ and } \|x\| < \varepsilon\right\} ,$$

where ε is small enough so that $\pi - Q(x)$ is positive over the ball $\|x\| \leq \varepsilon$. In this section we will consider the squeezing properties of $S_{Q,\varepsilon}$ in the cylinder $W = \mathbf{R}^{2n} \times D$, where D is the unit disc in $\mathbf{R}^2$. Since Q may be diagonalized with respect to a symplectic basis of $\mathbf{R}^{2n}$, there is a symplectomorphism of the form $\Psi \times \mathbb{1}$ of W which takes $S_{Q,\varepsilon}$ to the corresponding set defined by the diagonalized form

$$\sum_{i=1}^{n} \pi a_i^2 (x_{2i-1}^2 + x_{2i}^2)$$

where $a_1 \geq \ldots \geq a_n > 0$. Therefore, we will assume that Q has this form. The analog of Theorem 1.3 for these slices is:

PROPOSITION 2.1. *(i) If $a_1 > 1$, then $S_{Q,\varepsilon}$ is locally squeezable by isotopy in W.*

(ii) If $a_1 = 1$, then $S_{Q,\varepsilon}$ is squeezable by isotopy but not locally squeezable in W.

(iii) If $a_1 < 1$, then $S_{Q,\varepsilon}$ is neither locally squeezable nor squeezable by isotopy in W.

Proof: Let (u, v) be rectangular coordinates of the $\mathbf{R}^2$-plane containing the base D of the cylinder W. If $a_1 > 1$ then one can rotate $S = S_{Q,\varepsilon}$ in the x_1, x_2, u, v-plane so that it does not project surjectively onto D for any $t > 0$. For example, one can use the matrix

$$\begin{pmatrix} \cos t & 0 & -\sin t & 0 \\ 0 & \cos t & 0 & -\sin t \\ \sin t & 0 & \cos t & 0 \\ 0 & \sin t & 0 & \cos t \end{pmatrix} .$$

This proves (i). If $a_1 = 1$ then the same rotation keeps S in W and will take S to a position in which the projection to D is not surjective when $\sin(t) > \varepsilon$. However, S is not locally squeezable in this case. To see this, suppose by contradiction that it were and let ψ_i be a sequence of embeddings of S into $\operatorname{Int} W$ which converge C^1 to the inclusion. We may suppose that the ψ_i are so close to the inclusion that their graphs in $-W \times W$ may be considered as sections of the cotangent bundle T^*W. The ψ_i therefore extend to embeddings of the whole ellipsoid

$$E_Q = \left\{(x, r, t) \in \mathbf{R}^{2n} \times \mathbf{R}^2 \mid r^2 \leq 1 - Q(x)\right\} ,$$

which also converge to the inclusion. Hence, for large enough i, these embeddings take E_Q into $\operatorname{Int} W$. But E_Q contains the unit ball, and this cannot be mapped strictly inside W by the Non-Squeezing theorem ([G],[LM1]).

Now consider case (iii). Since the sets S increase when the a_i decrease, S cannot be locally squeezable by (ii). Suppose, by contradiction, that it were squeezable by isotopy, and choose a symplectic isotopy $\phi_t : S \to W$ such that

$$\phi_1(S) \subset W_{2\delta} = M \times D(\pi - 2\delta)$$

for some $\delta > 0$. (Here $D(a)$ denotes the 2-disc in $\mathbf{R}^2$ of area a centered at the origin.) It is easy to see that there is some $\nu > \delta$ such that $S_{Q,\varepsilon}$ contains the product

$$P = B^{2n}(\nu) \times D(\pi - \delta) .$$

We claim that P cannot be isotoped in W to a subset of $W_{2\delta}$. The basic reason for this is that there is an element of the symplectic homology of W coming from a closed characteristic of ∂W which is non-zero in P but which vanishes on any subset of $W_{2\delta}$. To be precise, we will use the formulation of symplectic homology with $\mathbf{Z}_2$ coefficients given by Floer, Hofer, Wysocki ([FH],[FHW]). The result we want can almost be quoted directly from there, but to be complete we will give some details.

Symplectic homology is calculated using a complex of the form

$$\ldots \xrightarrow{d} \oplus_i(C_i, k; b_i) \xrightarrow{d} \oplus_j(C_j, k-1; b_j) \xrightarrow{d} \ldots$$

where C_i, C_j are vector spaces over $\mathbf{Z}_2$ generated by certain periodic orbits, the grading k is given by the index of the orbit and the level b_i is determined by the action of the orbits. The differential is defined, as in Floer homology, by counting connecting orbits satisfying some elliptic PDE. The group $S_k^{[a,b)}$ is then defined as the homology of the truncated quotient complex, obtained by ignoring all boundary operators with domain $(C_i, k; b_i)$ where $b_i \geq b$ and by quotienting out by all groups $(C_i, k; b_i)$ with $b_i < a$. For example, the terms with grading ≤ 4 in the complex for the 2-disc $D(\nu)$ of capacity ν are

$$\xrightarrow{0} (\mathbf{Z}_2, 4; 2\nu) \xrightarrow{\mathrm{Id}} (\mathbf{Z}_2, 3; \nu) \xrightarrow{0} (\mathbf{Z}_2, 2; \nu) \xrightarrow{\mathrm{Id}} (\mathbf{Z}_2, 1; 0) \to 0 .$$

Therefore, if $0 \leq \delta < \nu$ and $0 < \delta' \leq \nu$, the groups

$$S_k^{[\nu-\delta, \nu+\delta')}(D(\nu)) , \qquad k = 2, 3 ,$$

are non-zero. The complex for the $2n$-ball $B^{2n}(\nu)$ of capacity ν is similar except that there are $2n$ contributions at level ν with gradings going from $n+1$ to $3n$ and then further contributions at levels $j\nu$, $j > 1$ with gradings $\geq 3n+1$. Therefore, with δ, δ' as above, the groups

$$S_{n+1}^{[\nu-\delta, \nu+\delta')}(B^{2n}(\nu)) \ \text{ and } \ S_{3n}^{[\nu-\delta, \nu+\delta')}(B^{2n}(\nu))$$

are non-zero.

The groups for the product P are calculated by the tensor product of the complexes for the factors where

$$(C, k; b) \otimes (C', k'; b') = (C \otimes C', k + k'; b + b')$$

with the differential $1\!\!1 \otimes d' + d \otimes 1\!\!1$. Therefore, if $P = B^{2n}(\nu) \times D(\alpha)$, the complex in degrees $n+1, n+2, n+3$ is

$$\ldots \to (\mathbf{Z}_2, n+3, \nu) \oplus (\mathbf{Z}_2, n+3, \nu+\alpha)$$
$$\xrightarrow{d_{n+3}} (\mathbf{Z}_2, n+2, \nu) \oplus (\mathbf{Z}_2, n+2, \alpha) \to (\mathbf{Z}_2, n+1; \nu) \oplus (\mathbf{Z}_2, n+1; 0) \to \ldots .$$

Here the terms at level ν are products of entries in the complex for the ball with the lowest term $(\mathbf{Z}_2, 1; 0)$ for the 2-disc, and so the differential has the form $d \otimes 1\!\!1$ on them. However the term at level $\nu + \alpha$ is the product

$$(\mathbf{Z}_2, n+1, \nu) \otimes (\mathbf{Z}_2, 2; \alpha)$$

and so d_{n+3} takes its generator 1 to the element $1 \oplus 1$. Thus d_{n+3} is surjective if the window $[a, b)$ contains both α and $\nu + \alpha$. However, if

$$0 < a < \alpha \leq \pi , \quad \pi < b < \min(\nu + \alpha, 2\alpha) ,$$

then $S_{n+2}^{[a,b)}(P(\nu, \alpha)) = \mathbf{Z}_2$.

Now, let $\alpha_{t \in [0,1]}$ increase from $\alpha_0 = \alpha$ to $\alpha_1 = \pi$, let a, b be as above, and consider the restriction map

$$S_{n+2}^{[a,b)}(P(\nu, \alpha_t)) \to S_{n+2}^{[a,b)}(P(\nu, \alpha)) .$$

This is the identity map when $t = 0$, and as α_t increases none of the levels in the complexes we are considering crosses the given window. Hence by [FHW, Lemma 18], it is an isomorphism when $t = 1$.

Next let $\nu_{t\in[0,1]}$ increase from $\nu_0 = \nu$ to some large value $\nu_1 = \kappa$, and consider the restriction map

$$S_{n+2}^{[a,b)}(P(\nu_t, \pi)) \to S_{n+2}^{[a,b)}(P(\nu, \alpha)) .$$

As ν_t increases, some levels of the part of the complex coming from $B^{2n}(\nu_t)$ cross the window. However, the part of the complex used to calculate $S_{n+2}^{[a,b)}$ for the given a, b does not change. Therefore, as before, because this is the identity map when $t = 1$ it is the identity map for all t.

Now suppose that $P(\nu, \pi - \delta)$ can be isotoped in W into $W_{2\delta}$. Choose κ so that this isotopy takes place in $P(\kappa, \pi)$. and suppose that

$$0 < \pi - 2\delta < a < \pi - \delta , \quad \pi < b < \pi + \nu .$$

Then, by the above, the restriction map

$$S_{n+2}^{[a,b)}(P(\kappa, \pi)) \to S_{n+2}^{[a,b)}(P(\nu, \pi - \delta))$$

is non-zero. But, because it is invariant under isotopy in $P(\kappa, \pi)$ it equals the restriction map

$$S_{n+2}^{[a,b)}(P(\kappa, \pi)) \to S_{n+2}^{[a,b)}(P(\nu, \pi - 2\delta)) .$$

But this is the zero map because $a > \pi - 2\delta$. A contradiction. □

Remark 2.2: The case $Q(x) = \|x\|^2$ is borderline. Although the slice $S_{Q,\varepsilon}$ of the ball considered above is squeezable by isotopy inside the whole cylinder $\mathbf{R}^{2n} \times D$ it is very likely that it does not squeeze by isotopy inside a very short cylinder $B^{2n}(\delta) \times D$ when δ is only just larger than ε. (One could also compactify $B^{2n}(\delta)$ to a complex projective space $\mathbf{C}P^n$ of small volume if one wants to assume that M is in some sense complete.)

2.2 General slices. We now consider quadratic non-negative Hamiltonians Q_t which are 1-periodic in t, and let A_t be the corresponding flow. We write S_A for the germ along the 2-disc $D_0 = 0 \times D$ of the corresponding slice $S_{Q_t,\varepsilon}$.

LEMMA 2.3. *(i) If two positive paths A_t and B_t have the same endpoint $A_1 = B_1$ and are homotopic in* $\mathrm{Sp}(2n, \mathbf{R})$ *rel endpoints, there exists a symplectic diffeomorphism*

$$\Phi : S_A \to S_B .$$

(ii) Suppose that $A^s_{t\in[0,1]}$ where $s \in [0,1]$ is a smooth family of positive paths with fixed endpoints. Then there is a symplectic isotopy

$$\Phi^s : S_{A^0} \to S_{A^s} , \quad 0 \le s \le 1 .$$

Proof: (*i*) If A_t, B_t are generated by $-\mathcal{Q}_t, -\mathcal{K}_t$ respectively, then

$$S_A = \{(x,c,t) : c \le \pi - \mathcal{Q}_t(x)\ ,\ \|x\| \text{ small}\}$$
$$S_B = \{(x,c,t) : c \le \pi - \mathcal{K}_t(x)\ ,\ \|x\| \text{ small}\}\ ,$$

and it is easy to check that the map

$$\Psi : (x,c,t) \mapsto \left(B_t A_t^{-1}x, c - \mathcal{K}_t(B_t A_t^{-1}x) + \mathcal{Q}_t(x), t\right)$$

is a symplectomorphism defined on some neighborhood V_A of ∂S_A in S_A which takes ∂S_A to ∂S_B. (Note that S_A, S_B are germs along D_0, so that we can make them smaller as necessary.) Moreover, since $\mathcal{Q}_t(0) = \mathcal{K}_t(0) = 0$, we may extend Ψ to D_0 by the identity.

Since the paths A_t, B_t are homotopic, Ψ can be extended further to a smooth diffeomorphism of S_A onto S_B. The pull-back $\Omega = \Psi^*(\omega \oplus \sigma)$ coincides with $\omega \oplus \sigma$ on V_A and is such that $\Omega\ |_{D_0} = \sigma$. After a slight perturbation of Ψ in the transversal direction along D_0, we may assume that

$$\Omega = \omega \oplus \sigma$$

on the full tangent space $T_{(x,c,t)}(\mathbf{R}^{2n} \times D)$ at all points of $V_A \cup D_0$. But then the 1-parameter family of closed 2-forms

$$\Omega_\lambda = (1-\lambda)(\omega \oplus \sigma) + \lambda\Omega\ , \quad \lambda \in [0,1]$$

is a symplectic isotopy in some small neighbourhood of $V_A \cup D_0$, which always equals the split form in V_A and on D_0. By Moser's argument, there is a diffeotopy Φ_λ on a neighbourhood N of $V_A \cup D_0$ which is the identity on $V_A \cup D_0$ and is such that Φ_1 pulls Ω back to the split form $\omega \oplus \sigma$.

This proves (*i*). Statement (*ii*) is obvious since all choices can be made to depend smoothly on s. □

COROLLARY 2.4. *Proposition 1.5 holds.*

Proof: Let $A_{t\in[0,1]}$ be a short positive path with endpoint $A_1 \in \mathcal{U}$. Then $A_1 \in \mathcal{U}$ is the endpoint of a short positive autonomous path B_t. By Theorem 1.6 (*iii*), there exists a 1-parameter family of short positive autonomous paths B_t^s and a 1-parameter family of short positive paths A_t^s such that $A_t^1 = A_t$, $B_t^1 = B_t$, $A_1^s = B_1^s (\in \mathcal{U})$ for all $s \in [0,1]$, and $A_t^0 = B_t^0$.

We first wish to define a smooth 1-parameter family of symplectic automorphisms (with respect to the standard symplectic structure) $f^s : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ with the following property: if Q^s denotes the autonomous Hamiltonian that generates B_1^s, the pull-back $Q^s \circ f^s$ is diagonal in the standard basis for each $0 \le s \le 1$. Here of course "diagonal" means that $Q^s \circ f^s$ is of the form

$$Q^s \circ f^s = \sum_{i=1}^{n} \pi a_i^2 (x_{2i-1}^2 + x_{2i}^2) \ .$$

To do so, note first that since our argument is purely homotopical, we have complete freedom to jiggle or reparametrize the matrices B_1^s along time s. So first jiggle the path $B_1^{s\in[0,1]}$ so that

(i) the endpoints B_1^0 and B_1^1 are fixed, and

(ii) the autonomous positive quadratic forms Q^s corresponding to B_1^s are such that there exists only a finite number of points $s \in [0,1]$ where Q^s is not generic (that is to say: having a $2k$-dimensional space over which Q^s is equal to const$\|v\|^2$, for $k \geq 2$).

This is possible because the set of non-generic quadratic forms is of codimension at least one. Of course, one jiggles accordingly the paths A_t^s so that $A_1^s = B_1^s$ for all s. Let $\mathcal{I} \subset [0,1]$ denote the finite subset referred to in (ii) above.

Now we can reparametrize the path B_1^s so that the following holds: for each $s_k \in \mathcal{I}$, there exists an interval $I_k \subset [0,1]$ of non-zero measure such that all matrices B_1^s with s in this interval are equal. In this case, although the choice of a symplectic ordered basis $\mathcal{B}_s$ of oriented eigen-2-planes of each Q^s is not unique, there exists some choice of $\mathcal{B}_s = (P_{1,s}, \dots, P_{n,s})$ which varies smoothly with s. The reason is that the unordered basis of nonoriented eigen-2-planes is unique when s is not in one of the intervals I defined above. And when s belongs to one of them, the above condition makes it possible to move the unordered basis smoothly from the one needed at the left end of the interval I_k to the one needed at the other end. This defines a smooth path of unordered and nonoriented bases: we then choose an order and an orientation at time $s = 0$ and extend this uniquely over all $s \in [0,1]$. Finally, since each $P_{i,s}$ is oriented, there exists a continuous lift of $\mathcal{B}_s$ to f_s, obtained by choosing a pair of symplectic vectors in each $P_{i,s}$. Such a lift obviously exists because we need to define it only over a path.

Now we conjugate both paths $A^s_{t\in[0,1]}, B^s_{t\in[0,1]}$ by f^s. This gives new paths, that we still denote in the same way, but which are such that the endpoints $A_1^s = B_1^s$ are all diagonalizable with respect to the standard symplectic basis. By the last lemma, there exist symplectic diffeomorphisms

$$\Phi^s : S_{B^s} \to S_{A^s} \ , \quad s \in [0,1]$$

with $\Phi^0 = \mathrm{id}$. But since all B^s are generated by diagonalized time-independent quadratic Hamiltonians, the sets S_{B^s} are ellipsoids with principal axes independent of t and s. Hence $\cap_{s\in[0,1]} S_{B^s}$ contains $S_{K,\varepsilon}$ for

$$\mathcal{K} = \sum_{i=1}^{n} \pi a_i^2 (x_{2i-1}^2 + x_{2i}^2)$$

where $a_i = \max_s a_i^s$. Note that the path generated by $\mathcal{K}$ is still short. The restriction of Φ^s to $S_{\mathcal{K},\varepsilon}$ gives the desired isotopy. □

In order to prove part (*ii*) of Theorem 1.3 we need the following lemmas. Recall that if $W = M \times D(\pi)$ we write W_δ for $M \times D(\pi - \delta)$.

LEMMA 2.5. *If S is locally squeezable in W, there is some $\delta > 0$ such that every compact subset X of $S \cap \operatorname{Int} W$ is squeezable by isotopy into W_δ.*

Proof: Given $\psi : S \to W$ consider the associated map $\tilde{\psi} : \partial W \cup X \to W$ which equals ψ on X and the identity on ∂W. If ψ is so close to the inclusion that its graph may be identified with a partial Lagrangian section of T^*W, then $\tilde{\psi}$ extends over W and is isotopic to the identity by maps which fix all points of ∂W. Note that this statement holds whatever X is provided that $X \cap \partial W = \emptyset$. The result now follows by applying this to a local squeezing ϕ_i of S. □

LEMMA 2.6. *Suppose that S is squeezable by isotopy in W and that it intersects ∂W in a closed characteristic $x_0 \times \partial D$ which is isolated among the closed characteristics on ∂S. Then the germ of S along $x_0 \times D$ is isotopic in W by an isotopy which fixes $x_0 \times \partial D$ to a slice germ which is locally squeezable in W.*

Proof: Suppose that $\psi_t : S \to W$ is an isotopy such that $\psi_1(S) \subset M \times \operatorname{Int} D$. Let γ be the closed characteristic $x_0 \times \partial D$ and consider the set

$$\mathcal{I} = \left\{ t \in [0,1] : \psi_t(\gamma) \subset M \times \partial D \right\} .$$

If $T = \inf\{t \in [0,1] : t \notin \operatorname{Int} \mathcal{I}\}$, then $T < 1$ since S is squeezable by isotopy. Further, since $\psi_t(\gamma)$ must always be a flat circle $x \times \partial D$ for $t \leq T$ we may compose ψ_t with maps of the form $g_t \times h_t$ so that ψ_t fixes all points of γ for $t \leq T$. We claim that $\psi_T(S)$ is locally squeezable. For, by hypothesis there is a sequence $\varepsilon_i \to 0^+$ such that $\psi_{T+\varepsilon_i}(S)$ does not intersect $M \times D$ in a closed characteristic, and so these sets can be pushed inside $M \times D$ by Lemma 1.2. □

Let $A_{t\in[0,1]}$ be a positive path, and $A^T = A_{t\in[0,T]}$ any subpath. We define S_{A^T} as the slice associated to the positive path A_{tT}, $t \in [0,1]$.

LEMMA 2.7. *If a positive path $A_{t\in[0,1]}$ is such that S_A is neither locally squeezable nor squeezable by isotopy, the same is true of any subpath $A^T = A_{t\in[0,T]}$.*

Proof: This is obvious. We may extend the subpath by setting $A_t = A_T$ for $t \geq T$, which corresponds to setting $Q_t = 0$ for $t \geq T$. Denote by $A'_{t\in[0,1]}$ that path. Clearly, $S_{A'}$ is locally squeezable (or squeezable by isotopy) if and only if the same is true of S_{A^T}. (The non-smoothness of $S_{A'}$ is irrelevant here.) But the corresponding set $S_{A'}$ clearly contains S_A, and so if it were locally squeezable or squeezable by isotopy, the same would be true of S_A.□

Proof of Theorem 1.3(ii): Let S be any slice germ along $D_0 = x_0 \times D$ whose characteristic flow is given by a short positive path $A_{t\in[0,1]}$. Since we are only looking at the germ we may suppose that M is just a small neighborhood of x_0 and so identify it with an open neighborhood U of $\{0\}$ in $\mathbf{R}^{2n}$. Further, by Theorem 1.6(ii) and Lemma 2.7 we may suppose that $A_1 \in \mathcal{U}_0$ so that we can apply Proposition 1.5. We may therefore isotop S in $U \times D$ by Φ_s, fixing D_0, so that $\Phi_1(S)$ is an ellipsoidal slice S_E, say. By Proposition 2.1 the latter slice is neither locally squeezable nor squeezable by isotopy in $U \times D$. It follows that S is not locally squeezable by isotopy in $M \times D$. However, we want to prove more: namely that S is neither locally squeezable nor squeezable by isotopy.

To see that S is not locally squeezable, observe that for all $\delta > 0$ there are subsets X' of $S_E \cap U \times D$ which cannot be squeezed by isotopy into $U \times D(\pi - \delta)$: namely the polydiscs $P(\varepsilon, \alpha)$ where $\varepsilon + \alpha > \pi$. Therefore, a similar statement holds for $X = \Phi_1^{-1}(X')$, and the desired conclusion follows from Lemma 2.5.

To see that S is not squeezable by isotopy in the original cylinder $M \times D$ we apply Lemma 2.6. Note first that because S is the set associated to a (time-dependent) quadratic positive non-degenerate form, the hypothesis of Theorem 1.3(ii) on the nonexistence of non-trivial closed trajectories of A_t implies in particular that there is no closed characteristic of ∂S near $x_0 \times \partial D$. This is because in the quadratic case, the flow A_t is given by the characteristic foliation of ∂S. Now we can apply Lemma 2.6: it implies that if S were squeezable by isotopy, there would be an isotopic set $S' \subset M \times D$ which was locally squeezable. But the characteristic flow of S' near γ is the same as that of S, and so such a set S' cannot exist by which we have just proved. □

3. Relations with Hofer's Geometry and Proof of Part (i) of the Main Theorem

As we mentioned earlier, the problem of (in)stability in Hofer's metric and the problem of local (un)squeezability of subsets of cylinders are essentially equivalent: the necessary condition for stability is equivalent to part (i) of

Theorem 1.3 and the sufficient condition for stability is equivalent to part (*ii*) of that theorem. We established a necessary condition for stability in [LM2] and we will use it below to prove part (*i*) of the main Theorem on the squeezability of sets $S \subset W$. Concerning the sufficient condition for stability or unsqueezability, we proceed in the reverse direction: we have just proved part (*ii*) of the main theorem and we will show below that it implies the sufficiency of the condition for the stability of geodesics.

3.1 Proof of part (*i*) of the Main Theorem. Here is the rough outline of the proof. Consider the part of ∂S inside a thin cylindrical annulus $W - W_\delta$. There, ∂S is the graph of a partially defined Hamiltonian and we can therefore apply the curve-shortening construction in the proof of Theorem 1.6 of [LM2] to move the graph away from ∂W. Since this construction is localized near the extrema, we can attach the shortened curves to the set $S \cap W_\delta$ and obtain a squeezing isotopy of S inside W.

So choose $\delta > 0$ small enough so that $\partial S \cap (W - W_\delta)$ is the graph of a function $G = \pi - H : E \to \mathbf{R}$, where E is some compact (not necessarily connected) subset of $\mathbf{R}^{2n} \times S^1$. Of course E contains a neighbourhood of the set of all maxima of G. We will suppose that G is normalized so that its maximum value is π. Let $\tilde{G} : \tilde{E} \to \mathbf{R}$ denote the pull-back of G by the map $\mathrm{id} \times h : \mathbf{R}^{2n} \times [0,1] \to \mathbf{R}^{2n} \times S^1$, where h identifies 0 and 1. Thus

$$\mathrm{graph}(\tilde{G}) \subset \tilde{E} \times [0,\pi] \subset \mathbf{R}^{2n} \times [0,1] \times [0,\pi] .$$

Now, by the proof of Theorem 1.6 of [LM2], there exists a smooth isotopy

$$\tilde{\phi}_\lambda : \mathrm{graph}(\tilde{G}) \to \tilde{E} \times \mathbf{R} , \quad \lambda \in [0,\varepsilon]$$

which begins with the identity and satisfies for all $\lambda > 0$:

(*i*) $\tilde{\phi}_\lambda$ is a symplectic diffeomorphism onto its image and the restriction of $\tilde{\phi}_\lambda$ to $\tilde{G}^{-1}([\pi - \delta, \pi - \delta/2])$ is the identity, as well as its restriction to some time intervals containing $\{t = 0\}$ and $\{t = 1\}$;

(*ii*) the set $\tilde{\phi}_\lambda(\mathrm{graph}(\tilde{G}))$ sits inside $\tilde{E} \times [0,\pi]$ and its restriction to some time interval $I \subset (0,1)$ sits inside $\tilde{E} \times [0,\pi)$; here I is independent of λ.

To see this, one first extends $\tilde{G}$ to a compactly supported Hamiltonian $\mathbf{R}^{2n} \times [0,1] \to \mathbf{R}$ whose values outside $\tilde{E}$ are in $[0, \pi - \delta/2)$. One then constructs the desired isotopy as in section 4.2 of [LM2].

Because $\tilde{\phi}_\lambda$ is the identity on some time intervals containing $\{t = 0\}$ and $\{t = 1\}$, it descends to an isotopy

$$\phi_\lambda : \mathrm{graph}(G) = \partial S \cap (W - W_\delta) \to W$$

which trivially extends to an isotopy $\partial S \to W$. By condition (*ii*) above, this isotopy, that we still denote ϕ_λ, has for all $\lambda > 0$ an image whose projection on the base D of the cylinder is *not* onto. Extend this isotopy to a symplectic isotopy defined on some interior collar neighbourhood of ∂S in S. Let $\bar{\phi}_\lambda$ be its lift to S. Then the pull-back of the standard symplectic form $\bar{\phi}_\lambda^*(\Omega)$ is a symplectic isotopy rel ∂S for all sufficiently small λ. The relative Moser argument then yields a symplectic isotopy $\psi_\lambda : S \to W$ beginning with the identity, which is such that $\mathrm{Im}(\psi_\lambda)$ does not project onto the base D^2 of the cylinder W when $\lambda > 0$. Then composing with an appropriate area preserving map of the base gives a local squeezing of S by isotopy. □

3.2 Sufficient condition for the stability of geodesics. We show that Theorem 1.3 (*ii*) proved in § 2 implies the sufficient condition for the stability of geodesics in Hofer's metric.

As in the characterisation of geodesics established in [LM3], we need the gluing-along-monodromy construction. However, we must use a slightly different normalization here, and so we begin by repeating the main constructions of [LM3, §2] in modified form. For now, we will assume that M is closed and that ω has been rescaled so that $\mathrm{vol}\, M = 1$.

Let $H_{t\in[0,1]}$ be a regular path in $\mathrm{Ham}^c(M)$ which has a fixed minimum p and a fixed maximum P at which the linearized flows have no non-trivial closed orbits in time less than or equal to 1. Proceeding by contradiction, we assume that this is not a stable geodesic, and will show that this contradicts Theorem 1.3 (*ii*). By replacing H_t by $H_t - H_t(p)$, we may assume that $\min H_t = 0$ for all t. Since $H_{t\in[0,1]}$ is regular, its maximum value $m_H(t) = H_t(P)$ is strictly greater than its minimum value $H_t(p)$ for all t. We reparametrize all flows which we consider, i.e. both $H_{t\in[0,1]}$ and the nearby flows $K_{t\in[0,1]}$, so that they are generated by Hamiltonians which vanish along with all their derivatives when $t = 0, 1$. Since the reparametrized flow $\phi_{\beta(t)}$ is generated by the Hamiltonian $\beta'(t)H_{\beta(t)}$ this does not change the length of the isotopy. Moreover, if we use the same reparametrization function β for all flows, it is easy to check this will not affect the stability properties of the path $H_{t\in[0,1]}$. The maximum of H_t will be denoted $m_H(t) = H_t(P)$. So our conventions imply that $m_H(t)$ is a smooth function of t which is infinitely tangent to 0 at $t = 0, 1$ and is > 0 on $(0, 1)$.

Denote by $\tilde{R}_H^-$ and $\tilde{R}_H^+$ the parts under and over the graph of H:

$$\tilde{R}_H^- = \{(x, s, t) \in M \times \mathbf{R}^2 : 0 \le s \le H_t(x)\}$$
$$\tilde{R}_H^+ = \{(x, s, t) \in M \times \mathbf{R}^2 : H_t(x) \le s \le m_H(t)\} ,$$

and define

$$U_H = \{(s, t) \in \mathbf{R}^2 : 0 \le s \le m_H(t)\} .$$

By assumption on $m_H(t)$, the set U_H has area $\mathcal{L}(H)$ and lies between two curves $s = 0$ and $s = m_H(t)$ which are infinitely tangent at their endpoints but are otherwise disjoint. Similarly, both $\tilde{R}_H^-$ and $\tilde{R}_H^+$ are regions lying between two hypersurfaces which are infinitely tangent along $M \times 0 \times \{t = 0,1\}$. One of these hypersurfaces is the graph of H with monodromy ϕ_1, and the other (either $s = 0$ or $s = m_H(t)$) has trivial monodromy.[1] Note that these hypersurfaces will touch at points where $H_t(x)$ is either 0 or $m_H(t)$. However, this does not matter because we will be interested only in the piece of $\tilde{R}_H^-$ near P (or the piece of $\tilde{R}_H^+$ near p). The gluing of both halves $\tilde{R}_H^-$ and $\tilde{R}_H^+$ by the identification of their common side, which matches the characteristic foliations, gives back simply the product $M \times U_H$ of area $\mathcal{L}(H)$.

Now let $K_{t\in[0,1]}$ be another path that we may assume to be homotopic and C^∞-close to $H_{t\in[0,1]}$. As above, $K_t = 0$ when $t = 0,1$ and is infinitely tangent to 0 there, but, because $K_{t\in[0,1]}$ need not have a fixed minimum, we cannot necessarily normalize K_t, keeping it a smooth function of t, so that $\min K_t = 0$ for each t. Therefore we define ε-thickenings of $\tilde{R}_K^\pm$ as follows:

$$\tilde{R}_{K,\varepsilon}^- = \{(x,s,t) \in M \times \mathbf{R}^2 : \lambda_1(t) \le s \le K_t(x)\}$$
$$\tilde{R}_{K,\varepsilon}^+ = \{(x,s,t) \in M \times \mathbf{R}^2 : K_t(x) \le s \le \lambda_2(t)\}\ ,$$

where $\lambda_1(t) < \min K_t$ and $\lambda_2(t) > \max K_t$ are smooth functions which are infinitely tangent to 0 at $t = 0,1$, and so close to the minimum and maximum of K_t that

$$\mathrm{vol}(\tilde{R}_{K,\varepsilon}^+ \cup \tilde{R}_{K,\varepsilon}^-) = \mathcal{L}(K) + \varepsilon\ .$$

Since the flow $\psi_{t\in[0,1]}$ of $K_{t\in[0,1]}$ has endpoint $\psi_1 = \phi_1$, we can use the map

$$\Phi_{K,H} : \tilde{R}_{K,\varepsilon}^+ \to \tilde{R}_H^-$$
$$\Phi_{K,H}(x,s,t) = \left(\phi_t \circ \psi_t^{-1}(x), s - K_t(x) + H_t(\phi_t \circ \psi_t^{-1}(x)),\ t\right)\ .$$

to glue $\tilde{R}_{K,\varepsilon}^+$ to $\tilde{R}_H^-$ to get a subset of $M \times \mathbf{R}^2$ which we call $\tilde{R}_{H,K,\varepsilon}$. The set

$$\tilde{R}_{K,\varepsilon,H} = \tilde{R}_{K,\varepsilon}^- \cup \tilde{R}_H^+$$

is defined similarly. These sets $\tilde{R}_{H,K,\varepsilon}, \tilde{R}_{K,\varepsilon,H}$ have the same basic shape as $\tilde{R}_H^\pm$, i.e. they are manifolds except for the fact that their front and back faces are infinitely tangent along $M \times \{0\} \times \{t = 0,1\}$. But now the ε-thickening prevents the front and back faces from touching each other. Recall from [LM3] that the *area* of a set such as $\tilde{R}_H^+$ is the number A defined by:

$$\mathrm{vol}(\tilde{R}_H^+) = A\,\mathrm{vol}(M)\ .$$

[1]The monodromy of a hypersurface diffeomorphic to $M \times [0,1]$ is the (partially defined) map $M \to M$ which takes the point $x \in M$ to y, where $y \times 1$ is the endpoint of the leaf of the characteristic foliation which goes through $x \times 0$.

Since we are assuming that $\mathrm{vol}(M) = 1$, A is just $\mathrm{vol}(\tilde{R}_H^+)$.

Before beginning the proof, we must deal with the smoothing/normalization problem. We need a canonical way to thicken sets like U_H, and have to be careful because there is no extra room to play with. Fix a real number $\lambda > 0$ and denote by

$$\tilde{R}_H^-(\lambda) = \{(x,s,t) : 0 \leq s \leq \lambda + H_t(x)\}$$
$$\tilde{R}_H^+(\lambda) = \{(x,s,t) : H_t(x) \leq s \leq m_H(t) + \lambda\}$$
$$U_H(\lambda) = \{(s,t) : 0 \leq s \leq \lambda + m_t\} .$$

Hence, for instance, $U_0(\lambda)$ is simply the square $[0,\lambda] \times [0,1]$. Correspondingly, we may thicken $\tilde{R}_{H,K,\varepsilon}$ to

$$\tilde{R}_{H,K,\varepsilon}(\lambda) = \tilde{R}_H^-(\lambda) \cup \tilde{R}_{K,\varepsilon}^+ ,$$

where $\tilde{R}_{K,\varepsilon}^+$ is translated by λ in the s direction so that it fits together with $\tilde{R}_H^-(\lambda)$.

All symplectic embeddings

$$\tilde{R}_{H,K,\varepsilon}(\lambda) \to M \times \mathbf{R}^2$$

which we will later consider will be infinitely tangent to the inclusion along the three sides $s = 0$, $t = 0, 1$. We will call such maps *normalized*.

Now let us begin the proof. If H_t is not a stable geodesic, there is a sequence K_t^i of Hamiltonians which converge to H_t and are such that

$$\mathcal{L}(K_t^i) < \mathcal{L}(H_t) = m$$

for all i. Because

$$\mathrm{area}\,(\tilde{R}_{H,K,\varepsilon}(\lambda)) + \mathrm{area}\,(\tilde{R}_{K,\varepsilon,H}(\lambda)) = \mathrm{area}\,(\tilde{R}_H(\lambda)) + \mathrm{area}\,(\tilde{R}_{K,\varepsilon}(\lambda))$$
$$= \mathcal{L}(H) + \mathcal{L}(K) + 2(\lambda+\varepsilon) ,$$

this can happen only if there is a sequence of positive real numbers ε_i converging to 0 such that either

$$\mathrm{area}\,(\tilde{R}_{H,K^i,\varepsilon_i}(\lambda)) < \mathcal{L}(H) + \lambda = \mathrm{area}\,(U_H(\lambda))$$

or

$$\mathrm{area}\,(\tilde{R}_{K^i,\varepsilon_i,H}(\lambda)) < \mathcal{L}(H) + \lambda = \mathrm{area}\,(U_H(\lambda)) .$$

We will suppose the former. (The latter case is of course symmetric and would be handled in the same way.) Denoting by a_i the area of $\tilde{R}_{H,K^i,\varepsilon_i}$, we then have: $a_i + \lambda < m + \lambda$ for all i.

In order to deduce Theorem 1.7 from Theorem 1.3, the key technical lemma needed is the following:

LEMMA 3.1. *Let the regular path $H_{t\in[0,1]}$ satisfy the hypothesis of the Stability Theorem 1.7 and suppose that it is not a stable geodesic from $\mathbb{1}$ to*

ϕ. Assume that K_i is a sequence of Hamiltonians with time-1 map ϕ which converge C^∞ to H_t and are such that

$$\text{area}\left(\tilde{R}_{H,K^i,\varepsilon_i}(\lambda)\right) < \mathcal{L}(H) + \lambda \ .$$

Let B be a closed neighbourhood of P such that $H_t(x) \geq \frac{1}{2}H_t(P)$ for $x \in B$, and define

$$\tilde{S} = \tilde{R}_H^-(\lambda) \cap (B \times \mathbf{R}^2) \ ,$$
$$\tilde{W} = M \times U_H(\lambda) \ .$$

Then there exists a sequence of symplectic embeddings

$$f_i : \tilde{S} \to \tilde{W} \ , \quad i = 1, 2, ...$$

which satisfies:

- *(i) each map f_i coincides with the inclusion on the three sides $s = 0$, $t = 0, 1$ and has contact of infinite order with the inclusion there,*
- *(ii) the sequence converges in the C^∞-topology to the inclusion $\tilde{S} \hookrightarrow \tilde{W}$, and*
- *(iii) for each i, the composite*

$$\tilde{S} \xrightarrow{f_i} \tilde{W} \xrightarrow{\pi} U_H(\lambda)$$

is not onto (where the second map is the projection).

Indeed, with this, it is then an easy matter to complete the proof:

COROLLARY 3.2. *Theorem 1.3 (ii) implies the Stability Theorem 1.7.*

Proof: The set $\tilde{S}$ is the region under the graph of a Hamiltonian $G_{t\in[0,1]} : U \to [\lambda, \infty)$ which equals the constant map λ at $t = 0, 1$ and is infinitely tangent to it there. Since this Hamiltonian is obtained from the initial Hamiltonian $H_{t\in[0,1]}$ by reparametrisation and addition of a constant map, its linearised flow at P is the same as the one of H, and therefore it has no non-trivial closed trajectory in time less than or equal to 1. Now let $\Phi : U_H(\lambda) \to [0, m + \lambda] \times [0, 1]$ be an area preserving map of the form $(s, t) \mapsto (a(s, t), b(t))$ (thus preserving the fibers $t =$ const). Then the maps

$$g_i = (\text{id} \times \Phi) \circ f_i \circ (\text{id} \times \Phi)^{-1} : \bar{S} = (\text{id} \times \Phi)(\tilde{S}) \to M \times [0, m + \lambda] \times [0, 1]$$

are symplectic embeddings, converge to the inclusion, are tangent to infinite order to the inclusion on $s = 0$, and $t = 0, 1$, and do not have surjective projection onto $[0, m + \lambda] \times [0, 1]$. Further, $\bar{S}$ is the graph of a Hamiltonian obtained from G by reparametrisation, and therefore its linearised flow at P has no non-trivial closed trajectory in time ≤ 1. But the condition on the tangency of all maps g_i on the three sides of $\bar{S}$ shows that these maps descend to symplectic embeddings

$$h_i : S = (\text{id} \times \phi)(\bar{S}) \to (\text{id} \times \phi)\left(M \times [0, m + \lambda] \times [0, 1]\right) = M \times D^2(m + \lambda)$$

where $D^2(m+\lambda)$ is the standard closed disk of area $m+\lambda$ and where ϕ is the map taking the (s,t) coordinate to the action-angle coordinates $(c = s, t)$ $(c = \pi r^2,\ 2\pi t = \theta)$. Thus the family h_i gives a local squeezing of the set S inside the split round cylinder, although this set does satisfy the hypothesis on non-existence of closed trajectories of Theorem 1.3. This contradicts the latter theorem. □

Proof of Lemma 3.1: We need first the following definition.

DEFINITION 3.3: For $a \geq 0$, choose a smooth family of functions $\mu_a : [0,1] \to [0,\infty)$ which

(i) increase with a,
(ii) map $(0,1)$ into $(0,\infty)$ and are infinitely tangent to 0 at $t = 0, 1$, and
(iii) are such that the set

$$U_a = \{(s,t) : 0 \leq s \leq \lambda + \mu_a(t)\ ,\ 0 \leq t \leq 1\}$$

has area $\lambda + a$.

Then we will say that $\tilde{R}_{H,K,\varepsilon}(\lambda)$ is a *square cylinder* of area a if there is a smooth normalized symplectomorphism

$$\Phi_{H,K} : (\tilde{R}_{H,K,\varepsilon}(\lambda),\ \omega \oplus \sigma) \to (M \times U_a,\ \omega \oplus \sigma)\ .$$

This is possible only if $a = \operatorname{vol} \tilde{R}_H^- + \operatorname{vol} \tilde{R}_{K,\varepsilon}^+$ (recall that $\operatorname{vol} M$ has been set equal to 1), and if $\tilde{R}_{H,K}$ has trivial monodromy (or, equivalently, that the time 1 maps of the flows of H_t and K_t are the same).

The *front face* of a square cylinder consists of the points which map onto

$$\{(x, \lambda + \mu_a(t), t) \mid t \in [0,1]\ ,\ x \in M\}\ .$$

The following lemma is an adaptation of [LM3, Lemma 2.6] to the present context.

LEMMA 3.4. *Fix $\phi \in \operatorname{Ham}^c(M)$ and let H_t, K_t be Hamiltonians with flows ϕ_t, ψ_t from* id *to ϕ normalized as above. Then, there is a C^1-neighbourhood $\mathcal{U}$ of* id *in $\operatorname{Ham}^c(M)$ such that $\tilde{R}_{H,K,\varepsilon}(\lambda)$ is a square cylinder whenever $\phi_t \circ \psi_t^{-1} \in \mathcal{U}$ for all t.*

Proof: First observe that because $\phi_1 = \psi_1$, the gluing map $\Psi_{K,H}$ defined above by

$$\Psi_{K,H}(x,s,t) = \left(\phi_t \circ \psi_t^{-1}(x), s + \lambda - K_t(x) + H_t(\phi_t \circ \psi_t^{-1}(x)),\ t\right)$$

equals the identity when $t = 1$. Therefore, if we extend it by the identity, it defines a normalized map

$$\Psi_K : \partial \tilde{R}_{H,K,\varepsilon}(\lambda) \to M \times \partial U_a$$

where $a = \mathrm{vol}\,\tilde{R}_H^- + \mathrm{vol}\,\tilde{R}_{K,\varepsilon}^+$. We take $\mathcal{U}$ to be a star-shaped neighbourhood of *id* in $\mathrm{Ham}^c(M)$ consisting of Hamiltonian diffeomorphisms ψ whose graphs lie close enough to the diagonal *diag* in $(M \times M, -\omega \oplus \omega)$ to correspond to graphs of 1-forms $\rho(\psi)$ in $(T^*M, -d\,\lambda_{\mathrm{can}})$. Then, if K_t is so close to H_t that the corresponding paths $\{\psi_t\}, \{\phi_t\}$ satisfy $\phi_t \circ \psi_t^{-1} \in \mathcal{U}$ for all t, there is a unique choice of retracting homotopy $f_{c,t}$ from $f_{0,t} = \mathrm{id}$ to $f_{1,t} = \phi_t \circ \psi_t^{-1}$ defined by

$$\rho(f_{c,t}) = c\,\rho(\phi_t \circ \psi_t^{-1}) \ .$$

It will be convenient to parametrize this homotopy a little differently, by the points of $U_a \cap \{s \geq \lambda\}$ instead of the (c,t) square. In fact, because H_t and K_t are both infinitely tangent to 0 at $t = 0, 1$, there is for each K_t a smooth map

$$g_K : U_a \cap \{s \geq \lambda\} \to \mathrm{Ham}^c(M)$$

such that

$$g_K(0,t) = \mathrm{id}\ , \quad g_K(s,t) = f\big(c(s,t),t\big)\ , \quad g_K\big(\mu_a(t),t\big) = \phi_t \circ \psi_t^{-1}\ .$$

(Note that the reparametrization map $c(s,t)$ itself need not be smooth at $t = 0, 1$.) Moreover, we may assume that $g_K(s,t)$ is infinitely tangent to the identity along the line $s = 0$. Then g_K may be used to extend Ψ_K to a smooth normalized map

$$\widetilde{\Psi}_K : \tilde{R}_{H,K,\varepsilon}(\lambda) \to M \times U_a\ ,$$

which has the form

$$(x,s,t) \mapsto \big(g_K(s,t)x, s'(x,s,t), t\big)$$

on $\tilde{R}_{H,K,\varepsilon} \cap \{s \geq 0\}$ for some suitable function s'. When $\mathcal{U}$ is sufficiently C^1-small (that is when K_t is C^2-close to H_t), the push-forward

$$\widetilde{\Omega} = (\widetilde{\Psi}_K)_*(\omega \oplus \sigma)$$

on $M \times U_a$ restricts to an area form on each flat disc $pt \times U_a$. The standard Moser method now shows that there is an isotopy f_t of $M \times U_a$ which is the identity on $M \times \partial U_a$ (and is infinitely tangent to the identity on the sides $s = 0$, $t = 0, 1$) such that $f_1^*(\widetilde{\Omega}) = \omega \oplus \sigma$. See [LM1, Lemma 2.3]. To make f_t have the required properties near $M \times \partial U_a$, one should first adjust $\widetilde{\Omega}$ near the front face and then adjust it inside. Further details are left to the reader. □

LEMMA 3.5. *Let H_t be as in Lemma 3.4, and recall that $\tilde{S} = \tilde{R}_H^-(\lambda) \cap B \times \mathbf{R}^2$, where $B \subset M$ is a neighbourhood of the fixed maximum P on which H_t is $\geq m_H(t)/2$. Then, given the sequences K_t^i converging C^∞ to H_t and ε_i converging to 0, we may construct the normalized maps $\widetilde{\Psi}_{K^i}$ so that their* *restrictions to $\tilde{S}$ converge to the inclusion*

$$\iota : \tilde{S} \hookrightarrow \tilde{R}_H^-(\lambda) \subset M \times U_m$$

where $m = \mathcal{L}(H)$.

Proof: The reader may check that at each point of the above construction the distance of the map from ι is dominated by the distance of K_t from H_t. We need to assume that $H_t(x)$ is bounded away from 0 on B in order that the function s' which gives the s-coordinate of $\widetilde{\Psi}_K$ is well-behaved (and independent of ε). Note also that the size of the isotopy provided by Moser's method depends only on the distance between the endpoints $\widetilde{\Omega}$ and $\omega \oplus \sigma$ of the isotopy of forms, which in the given situation can be assumed to tend to 0. □

Lemma 3.1 is a direct consequence of the preceeding two lemmas.

This completes the proof of Theorem 1.7 when M is closed. To get rid of this last hypothesis, assume that M is non-compact and without boundary. If $H_{t\in[0,1]}$ is not stable with respect to the (strong) C^∞-topology, this means that there is a compact set $X \subset M$ containing the support of H and a sequence of Hamiltonians $K^i_{t\in[0,1]}$ with support in X such that $K^i \to H$ on X with, say,

$$\text{area}\left(\tilde{R}_{H,K^i,\varepsilon}(\lambda)\right) < \text{area}\left(U_H(\lambda)\right) ,$$

for all i. Then the proof goes as before.

Finally, if (M, ω) is a manifold with boundary, one may slightly extend both M and ω to a small open collar neighbourhood V of ∂M. Then, if all Hamiltonian isotopies were the identity on some neighbourhood of ∂M, one would first extend them by the identity to V and apply the previous argument.

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