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L^p-ESTIMATES FOR OSCILLATORY INTEGRALS IN SEVERAL VARIABLES

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1. Introduction and Summary

This paper is concerned with Hörmander's problem on the behaviour of the operators

$$T_N f(x) = \int e^{iN\varphi(x,y)} a(x,y) f(y) dy \tag{1.1}$$

where $a \in C_0^\infty(\mathbb{R}^{2d-1})$ and $\varphi \in C^\infty(\mathbb{R}^{2d-1})$ is a real valued function satisfying the conditions for $(x,y) \in \operatorname{supp} a$

$$\operatorname{rank} \partial^2\varphi/\partial x\, \partial y = d-1 \tag{1.2}$$

$$\frac{\partial}{\partial y}\left\langle \frac{\partial\varphi}{\partial x}, t\right\rangle = 0\ , \quad 0 \neq t \in \mathbb{R}^d \Rightarrow \det\left(\frac{\partial^2}{\partial y^2}\left\langle \frac{\partial\varphi}{\partial x}, t\right\rangle\right) \neq 0\ . \tag{1.3}$$

The second condition (1.3) means that the map $y \mapsto \left\langle \frac{\partial\varphi}{\partial x}, t\right\rangle$ has only non-degenerated critical points.

In [H], Hörmander considers the problem whether under the above hypothesis, there is an inequality

$$\|T_N f\|_q \leq c_{d,q,r} N^{-d/q} \|f\|_r \tag{1.4}$$

for

$$q > \frac{2d}{d-1} \quad \text{and} \quad \frac{d+1}{(d-1)q} + \frac{1}{r} \leq 1\ . \tag{1.5}$$

If $d=2$, the answer is affirmative. This fact is proved in [H] and is essentially due to Carleson and Sjölin [CS]. The interest of the previous conjecture is in particular a positive solution to the Bochner Riesz summability problem, i.e. the fact that the multiplier ($\lambda > 0$)

$$\begin{cases} m_\lambda(\xi) = (1-|\xi|^2)^\lambda & \text{if} \quad |\xi| \le 1 \\ \qquad\quad = 0 & \text{if} \quad |\xi| > 1 \end{cases}, \tag{1.6}$$

defines a bounded Fourier multiplier on $L^p(\mathbf{R}^d)$ if

$$\frac{2d}{d+1+2\lambda} < p < \frac{2d}{d-1-2\lambda} . \tag{1.7}$$

(See [CS] and [H] for the relation between these conjectures.) For $d \ge 3$, it is shown in [St1] (see Th. 10) that Hörmander's conjecture is valid in the range $q \ge \frac{2(d+1)}{d-1}$. Similarly, the Bochner-Riesz summability conjecture was known to be true assuming $p \notin \left]\frac{2(d+1)}{d+3}, \frac{2(d+1)}{d-1}\right[$ (as a consequence of L^2-restriction theory) and the author recently narrowed this interval to $\left]\frac{2(d+1)-\varepsilon}{d+3-\varepsilon}, \frac{2(d+1)-\varepsilon}{d-1}\right[$ for some $\varepsilon = \varepsilon(d) > 0$ (see [B1], [B2]; in particular for $d = 3$, one has $\varepsilon = \frac{8}{75}$). Its full validity is at the present still undecided and depends on unsettled questions in geometric measure theory (see [B1]).

Besides an approach to Bochner-Riesz, Hörmander's conjecture also generalizes the so-called restriction conjecture, which is the special case of a phase function $\varphi(x,y)$ which is linear in x. In this case, the validity of (1.4) for $r = 2$, $q > \frac{2(d+1)}{d-1}$ is a result due to P. Tomas [T] and the case $r = 2$, $q \ge \frac{2(d+1)}{d-1}$ appears in [St] (Th. 3). This last result is the L^2-restriction theorem (which is a sharp result). Again in [B1], (1.4) was proved for $\varphi(x,y)$ linear in x and $r = q > \frac{2(d+1)}{d-1} - \varepsilon$, where $\varepsilon = \varepsilon(d) > 0$ ($\varepsilon(3) = \frac{2}{15}$ in particular, see [B2]). This statement is obviously not a complete solution but goes beyond the L^2-methods and involves new ideas of geometric nature. The problem reduces to phase functions of the form

$$\varphi(x,y) = x_1 y_1 + \cdots + x_{d-1} y_{d-1} + x_d \psi(y) \tag{1.8}$$

where

$$\det\left(\frac{\partial^2}{\partial y^2}\psi\right) \neq 0 . \tag{1.9}$$

Coming back to the general case, as observed in [H], one may take φ of the form

$$\varphi(x,y) = x_1y_1 + \cdots + x_{d-1}y_{d-1} + x_d \langle Ay, y\rangle + O\big(|x||y|(|x|^2+|y|^2)\big) \ , \quad (1.10)$$

where A is a symmetric matrix. In fact, by an additional coordinate change in the variables x, y, one gets

$$\varphi(x,y) = x_1y_1 + \cdots + x_{d-1}y_{d-1} + x_d \langle Ay, y\rangle + O\big(|x_d||y|^3 + |x|^2|y|^2\big) \quad (1.11)$$

($|x|, |y|$ are confined to a small neighborhood of 0). Condition (1.3) amounts to $\det A \neq 0$.

In this paper, we only consider the case $d = 3$.

Our first aim is to exhibit some simple examples showing that under hypothesis (1.2), (1.3) inequality (1.4) may fail for all $q < 4$ and $f \in L^\infty$. Hence, even for $r = \infty$, Th. 10 of [St1] is optimal. It turns out that in fact (1.5) does *not* imply (1.4) for a generic phase function ϕ. (This does not include however those functions appearing in the context of the Bochner-Riesz problem described above.) The argument here is more elaborate and involves geometric considerations related to the Kakeya phenomenon. There are similarities with the approach in [Fe]. This discussion will show that the presence of the $o\big(|x|^2|y|^2\big)$-term in (1.11) is significant and in some sense the case of linearity in x is special. Essentially speaking, there is a difference between straight tubes and "distorted" tubes with respect to the Kakeya compression phenomenon, which is roughly the main point in these considerations. Finally, it is shown that (1.4) holds for certain $q < 4$ and $r = \infty$, for "most" real analytic phase functions φ of the form

$$\varphi(x,y) = x_1y_1 + x_2y_2 + x_3 \langle Ay, y\rangle + \psi(x_3, y) \ , \quad (1.12)$$

where $\psi(x_3, y) = O\big(|x_3||y|^2(|x_3| + |y|)\big)$. See the theorem at the end of section 6. The argument is closely related to section 5 of [B1].

The main difficulty comes from the fact that one has to deal with a Kakeya maximal function defined from certain curves rather than straight lines. Conditions (1.2) and (1.3) do not exclude that the corresponding curves may be pushed by a y-translation in a 2-dimensional surface, also for phase functions of the form (1.12) (this is however a non-generic behaviour). In proving (1.4) for $q < 4$, we do not want the previous phenomenon to happen.

This work aims to get some better understanding of the oscillatory integral problems in higher dimension. Many natural questions are only very partially solved. It shows, however, the importance of certain geometric structures and significant differences between the two and higher dimensional situations.

The author wishes to thank T. Wolff for discussions on the subject.

2. An Example

Consider the following phase function ($d = 3$)

$$\phi(x,y) = \phi(x_1,x_2,x_3,y_1,y_2) = x_1y_1 + x_2y_2 + x_3y_1y_2 + \tfrac{1}{2}x_3^2y_1^2 \ . \tag{2.1}$$

Thus

$$\frac{\partial^2\varphi}{\partial x\,\partial y} = \begin{pmatrix} 1 & 0 & y_2 + 2x_3y_1 \\ 0 & 1 & y_1 \end{pmatrix} \tag{2.2}$$

has rank 2. Further, assuming $\frac{\partial}{\partial y}\left\langle \frac{\partial\varphi}{\partial x}, t\right\rangle = 0$, $t \neq 0$, i.e.

$$\begin{cases} t_1 + t_3(2x_3y_1 + y_2) = 0 \\ t_2 + t_3 = 0 \end{cases} \tag{2.3}$$

$$\det\left(\frac{\partial^2}{\partial y^2}\left\langle\frac{\partial\varphi}{\partial x}, t\right\rangle\right) = \begin{vmatrix} 2x_3t_3 & t_3 \\ t_3 & 0 \end{vmatrix} = -t_3^2 \neq 0$$

(assuming $|x|$ sufficiently small).

The operators (1.1) is applied to the function

$$f(y) = e^{\frac{N}{2}iy_2^2} \tag{2.4}$$

and one finds the expression

$$T_Nf(x) = \int e^{iN\{x_1y_1+x_2y_2+\frac{1}{2}[y_2+x_3y_1]^2}a(x,y)dy \ . \tag{2.5}$$

Consider the surface

$$S = \{x_1 = x_2x_3 \mid |x| < 1\} \tag{2.6}$$

and denote by S_δ the set $\{|x_1 - x_2x_3| < \delta \mid |x| < 1\}$.

On S, one has

$$T_N f(x) = \int e^{iN\{x_2(y_2+x_3y_1)+\frac{1}{2}(y_2+x_3y_1)^2\}} a(x,y)dy \ . \tag{2.7}$$

Putting

$$z = y_2 + x_3y_1 \tag{2.8}$$

the phase function becomes

$$x_2z + \tfrac{1}{2}z^2 \tag{2.9}$$

which has a critical point at $z = -x^2$. Hence the expected size of $T_N f$ on S is thus $\sim N^{-1/2}$. Since also clearly

$$|\nabla_x T_N f| < CN^{1/2} \ , \tag{2.10}$$

it follows that for $\delta \sim N^{-1}$

$$|T_N f| \sim N^{-1/2} \quad \text{on} \quad S_\delta \ . \tag{2.11}$$

Thus, there is a lower estimate

$$\|T_N f\|_q > cN^{-1/2}|S_\delta|^{1/q} \sim N^{-\frac{1}{2}-\frac{1}{q}} \ . \tag{2.12}$$

The validity of inequality (1.4) for $r = \infty$ thus requires

$$CN^{-3/q} > N^{-\frac{1}{2}-\frac{1}{q}} \ , \tag{2.13}$$

hence $q \geq 4$.

Remark 1: In the next section, we will develop a different method of disproving the validity of inequality (1.4) with $r = \infty$ and q sufficiently close to 3. Those considerations, related to [Fe], make the connection with "Kakeya type" phenomena and will permit us to disprove the conjecture for most phase functions $\varphi(x, y)$ of the form (1.11).

Remark 2: For the behaviour of exponential integrals, we refer the reader to [St1] or to [I], Chapter 2.

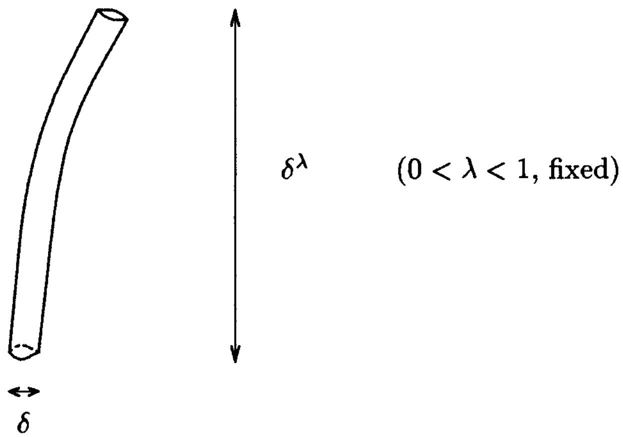
Remark 3: The example described above may be adjusted to show that for d *odd*, the $g \geq \frac{2(d+1)}{d-1}$ condition may be necessary for (1.4) to hold, letting $r = \infty$. The situation d *even* is likely to be different, but this will be investigated elsewhere.

3. Generic Failure

We first explain the general pattern of the construction. Given $\varphi(x,y)$, we are concerned with the inequality

$$\int \left| \int g(x) e^{iN\varphi(x,y)} a(x,y) dx \right| dy \leq CN^{-3/q} \|g\|_{q'} \ , \tag{3.1}$$

for $3 < q$, which is the dual form of (1.4) for $r = \infty$. Let $\delta \sim N^{-1/2}$ and consider a δ-net $\{y(\alpha)\}$ in the (y_1, y_2)-variable. To each α, we associate a "tube" T_α of the form



and let

$$g(x) = g_\varepsilon(x) = \sum_\alpha \varepsilon_\alpha e^{-iN\varphi(x,y_\alpha)} \chi_{T_\alpha}(x) \ , \tag{3.2}$$

where χ_{T_α} stands for the indicator function of T_α and the ε_α are random ± 1 signs. We also let

$$A = \bigcup_\alpha T_\alpha \ . \tag{3.3}$$

Assuming (3.1) valid and integrating on $\varepsilon_\alpha = \pm 1$, it follows

$$\int \max_\alpha \left| \int_{T_\alpha} e^{iN[\varphi(x,y)-\varphi(x,y_\alpha)]} a(x,y) dx \right| dy \leq$$

$$\int \Big(\sum_\alpha \left| \int_{T_\alpha} e^{iN[\varphi(x,y)-\varphi(x,y_\alpha)]} a(x,y) dx \right|^2 \Big)^{1/2} dy \leq$$

$$CN^{-3/q} \int \|g_\varepsilon\|_{q'} d\varepsilon \leq$$

$$CN^{-3/q}\Big(\int \big(\sum \chi_{T_\alpha}\big)^{q'/2} dx\Big)^{1/q'} . \tag{3.4}$$

From (3.3) and Hölder's inequality, estimate further

$$\int \big(\sum \chi_{T_\alpha}\big)^{q'/2} \le |A|^{1-\frac{q'}{2}} \big(\sum |T_\alpha|\big)^{q'/2} \tag{3.5}$$

(| | denotes "measure") which yields following bound on (3.4)

$$C \cdot N^{-3/q} |A|^{\frac{1}{2}-\frac{1}{q}} \big(\sum |T_\alpha|\big)^{1/2} . \tag{3.6}$$

Write next for $|y - y_\alpha| < \delta$

$$\varphi(x,y) = \varphi(x,y_\alpha) + \langle \nabla_y \varphi(x,y_\alpha), y - y_\alpha\rangle + O(\delta^2) \tag{3.7}$$

and assume there is a function $\Omega = \big(\Omega_1(y), \Omega_2(y)\big)$ satisfy

$$\big|\nabla_y \varphi(x,y_\alpha) - \Omega(y_\alpha)\big| < \delta \quad \text{for} \quad x \in T_\alpha . \tag{3.8}$$

On gets from (3.7), (3.8) and the choice of δ

$$\varphi(x,y) = \varphi(x,y_\alpha) + \langle \Omega(y_\alpha), y - y_\alpha\rangle + o\left(\frac{1}{N}\right) , \tag{3.9}$$

if $x \in T_\alpha$.

The left member of (3.4) is thus at least

$$\sum_\alpha \int_{|y-y_\alpha|<\delta} \left| \int_{T_\alpha} e^{iN[\varphi(x,y)-\varphi(x,y_\alpha)]} a(x,y) dx \right| \sim \delta^2 \sum |T_\alpha| , \tag{3.10}$$

assuming $a = 1$ if $x \in T_\alpha$ and $|y - y_\alpha| < \delta$.

By (3.10), (3.6) we have

$$\big(\sum |T_\alpha|\big)^{1/2} \le CN^{-3/q} \delta^{-2} |A|^{\frac{1}{2}-\frac{1}{q}} . \tag{3.11}$$

Taking the shape of the T_α into account, this yields

$$\delta^{\lambda/2} \le CN^{-3/q} \delta^{-2} |A|^{\frac{1}{2}-\frac{1}{q}} \tag{3.12}$$

$$|A| > C_q N^{-\frac{1+\frac{\lambda}{4}-\frac{3}{q}}{\frac{1}{2}-\frac{1}{q}}} \tag{3.13}$$

and letting $q \overset{>}{\to} 3$

$$|A| > N^{-\frac{3}{2}\lambda - \varepsilon} . \tag{3.14}$$

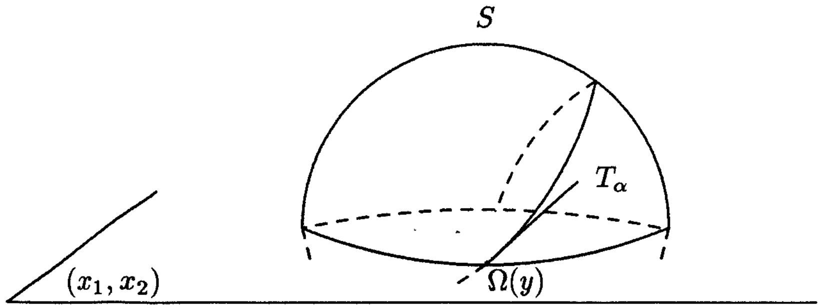
Remark 3.15: Consider the case of $\varphi(x,y)$ which is linear in x, for instance

$$\varphi(x,y) = x_1y_1 + x_2y_2 + \tfrac{1}{2}x_3(y_1^2 + y_2^2)\ . \tag{3.15}$$

Condition (3.8) then becomes (writing y for y_α)

$$\begin{cases} \big|x_1 + x_3y_1 - \Omega_1(y)\big| < \delta \\ \big|x_2 + x_3y_2 - \Omega_2(y)\big| < \delta \end{cases} \quad \text{for} \quad \alpha \in T_\alpha\ . \tag{3.16}$$

Eqs. (3.16) define a tube in direction $(y_1, y_2, 1)$ subject to some translation $\Omega(y)$. The width of the tube is δ and the length given by the bound on $|x_3|$, thus δ^λ for T_α. Inequality (3.14) for $\lambda = 0$ is simply the fact that the restriction conjecture implies that Kakeya sets in $\mathbb{R}^3$ have Hausdorff dimension 3 (or, more precisely, the equivalent entropy statement). This statement appears to be an open problem and the best lower bound the author knows of presently is 7/3 (see [B1]). Consider next for S_δ a δ-shell of the unit sphere.



Every T_α-tube may be translated to be contained in S_δ, if we let $\lambda = \frac{1}{2}$ (the line $(-y_1, -y_2, 1)$ should lie in the corresponding tangent plane at $\Omega(y)$). For such $A \subset \big\{x \in S_\delta \ \big|\ |x_3| < \delta^\lambda\big\}$, one has $|A| \sim \delta^{1+\lambda}$ and (3.14) gives $\frac{3}{2}\lambda \geq \frac{1}{2}(1+\lambda)$, i.e. $\lambda \geq \frac{1}{2}$ (equality). This shows the relevance of (3.14) and the fact that $\lambda = \frac{1}{2}$ is optimal in the previous set-up, if one takes (1.4), $r = \infty$, $q > 3$, for valid. The main idea is disproving (1.4) for certain phase functions φ is to achieve the above construction for a suitable surface S and certain $\lambda < \frac{1}{2}$.

Consider a general phase function φ and the normal form (1.11)

$$\psi(x,y) = \varphi(\overline{x} + x, \overline{y} + y) = x_1y_1 + x_2y_2 + x_3\,\langle Ay, y\rangle + O\big(|x_3||y|^3 + |x|^2|y|^2\big) \tag{3.17}$$

in a neighborhood of the point $(\overline{x}, \overline{y})$. In fact, we assume

$$|x| < \delta^\tau \ , \quad |x_3| < \delta^{1/3} \ , \quad |y| < \delta^\tau \ , \tag{3.18}$$

where $\delta \sim N^{-1/2}$ and τ is any number $< 1/3$.

Condition (3.8) becomes a δ-estimate on both expressions

$$\begin{cases} x_1 + a_{11}x_3y_1 + a_{12}x_3y_2 + x_3\partial_{y_1}K + x_1\big(L^1_{11}(x)y_1 + L^1_{12}(x)y_2\big) + \\ x_2(L^2_{11}(x)y_1 + L^2_{12}(x)y_2) + x_3^2(q_{11}y_1 + q_{12}y_2) - \Omega_1(y) \\ \\ x_2 + a_{12}x_3y_1 + a_{22}x_3y_2 + x_3\partial_{y_2}K + x_1(L^1_{12}y_1 + L^1_{22}y_2) + \\ x_2(L^2_{12}y_1 + L^2_{22}y_2) + x_3^2(q_{12}y_1 + q_{22}y_2) - \Omega_2(y) \end{cases} \tag{3.19}$$

where

$$\begin{aligned} \psi(x,y) = {} & x_1y_1 + x_2y_2 + x_3\,\langle Ay, y\rangle + x_3K(y) + \\ & x_1\big[\tfrac{1}{2}L^1_{11}y_1^2 + L^1_{12}y_1y_2 + \tfrac{1}{2}L^1_{22}y_2^2\big] + \\ & x_2\big[\tfrac{1}{2}L^2_{11}y_1^2 + L^2_{12}y_1y_2 + \tfrac{1}{2}L^2_{22}y_2^2\big] + \\ & x_3^2\big(\tfrac{1}{2}q_{11}y_1^2 + q_{12}y_1y_2 + \tfrac{1}{2}q_{22}y_2^2\big) + \\ & O\big(|x_3||y|^4 + |x|^2|y|^3 + |x|^3|y|^2\big) \end{aligned} \tag{3.20}$$

and $A = \begin{pmatrix} \frac{1}{2}a_{11} & a_{12} \\ a_{12} & \frac{1}{2}a_{22} \end{pmatrix}$, K is a cubic function of y and the L^i_{jk} are linear functions of x. Notice that the error terms of (3.20) do not enter in (3.19) because of (3.18), assuming

$$4\tau > 1 \ . \tag{3.21}$$

Since $|\nabla K| = O\big(|y|^2\big)$, a change of coordinates in the y-variable of the form $I + O\big(|y|\big)$ permits the elimination of $x_3\partial_{y_i}K$-terms in (3.19). The expressions may be rewritten as

$$\begin{cases} x_1(1 + L^1_{11}y_1 + L^1_{12}y_2) + x_2(L^2_{11}y_1 + L^2_{12}y_2) + x_3(a_{11}y_1 + a_{12}y_2) + \\ x_3^2(q_{11}y_1 + q_{12}y_2) - \Omega_1(y) \\ \\ x_2(1 + L^2_{12}y_1 + L^2_{22}y_2) + x_1(L^1_{12}y_1 + L^1_{22}y_2) + x_3(a_{12}y_1 + a_{22}y_2) + \\ x_3^2(q_{12}y_1 + q_{22}y_2) - \Omega_2(y) \end{cases} \tag{3.22}$$

Here Ω_1, Ω_2 remain arbitrary functions of (y_1, y_2), $\Omega(0) = 0$.

Consider the following transformation $F = F_{x_3,y}$ of the (x_1, x_2)-variable

$$F(x_1,x_2) = \big((1+L^1_{11}y_1+L^1_{12}y_2)x_1 + (L^2_{11}y_1+L^2_{12}y_2)x_2,\\ (L^1_{12}y_1+L^1_{22}y_2)x_1+(1+L^2_{12}y_1+L^2_{22}y_2)x_2\big)\ , \tag{3.23}$$

clearly of the form $Id + O\big(|y|(|x_1|+|x_2|)\,(|x_1|+|x_2|+|x_3|)\big)$. Hence, up to $o(\delta)$-error terms

$$F^{-1}\big(-x_3(a_{11}y_1+a_{12}y_2) - x_3^2(q_{11}y_1+q_{12}y_2)+\Omega_1(y),\\ -x_3(a_{12}y_1+a_{22}y_2)-x_3^2(q_{12}y_1+q_{22}y_2)+\Omega_2\big)$$

has the form

$$\begin{cases} -x_3(a_{11}y_1+a_{12}y_2)-x_3^2(q_{11}y_1+q_{12}y_2)+\Omega_1(y)\\ +O\big((|\Omega_1|+|\Omega_2|)|y|(|y|+|x_3|)\big)\\ \\ -x_3(a_{12}y_1+a_{22}y_2)-x_3^2(q_{12}y_1+q_{22}q_2)+\Omega_2(y)\\ +O\big((|\Omega_1|+|\Omega_2|)|y|(|y|+|x_3|)\big)\end{cases} \tag{3.24}$$

The error terms in (3.24) have the form $O\big(|y|^2(|\Omega_1|+|\Omega_2|)\big)$ and $x_3\ \ O\big(|y|(|\Omega_1|+|\Omega_2|)\big)$. The second type of term may be absorbed in the $x_3(a_{11}y_1+a_{12}y_2)$, $x_3(a_{12}y_1+a_{22}y_2)$ terms and eliminated by a change of variable in the y-variable of the form $Id+O\big(|y||\Omega|\big)$. This change of variable depends on the Ω-functions. Thus up to δ-error terms, (3.24) becomes (denoting again y the new variable)

$$\begin{cases} -x_3(a_{11}y_1+a_{12}y_2)-x_3^2(q_{11}y_1+q_{12}y_2)+\Omega_1\big(y+O(|y||\Omega|)\big)\\ \\ -x_3(a_{12}y_1+a_{22}y_2)-x_3^2(q_{12}y_1+q_{22}y_2)+\Omega_2\big(y+O(|y||\Omega|)\big)\end{cases} \tag{3.25}$$

Since one considers functions in y vanishing at $y=0$, an appropriate choice of Ω_1,Ω_2 still permits the realization of arbitrary y-functions $\widetilde\Omega_1(y),\widetilde\Omega_2(y)$ for the last terms in (3.25). This follows from an implicit function argument and the assumption $|y|<\delta^\tau$.

The original condition (3.8) becomes

$$\begin{cases} \big|x_1+x_3(a_{11}y_1+a_{12}y_2)+x_3^2(q_{11}y_1+q_{12}y_2)-\widetilde\Omega_1(y)\big|<\delta\\ \\ \big|x_2+x_3(a_{12}y_1+a_{22}y_2)+x_3^2(q_{12}y_1+q_{22}y_2)-\widetilde\Omega_2(y)\big|<\delta\end{cases} \tag{3.26}$$

for x in a tube δ-tube $T = T_{y_1,y_2}$ as described above.

We will let $\widetilde{\Omega}$ be a linear function of y, i.e.

$$\widetilde{\Omega}(y) = (\omega_{11}y_1 + \omega_{12}y_2, \omega_{21}y_1 + \omega_{22}y_2) \ , \tag{3.27}$$

where the ω_{ij} will be suitably chosen. More precisely, we want the Jacobian determinant of the map

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{pmatrix} (\omega_{11} - x_3a_{11} - x_3^2q_{11})y_1 + (\omega_{12} - x_3a_{12} - x_3^2q_{12})y_2 \\ (\omega_{21} - x_3a_{12} - x_3^2q_{12})y_1 + (\omega_{22} - x_3a_{22} - x_3^2q_{22})y_2 \end{pmatrix} \tag{3.28}$$

to be $o(|x_3|^3)$. This will permit us to conclude that for fixed $|x_3| < \delta^{1/3}$ (cf. 3.18) the map $(y_1, y_2) \mapsto (x_1, x_2)$ given by (3.28) ranges in a δ-neighborhood of a line. Keeping $|y| < \delta^\tau$ fixed and varying x_3 in the interval $[0, \delta^{1/3}]$ will yield a curve Γ_y that, by previous considerations, is contained in a set of measure

$$|\Gamma| \sim \delta^{\tau+1+\frac{1}{3}} \tag{3.29}$$

and which is a union of δ-balls. If T_y is the tube obtained as δ-neighborhood of Γ_y, (3.26) clearly holds and T_y is contained in a set A of measure

$$|A| \sim \delta^{\tau+\frac{4}{3}} \ . \tag{3.30}$$

It remains to examine under what conditions on $A = (a_{ij})_{1\le i,j\le 2}$ and $Q = (q_{ij})$ the matrix $\widetilde{\Omega}$ may be chosen such that

$$\det(\widetilde{\Omega} - x_3A - x_3^2Q) = O(|x_3|^3) \ . \tag{3.31}$$

Since $\det A \neq 0$, $\widetilde{\Omega}$ may be replaced by $\overline{\Omega} = (\overline{\omega}_{ij}) = A^{-1}\widetilde{\Omega}$ and for $\overline{Q} = A^{-1}Q = (\overline{q}_{ij})$, (3.31) becomes

$$\det(\overline{\Omega} - x_3I - x_3^2\overline{Q}) = O(|x_3|^3) \ . \tag{3.32}$$

We have to satisfy the following conditions for $\overline{\Omega}$

$$\begin{cases} \det \overline{\Omega} = 0 \\ \operatorname{tr} \overline{\Omega} = 0 \\ 1 = \overline{\omega}_{11}\overline{q}_{22} + \overline{\omega}_{22}\overline{q}_{11} - \overline{\omega}_{21}\overline{q}_{12} - \overline{\omega}_{12}\overline{q}_{21} \end{cases} \tag{3.33}$$

In order to satisfy the first 2 conditions in (3.33), let

$$\overline{\omega}_{11} = \omega \ , \quad \overline{\omega}_{22} = -\omega \ , \quad \overline{\omega}_{12} = \gamma\omega \ , \quad \overline{\omega}_{21} = -\frac{1}{\gamma}\omega \ , \tag{3.34}$$

where ω, γ are parameters. The last condition then becomes

$$1 = \omega\left(\overline{q}_{22} - \overline{q}_{11} - \gamma\overline{q}_{21} + \frac{1}{\gamma}\overline{q}_{12}\right) . \tag{3.35}$$

This equation may always be satisfied, except if $\overline{q}_{11} = \overline{q}_{22}$, $\overline{q}_{12} = 0 = \overline{q}_{21}$, i.e. Q is a multiple of A. To avoid this, we thus require

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\psi}{\partial x_3^2}\right)\Bigg|_{\substack{x=0\\y=0}} \quad \text{is not a multiple of} \quad \frac{\partial^2}{\partial y^2}\left(\frac{\partial\psi}{\partial x_3}\right)\Bigg|_{\substack{x=0\\y=0}} , \tag{3.36}$$

where ψ is given by (3.17), (3.20).

If (3.36) holds, one may thus associate to the points y, $|y| < \delta^\tau$, a tube T_y as described at the beginning of this section $\left(\lambda = \frac{1}{3}\right)$, contained in a set A of measure at most $\delta^{\tau+\frac{4}{3}}$.

Let $\varphi(x,y)$ be given by (1.11), i.e.

$$\varphi(x,y) = x_1y_1 + x_2y_2 + x_3\langle Ay, y\rangle + O\big(|x_3||y|^3 + |x|^2|y^2\big) .$$

Assume

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\varphi}{\partial x_3^2}\right)\Bigg|_{\substack{x=0\\y=0}} \quad \text{is not a multiple of} \quad \frac{\partial^2}{\partial y^2}\left(\frac{\partial\varphi}{\partial x_3}\right)\Bigg|_{\substack{x=0\\y=0}} . \tag{3.37}$$

Take in (3.17) $\overline{x} = 0$ and let $\overline{y}$ range over a δ^τ-net in a neighborhood of 0 (of cardinality $\sim \delta^{-2\tau}$). In order to bring $\varphi(x, \overline{y} + y)$ in the form (3.17), there is a coordinate change required $(x_1, x_2, x_3) \to (x_1', x_3', x_3)$, which is a C^∞-perturbation of the identity, taking $|\overline{y}|$ small.There will also be a small perturbation of the matrix A. Clearly, the matrices appearing in (3.26) are perturbations of those in (3.37) and hence (3.37) yields (3.36) for $\psi(x', y')$ corresponding to $(0, \overline{y})$, if we let $|\overline{y}|$ be sufficiently small.

For each $\overline{y}$ in the net, a system $\zeta_{\overline{y}}$ of tubes $\{T_y \ ; \ |y - \overline{y}| < \delta^\tau\}$ is obtained, contained in a set $A_{\overline{y}}$ of measure at most $\delta^{\tau+\frac{4}{3}}$ and satisfying condition (3.8), i.e.,

$$\big|\partial_y\varphi(x,y) - \Omega(y)\big| < \delta \quad \text{if} \quad x \in T_y \ , \tag{3.38}$$

for a certain function Ω of y.

Define

$$A = \bigcup_{\overline{y}} A_{\overline{y}}\ , \quad |A| < \delta^{\frac{4}{3}-\tau}\ . \tag{3.39}$$

It follows from (3.13), (3.39) that validity of (1.4) for $r = \infty$ and a given $q > 3$ implies

$$\frac{2}{3} - \frac{\tau}{2} \leq \frac{\frac{13}{12} - \frac{3}{q}}{\frac{1}{2} - \frac{1}{q}}\ . \tag{3.40}$$

Here τ is subject to (3.21), i.e. $\tau > \frac{1}{4}$. Inequality (3.40) thus implies

$$q \geq \frac{118}{39} > 3\ , \tag{3.41}$$

If $\varphi(x, y)$ given by (1.10) satisfies (3.37), i.e.

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \varphi}{\partial x_3^2}\right) \quad \text{is not a multiple of} \quad \frac{\partial^2}{\partial y^2}\left(\frac{\partial \varphi}{\partial x_3}\right)\ , \tag{3.42}$$

at $x = 0$, $y = 0$.

Remark 3.43. The function $\varphi(x, y)$ appearing in the context of the Bochner-Riesz problem ($d = 3$) is obtained by considering $|x - y|$ with y restricted to a plane, say $y = (y_1, y_2, 1)$. After obvious changes in the x-variable, one gets

$$\varphi(x, y) = \left(1 + |y|^2 - 2(x_1 y_1 + x_2 y_2) + x_3\right)\ . \tag{3.44}$$

One obtains the form (1.10) by a coordinate change in x given by

$$\begin{cases} x_1' = \partial_{y_1}\varphi\big|_{y=0} - \partial_{y_1}\varphi\big|_{\substack{x=0\\y=0}} \\ x_2' = \partial_{y_2}\varphi\big|_{y=0} - \partial_{y_2}\varphi\big|_{\substack{x=0\\y=0}} \end{cases}$$

and a change of coordinates in the y-variable $y \mapsto y + O(|y|^2)$ that may be ignored in verifying (3.42).

(3.46) yields

$$x_1' = \frac{-x_1}{(1 + x_3)^{1/2}} \qquad x_2' = \frac{-x_2}{(1 + x_3)^{1/2}} \tag{3.47}$$

and (3.44) becomes

$$\psi(x,y) = \left[1 + |y|^2 + x_3 + 2(1+x_3)^{1/2}(x_1y_1 + x_2y_2)\right]^{/2} . \tag{3.48}$$

In verifying (3.42), take $x_1 = 0 = x_2$. Clearly

$$\frac{\partial^2}{\partial y^2}\left[(1+|y|^2+x_3)^{1/2}\right]\Big|_{y=0} \tag{3.49}$$

is a multiple of identity and hence the criterion fails in this case.

4. L^2-estimates Revisited

The rest of the paper is devoted to proving some positive results (i.e. results for $q < 4$). The main idea is to adapt the argument in [B1], section 5. One considers the level sets

$$A_\lambda = \left\{|x| < N \quad \text{and} \quad \Big|\int e^{iN\varphi(\frac{x}{N},y)}a(y)f(y)dy\Big| > \lambda\right\} . \tag{4.1}$$

Under suitable conditions on φ and $f \in L^\infty$, one seeks for an estimate

$$|A_\lambda| < \lambda^{-q+\varepsilon} \tag{4.2}$$

for some $q < 4$. That will allow to get an inequality

$$\|T_N f\|_q \le CN^{-3/q}\|f\|_\infty , \tag{4.3}$$

where $T_N f$ is given by (1.4).

Let us summarize the method. There are essentially 3 steps

(i) Consider the L^2-case, thus $f \in L^2$ and $q = 4$. This case appears in [St1, Th. 10]. We will make a further observation, nl. the fact that $|A_\lambda| \sim \lambda^{-4}$ only may happen if $|A_\lambda \cap B_{\lambda^{-2}}| \sim \lambda^{-4}$ for some ball $B_{\lambda^{-2}} = B(z, \lambda^{-2})$ of radius λ^{-2}.

(ii) Assuming now $f \in L^\infty$, the study of the level set A_λ in a ball of given radius, in order to obtain (4.2) for some $q < 4$, is done combining the L^2-estimates and some estimates on Kakeya-type maximal operators in L^p, $p > 2$. These Kakeya operators are related to φ.

(iii) Proof of certain L^p-inequalities on these Kakeya maximal operators. If $p = 2$ and the excentricity is δ (i.e. we consider δ- neighbourhoods of the curves), a bound $\delta^{-1/2}$ is found. Thus, by interpolation, there is a bound $\delta^{-1/p}$ for $2 \le p \le \infty$. It turns out one just needs for some $p > 2$ to get an estimate $\delta^{-\gamma(p)}$, for some $\gamma(p) < \frac{1}{p}$.

Redefine $T_N f(x) = \int e^{iN\varphi(\frac{x}{N},y)} a(y) f(y) dy$.

In this section, we carry out the first part of the program. Let $0 < \lambda < 1$ and A_λ defined as in (4.1), assuming $\int |f|^2 \le 1$. Fix $R > 1$ and consider a collection A'_α of subsets of $A = A_\lambda$ such that

$$\begin{cases} \operatorname{diam} A'_\alpha \le R & (4.4) \\ \operatorname{dist}(A'_\alpha, A'_\beta) > R \quad \text{for} \quad \alpha \neq \beta & (4.5) \\ \left| \bigcup_\alpha A'_\alpha \right| > c|A| \, . & (4.6) \end{cases}$$

Denote χ_α the indicator function of A'_α. One has (or at least may assume)

$$\operatorname{Re} \sum \langle T_N f, \chi_\alpha \rangle > c\lambda |A| \tag{4.7}$$

hence

$$\Big\| \sum_\alpha T_N^* \chi_\alpha \Big\|_2^2 > c\lambda^2 |A|^2 \, . \tag{4.8}$$

Expand the left member square as

$$\sum_\alpha \|T_N^* \chi_\alpha\|_2^2 + \sum_{\alpha \neq \beta} \langle T_N T_N^* \chi_\alpha, \chi_\beta \rangle \tag{4.9}$$

$$\le \|T_N^*\|_{L^2_{(R)}}^2 \sum_\alpha |A_\alpha| + \sum_{\alpha \neq \beta} c(\alpha,\beta) |A_\alpha| |A_\beta| \tag{4.10}$$

where

$$\|T_N^*\|_{L^2_{(R)}} = \sup \left\| \int e^{iN\varphi(\frac{x}{N},y)} g(x) dx \right\|_{L^2(\mathrm{loc})} \tag{4.11}$$

the supremum being taken over functions g satisfying

$$\|g\|_2 \le 1 \quad \text{and} \quad \operatorname{diam} \operatorname{supp} g \le R \tag{4.12}$$

and $c(\alpha,\beta) = c(\rho)$, $\rho = \operatorname{dist}(A_\alpha, A_\beta)$, $c(\rho)$ denoting a uniform bound on the kernel $K(x,x')$ of the operators $T_N T_N^*$ for $|x - x'| > \rho$. Here

$$K(x,x') = \int e^{iN[\varphi(\frac{x}{N},y) - \varphi(\frac{x'}{N},y)]} a(y) dy \, . \tag{4.13}$$

The square of (4.11) corresponds to the L^2-norm of the operator

$$L(y,y') = \int e^{iN[\varphi(\frac{x}{N},y)-\varphi(\frac{x}{N},y')]} b_R(x-z)dx \ , \tag{4.14}$$

where $z \in \mathbf{R}^3$, $|z| < N$ and b_R a smooth function such that $b_R(x) = 1$ of if $|x| < R$, $b_R(x) = 0$ if $|x| > 2R$ and fulfils the obvious derivative estimates. This L^2-norm may be bounded by

$$\sup_y \int \big|L(y,y')\big| a(y')dy' \tag{4.15}$$

from interpolation and symmetry.

Recall (1.11), i.e.

$$\varphi(x,y) = x_1y_1 + x_2y_2 + x_3\,\langle Ay,y\rangle + o\big(|x_3||y|^3 + |x|^2 y|^2\big) \ . \tag{4.16}$$

Taking $|x| = o(1)$ in (4.16), it follows from the oscillatory integral theory (cf. [St]) that

$$\left|\int e^{iN[\varphi(x,y)-\varphi(x',y)]} a(y)dy\right| < c\frac{1}{N|x-x'|} \tag{4.17}$$

and hence

$$\big|K(x,x')\big| < c\frac{1}{|x-x'|} \tag{4.18}$$

from where

$$c(\rho) \lesssim \rho^{-1} \ . \tag{4.19}$$

Similarly, one gets on (4.14) a bound

$$\big|L(y,y')\big| \lesssim R\cdot R^2 \cdot \psi\big(R(y_1-y_1')\big)\cdot\psi\big(R(y_2-y_2')\big) \tag{4.20}$$

where

$$\psi(t) = \frac{1}{1+t^2} \ . \tag{4.21}$$

Consequently (4.15) $\lesssim R$, thus

$$\|T_N^*\|^2_{L^2_{(R)}} \lesssim R \ . \tag{4.22}$$

Substituting (4.19), (4.22) in (4.10), it follows from (4.8)

$$\lambda^2|A|^2 \lesssim R|A| + \sum_{\alpha\neq\beta} \frac{|A_\alpha||A_\beta|}{\mathrm{dist}(A_\alpha,A_\beta)} \tag{4.23}$$

and therefore, one gets.

PROPOSITION 4.24. *There is the following measure estimate on the level set A_λ given by (4.1)*

$$|A_\lambda| \lesssim \lambda^{-2}\Big[R + \sum_{R<\rho<N} \sup_{|z|<N} \rho^{-1}\big|A_\lambda \cap B(z,\rho)\big|\Big] \tag{4.25}$$

where R is a parameter.

If we let in particular

$$R \sim \lambda^{-2} , \tag{4.26}$$

it follows that

$$|A_\lambda| < C\lambda^{-4} . \tag{4.27}$$

Of course, (4.27) corresponds to the inequality

$$\|T_N f\|_{L^4(B(0,N))} \leq C\|f\|_2 , \tag{4.28}$$

given by Th. 10 of [St].

5. Distributional Estimates in a Ball of Given Radius

Consider for $|f| \leq 1$

$$T_N f(x) = \int e^{iN\varphi(\frac{x}{N},y)} a(y) f(y) dy , \tag{5.1}$$

for x in a ball $B(z,R)$, $1 < R \ll N$, $|z| \ll N$.

Define a new phase function $\psi(x,y)$ given by

$$\psi(x,y) = \frac{N}{R}\left[\varphi\left(\frac{z}{N} + \frac{R}{N}x, y\right) - \varphi\left(\frac{z}{N}, y\right)\right] , \tag{5.2}$$

which still satisfies conditions (1.2), (1.3). Their verification for ψ at (x,y) amounts to the verification for φ at $\left(\frac{z}{N} + \frac{R}{N}x, y\right)$.

We are thus concerned with the operator

$$U_R g(x) = \int e^{iR\psi(\frac{x}{R},y)} a(y) g(y) dy , \tag{5.3}$$

where $x \in B(0,R)$, $|g| \leq 1$.

Write next

$$U_R g(x) \sim R \int e^{iR\psi(\frac{x}{R},y)} G_x(y) a(y) dy \ , \tag{5.4}$$

where

$$G_x(y) = \int e^{iR[\psi(\frac{x}{R},y+y')-\psi(\frac{x}{R},y)]} a\big(\sqrt{R}y'\big) g(y+y') dy' \ . \tag{5.5}$$

Fix $2 \leq q \leq 4$. Write

$$\int_{B_R} |U_R g|^q dx \sim R^{-3/2} \int_{B_R} \left[\int_{B_{\sqrt{R}}} \big|U_R g(x+x')\big|^q dx' \right] dx \ . \tag{5.6}$$

In the definition of $U_R g(x+x')$, there is no harm in replacing $G_{x+x'}(y')$ by $G_x(y')$. We use here the fact that on $B_R \times B_{\sqrt{R}} \times B_1 \times B_{\frac{1}{\sqrt{R}}}$

$$R\left[\psi\left(\frac{x+x'}{R}, y+y'\right) - \psi\left(\frac{x+x'}{R}, y\right) - \psi\left(\frac{x}{R}, y+y'\right) + \psi\left(\frac{x}{R}, y\right)\right] \in L^\infty(x)\widehat{\otimes}L^\infty(x')\widehat{\otimes}L^\infty(y)\widehat{\otimes}L^\infty(y') \ , \tag{5.7}$$

since

$$R\left[\psi\left(\widetilde{x}+\frac{\widetilde{x}'}{\sqrt{R}}, y+\frac{\widetilde{y}'}{\sqrt{R}}\right) - \psi\left(\widetilde{x}+\frac{\widetilde{x}'}{\sqrt{R}}, y\right) - \psi\left(\widetilde{x}, y+\frac{\widetilde{y}'}{\sqrt{R}}\right) + \psi(\widetilde{x}, y)\right] \in L^\infty(\widetilde{x})\widehat{\otimes}L^\infty(\widetilde{x}')\widehat{\otimes}L^\infty(y)\widehat{\otimes}L^\infty(\widetilde{y}') \ , \tag{5.8}$$

for $|\widetilde{x}|, |\widetilde{x}'|, |y|, |\widetilde{y}'| \leq 1$ (as is easily seen by differential calculus). Thus by (5.4), (5.6) becomes after replacement of $G_{x+x'}(y)$ by $G_x(y)$

$$R^{-\frac{3}{2}+q} \int_{B_R} \left[\int_{B_{\sqrt{R}}} \left| \int e^{i\sqrt{R}\eta_x(\frac{x'}{\sqrt{R}},y)} e^{iR\psi(\frac{x}{R},y)} G_x(y) a(y) dy \right|^q dx' \right] dx \ , \tag{5.9}$$

where one introduced the phase-function

$$\eta_x(x', y) = \sqrt{R}\left[\psi\Big(\frac{x}{R} + \frac{1}{\sqrt{R}}x', y\Big) - \psi\Big(\frac{x}{R}, y\Big)\right] \ , \tag{5.10}$$

which again satisfies conditions (1.2), (1.3).

Considering the operator (for fixed x)

$$(V_{\sqrt{R}}h)(x') = \int e^{i\sqrt{R}\eta_x(\frac{x'}{\sqrt{R}},y)}h(y)a(y)dy \ , \tag{5.11}$$

one gets from the $L^2 - L^4$ result (see [St], Th. 10 and previous section)

$$\|V_{\sqrt{R}}h\|_{L^4(B_{\sqrt{R}})} \leq C\|h\|_2 \ . \tag{5.12}$$

Inequality (4.22) from the previous section gives the following L^2-L^2 estimate

$$\|V_{\sqrt{R}}h\|_{L^2(B_{\sqrt{R}})} \leq CR^{1/4}\|h\|_2 \ . \tag{5.13}$$

Writing

$$\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{4} \ , \tag{5.14}$$

interpolation of (5.12), (5.13) yields

$$\|V_{\sqrt{R}}h\|_{L^q(B_{\sqrt{R}})} \leq CR^{\frac{1}{q}-\frac{1}{4}}\|h\|_2 \ . \tag{5.15}$$

Application of (5.15) in (5.9) fixing x and letting $h(y) = e^{iR\psi(\frac{x}{R},y)}G_x(y)$ yields following bound

$$R^{-\frac{3}{2}+q+1-\frac{q}{4}} \int_{B_R} \left[\int |G_x(y)|^2 dy\right]^{q/2} dx =$$

$$R^{\frac{5}{2}+\frac{3q}{4}} \int_{B_1} \left[\int \left|G_{Rx}(y)\right|^2 dy\right]^{q/2} dx \ . \tag{5.16}$$

One has for $|y'| < \frac{1}{\sqrt{R}}$

$$\psi(x,y+y') - \psi(x,y) = \langle \nabla_y \psi(x,y), y'\rangle + O\left(\frac{1}{R}\right) \ . \tag{5.17}$$

One may therefore substitute in (5.16) $G_{Rx}(y)$ by

$$\int e^{iR\langle \nabla_y \psi(x,y),y'\rangle} a(\sqrt{R},y')g(y+y')dy' \ . \tag{5.18}$$

Define for given y

$$k_y(\xi) = \int e^{i\langle \xi, y'\rangle} a(\sqrt{R}y')g(y+y')dy' \ . \tag{5.19}$$

Obviously

$$\operatorname{supp} \widehat{k_y} \subset B_{\frac{1}{\sqrt{R}}} \Rightarrow \operatorname{supp} \widehat{|k_y|^2} \subset B_{\frac{2}{\sqrt{R}}} \ . \tag{5.20}$$

Thus, if $b \in \mathcal{S}$ satisfies $\widehat{b} = 1$ on $B_{\frac{2}{\sqrt{R}}}$, then $|k_y|^2 = |k_y|^2 * b$. Such a b may be majorized by the function

$$R \Big| \int e^{i\langle y', \xi\rangle} a(\sqrt{R}y')dy' \Big|^2 \ . \tag{5.21}$$

Hence

$$\big|k_y(\xi)\big|^2 < CR \int \Big| \int e^{i\langle \xi - \xi', y'\rangle} a(\sqrt{R}y')dy' \Big|^2 \big|k_y(\xi')\big|^2 d\xi' \ . \tag{5.22}$$

Observe that by (5.19) and Plancherel

$$\int \big|k_y(\xi')\big|^2 d\xi' < \frac{C}{R} \ . \tag{5.23}$$

We used the fact that $|g| \leq 1$.

Put $\xi = R\nabla_y \psi(x,y)$ in (5.22). It follows from previous considerations

$$\Big| \int e^{iR\langle \nabla_y \psi(x,y), y'\rangle} a(\sqrt{R}y')g(y+y')dy' \Big|^2 \lesssim$$

$$\int \Big| \int e^{i\langle R\nabla_y \psi(x,y) - \xi', y'\rangle} a(\sqrt{R}y')dy' \Big|^2 \big|k_y(\xi')\big|^2 d\xi' \ . \tag{5.24}$$

From (5.24), (5.16) gets estimated by

$$R^{\frac{5}{2} - \frac{q}{4}} \int_{B_1} \left[\int \Big| \int e^{i\sqrt{R}\langle \nabla_y \psi(x,y) - \omega(y), y'\rangle} a(y')dy' \Big|^2 dy \right]^{q/2} dx \ , \tag{5.25}$$

where $\omega(y)$ is a vector associated to y. This follows from a simple convexity argument. The inner integral is of course

$$\widehat{a}\big(\sqrt{R}(\nabla_y \psi(x,y) - \omega(y))\big) \ . \tag{5.26}$$

Define the operator

$$\mathcal{M}_\varepsilon f(y) = \sup_\omega \varepsilon^{-2} \int_{|\nabla_y \psi(x,y) - \omega| < \varepsilon} |f(x)| dx . \tag{5.27}$$

(5.25) is bounded by $L^{q/2}(B_1) - L^{(q/2)'}(B_1)$ duality. This gives for some $f \in L^{(q/2)'}(B_1)$ of norm 1 the bound

$$R^{\frac{5}{2} - \frac{3q}{4}} \left[\int \mathcal{M}_{1/\sqrt{R}} f(y) dy \right]^{q/2} . \tag{5.28}$$

Assume the following inequality proved for $r = \left(\frac{q}{2}\right)'$

$$\|\mathcal{M}_\varepsilon f\|_{L^1(B_1)} < \left(\frac{1}{\varepsilon}\right)^{\gamma(r)} \|f\|_{L^r(B_1)} , \tag{5.29}$$

for some $\gamma(r) > 0$.

(5.28) is then estimated by

$$R^{\frac{5}{2} - \frac{3q}{4} + \frac{q}{4}\gamma((\frac{q}{2})')} . \tag{5.30}$$

Consequently

$$\|T_N f\|_{L^q(B(z,R))} < C R^{\frac{5}{2q} - \frac{3}{4} + \frac{1}{4}\gamma((\frac{q}{2})')} \quad \text{if} \quad |f| \leq 1 . \tag{5.31}$$

implying a distributional inequality

$$\text{meas}\left[x \in B(z,R) \mid |T_N f(x)| > \lambda\right] < C R^{\frac{5}{2} - \frac{3}{4}q + \frac{q}{4}\gamma((\frac{q}{2})')} \lambda^{-q} , \tag{5.32}$$

on the ball $B(z,R)$.

Assume we prove for some $r > 2$

$$\gamma(r) < \frac{1}{r} . \tag{5.33}$$

One then clearly gets with the notations of (4.25)

$$\left|A_\lambda \cap B(z,\rho)\right| \lesssim \rho^{2 - \frac{q}{2} - \tau} \lambda^{-q} , \tag{5.34}$$

for some $\tau > 0$. Hence (4.25) is bounded by

$$\lambda^{-2}[R + R^{1 - \frac{q}{2} - \tau} \lambda^{-q}] \tag{5.35}$$

and thus

$$|A_\lambda| \lesssim \lambda^{-(2 + \frac{2}{1 + \frac{2\tau}{q}})} = \lambda^{-4 + \tau'} , \tag{5.36}$$

for some $\tau' > 0$.

This would be the desired estimate (4.2).

In the next section, we will prove under suitable conditions on φ an estimate (5.33).

6. Related Kakeya Type Maximal Inequalities

We will only consider phase functions $\varphi(x,y)$ of the form (1.12), say

$$\varphi(x,y) = x_1y_1 + x_2y_2 + \tfrac{1}{2}x_3(a_{11}y_1^2 + a_{12}y_2^2) + \psi(x_3,y)\ , \tag{6.1}$$

where

$$\psi(x_3,y) = O\big(|x_3||y|^2(|x_3|+|y|)\big) \tag{6.2}$$

and ψ is real analytic.

The difference with the general case (1.10) ($d=3$) in the absence of the x_1, x_2 variables in the $\psi(x,y)$ additional term. This fact simplifies the gradient equations

$$\begin{cases} x_1 + a_{11}x_3y_1 + \partial_{y_1}\psi(x_3,y) - \omega_1 = 0 \\ x_2 + a_{22}x_3y_2 + \partial_{y_2}\psi(x_3,y) - \omega_2 = 0 \end{cases} \tag{6.3}$$

appearing in the context of the maximal function $\mathcal{M}_\varepsilon$ defined by (5.27), in the sense that they are explicit in x_1, x_2. Thus the corresponding curves Γ_y parametrized by

$$\begin{cases} x_1 = -a_{11}x_3y_1 - \partial_{y_1}\psi(x_3,y) \\ x_2 = -a_{22}x_3y_2 - \partial_{y_2}\psi(x_3,y) \end{cases} \tag{6.4}$$

are translated according to $\omega = (\omega_1,\omega_2)$ but ω does not affect the shape of the y-curve. This will play a role in the considerations below. I believe however that it is possible to carry out a similar approach in the general case (besides the previous more principal difficulty there are systematic complications if the (6.3)-equations are implicit in x_1, x_2).

Recall the definition of $\mathcal{M}_\varepsilon$

$$\mathcal{M}_\varepsilon f(y) = \sup_{\omega=(\omega_1,\omega_2)} \int_{|\nabla_y\psi(x,y)-\omega|<\varepsilon} f(x)dx\ . \tag{6.5}$$

This amounts to averaging f over an ε-neighborhood of the ω-translate of the curve Γ_y defined in (6.4). The tangent vector at $(x_1,x_2,x_3)\in\Gamma_y$ is given by

$$v_y(x_3) = \big(-a_{11}y_1 - \partial_{x_3}\partial_{y_1}\psi(x_3,y), -a_{22}y_2 - \partial_{x_3}\partial_{y_2}\psi(x_3,y), 1\big) \tag{6.6}$$

(of course independent from the translate). Since, by (6.2),

$$|\partial_{x_3} D_y^2 \psi| = O(|x_3|) = o(1) \tag{6.7}$$

one clearly has at given x_3, for $y = (y_1, y_2)$, $y' = (y_1', y_2')$

$$\left| v_y(x_3) - v_{y'}(x_3) \right| > c|y - y'| \tag{6.8}$$

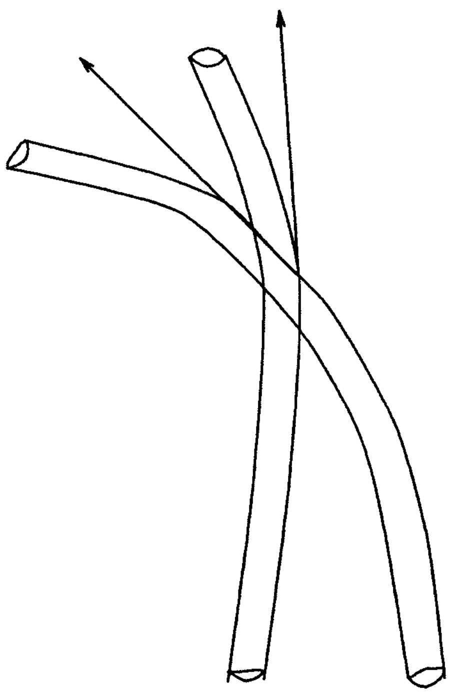
$(a_{11}, a_{22} \neq 0)$.

Inequality (6.8) easily leads to an L^2-estimate on $\mathcal{M}_\varepsilon$. For each y in a neighborhood of 0, let $\widetilde{\Gamma}_y$ stand for an ε-neighborhood of the $\omega(y)$-translate of Γ_y, i.e. the points $x = (x_1, x_2, x_3)$ satisfying

$$\begin{cases} \left| x_1 + a_{11} x_3 y_1 + \partial_{y_1} \psi(x_3, y) - \omega_1(y) \right| < \varepsilon \\ \left| x_2 + a_{22} x_3 y_2 + \partial_{y_2} \psi(x_3, y) - \omega_2(y) \right| < \varepsilon \, . \end{cases} \tag{6.9}$$

It clearly follows from (6.8) that

$$|\widetilde{\Gamma}_y \cap \widetilde{\Gamma}_{y'}| < c\beta(y, y') \frac{\varepsilon^3}{\varepsilon + |y - y'|} \tag{6.10}$$



defining

$$\beta(y,y') = 1 \quad \text{if} \quad \text{dist}(\Gamma_y, \Gamma_{y'}) < 2\varepsilon \ , \quad \beta(y,y') = 0 \text{ otherwise} \ . \qquad (6.11)$$

Checking the L^2-L^2 bound on $\mathcal{M}_\varepsilon$ essentially amounts to evaluating for $\sum_{y\in\mathcal{Y}} |a_y|^2 \leq 1$

$$\varepsilon \sum_{y\in\mathcal{Y}} a_y \left\langle \varepsilon^{-2}\chi_{\widetilde{\Gamma}_y}, f\right\rangle \leq \varepsilon^{-1}\Big\| \sum_{y\in\mathcal{Y}} a_y \chi_{\widetilde{\Gamma}_y}\Big\|_2 \qquad (6.12)$$

where $\mathcal{Y}$ denotes an ε-net in a neighborhood of 0 and $\|f\|_2 \leq 1$. Write

$$\Big\| \sum_{y\in\mathcal{Y}} a_y \chi_{\widetilde{\Gamma}_y}\Big\|_2^2 = \sum_{y,y'\in\mathcal{Y}} a_y \bar{a}_{y'} |\widetilde{\Gamma}_y \cap \widetilde{\Gamma}_{y'}| \ , \qquad (6.13)$$

and estimate the $\ell^2(\mathcal{Y})$-$\ell^2(\mathcal{Y})$ norm of the matrix $\left(|\widetilde{\Gamma}_y \cap \widetilde{\Gamma}_{y'}|\right)_{y,y'\in\mathcal{Y}}$ by Shur's lemma (i.e. the ℓ^1-ℓ^1,ℓ^∞-ℓ^∞ bound) and (6.10). This gives for fixed $y \in \mathcal{Y}$

$$\sum_{y'\in\mathcal{Y}} |\widetilde{\Gamma}_y \cap \widetilde{\Gamma}_{y'}| \leq c \sum_{y'\in\mathcal{Y}} \beta(y,y') \frac{\varepsilon^3}{\varepsilon + |y-y'|} \sim \varepsilon \int \frac{\beta(y,y')dy'}{\varepsilon + |y-y'|} \qquad (6.14)$$

(the integrals are restricted to a given neighborhood of 0).

Thus the right member of (6.12) is bounded by $\varepsilon^{-1/2}$, from what precedes, and hence

$$\|\mathcal{M}_\varepsilon\|_{2\to 2} < c \cdot \varepsilon^{-1/2} \ . \qquad (6.15)$$

Thus, in particular

$$\|\mathcal{M}_\varepsilon\|_{2\to 1} < c \cdot \varepsilon^{-1/2} \ . \qquad (6.16)$$

In fact, it also follows from the preceding that $\|\mathcal{M}_\varepsilon\|_{2-1}$ may be bounded by

$$\varepsilon^{-1/2} \left\{ \iint \frac{\beta(y,y')dy\,dy'}{\varepsilon + |y-y'|} \right\}^{1/2} \qquad (6.17)$$

hence

$$\varepsilon^{-1/2} \left(\log \frac{1}{\varepsilon} \right)^{1/4} \left\{ \iint \beta(y,y')dy\,dy' \right\}^{1/4} \ . \qquad (6.18)$$

For a given configuration of $\widetilde{\Gamma}_y$, (6.18) gives more information than (6.16). Indeed, assume $\theta_1 > 0$ a small number and

$$\Big\| \sum_{y \in \mathcal{Y}} \chi_{\widetilde{\Gamma}_y} \Big\|_2 > \Big(\frac{1}{\varepsilon}\Big)^{\frac{1}{2} - \theta_1} . \tag{6.19}$$

From the estimate (6.18), it then follows

$$\iint \beta(y, y') dy\, dy' > \varepsilon^{4\theta_1 +} . \tag{6.20}$$

Taking θ_1 small, this means that many pairs $(\widetilde{\Gamma}_y, \widetilde{\Gamma}_{y'})$ intersect each other. This fact will be exploited to replace the translation function ω by a C^∞-function, by imposing $\Gamma_y + \omega(y)$ to intersect 2 fixed translates $\Gamma_{y^1} + \omega(y^1)$ and $\Gamma_{y^2} + \omega(y^2)$. In the case of a smooth translation function, we then use differential calculus to get L^p-results, under suitable assumptions on φ. To make this general idea more precise will require however additional work.

Remark 6.21: The estimate (6.15) is the right estimate, in particular for straight lines. In this last case, the conjecture is that $(d = 3)$

$$\|\mathcal{M}_\varepsilon\|_{p \to p} \ll \varepsilon^{-(\frac{3}{p} - 1) - \tau} \qquad (\tau > 0) \quad \text{for} \quad p \leq 3 . \tag{6.22}$$

This is proved in [B1] if $p \leq \frac{7}{3}$. In this section, we consider $p > 2$, without seeking for a precise estimate but only an improvement over $\varepsilon^{-1/p}$ for $\|\mathcal{M}_\varepsilon\|_{p \to 1}$. Observe that in general (6.22) is not valid, as a consequence of the considerations made in section 3 of the paper.

Thus we have to consider for $q \leq 2$ the expression

$$\Big\| \int_\Omega \chi_{\widetilde{\Gamma}_y} dy \Big\|_q , \tag{6.23}$$

where Ω is some neighborhood of 0.

The variable x_3 ranges in a neighborhood of 0 which we partition in K subintervals I_j of length

$$|I_j| \sim \frac{1}{K} . \tag{6.24}$$

The number $K = K(\varepsilon)$ will be specified later.

For each j, we will decompose Ω as

$$\Omega = \bigcup \Omega^j_\alpha \cup \Omega^j_r \ , \tag{6.25}$$

where the number of Ω^j_α-components will be suitably bounded and

$$\left\| \int_{\Omega^j_r} \chi_{\widetilde{\Gamma}_y \cap (\mathbf{R}^2 \times I_j)} dy \right\|_2 < K^{-1/2} \varepsilon^{\frac{3}{2}+\theta_1} \ . \tag{6.26}$$

Hence, from (6.26)

$$\left\| \sum_j \int_{\Omega^j_r} \chi_{\widetilde{\Gamma}_y \cap (\mathbf{R}^2 \times I_j)} dy \right\|_2 < \varepsilon^{\frac{3}{2}+\theta_1} \ . \tag{6.27}$$

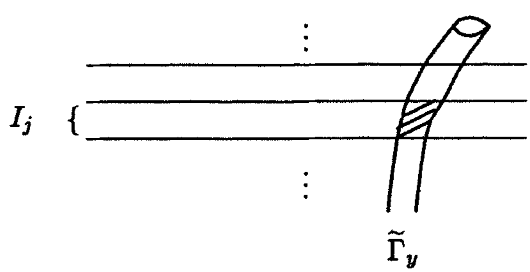
Thus there is the following estimate on (6.23)

$$\left\{ \sum_j \left(\sum_\alpha \left\| \int_{\Omega^j_\alpha} \chi_{\widetilde{\Gamma}_y \cap (\mathbf{R}^2 \times I_j)} dy \right\|_q \right)^q \right\}^{1/q} + \varepsilon^{2-\frac{1}{p}+\frac{2}{p}\theta_1} \ , \tag{6.28}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. The last term in (6.28) appears by interpolation between L^1 (the ε^2-estimate) and L^2 (estimated in (6.27)).

The main point is of course the construction of the Ω^j_α and its properties.

Fix j and denote $\Gamma_y \cap (\mathbf{R}^2 \times I_j)$ (resp. $\widetilde{\Gamma}_y \cap (\mathbf{R}^2 \times I_j)$) again by Γ_y (resp. $\widetilde{\Gamma}_y$). The number K will be chosen small w.r.t. $\frac{1}{\varepsilon}$



If $I_j = \left[a_j, a_j + \frac{1}{K}\right]$, we make a change of variable

$$x_3 \to x_3 - a_j \ . \tag{6.29}$$

Consider the equations (6.3). Introducing the new variable x_3 and making a coordinate change in (y_1, y_2), one gets equations of the form

$$\begin{cases} x_1 + a_{11}x_3y_1 + \eta_1(x_3, y) - \omega_1' = 0 \\ x_2 + a_{22}x_3y_2 + \eta_2(x_3, y) - \omega_2' = 0 \end{cases} \tag{6.30}$$

where $0 \le x_3 < \frac{1}{K}$ and

$$\big|\eta_i(x_3, y)\big| = O\big(|x_3|^2|y|\big) \; . \tag{6.31}$$

Here

$$\omega_1' = \omega_i - a_{ii} \cdot a_j \cdot y_i - \partial_{y_i}\psi(a_j, y) \; , \tag{6.32}$$

and the coordinate change in the y-variable given by

$$y_i' = y_i + a_{ii}^{-1}\big[\partial_{x_3}\partial_{y_i}\psi(a_j, y)\big] \; . \tag{6.33}$$

It follows from (6.2) that this last change of variable is of the form

$$\mathrm{Id} + \pi + O\big(|y|^2\big)$$

where π is affine and $\|\pi\| \lesssim \kappa$ where $[-\kappa, \kappa]$ is the domain of x_3.

Let $\Omega_0 \subset \Omega$ and suppose (6.26) does not hold for $\Omega_r^j = \Omega_0$. Thus with the new notation

$$\left\| \int_{\Omega_0} \chi_{\widetilde{\Gamma}_y} dy \right\|_2 > K^{-1/2}\varepsilon^{\frac{3}{2}+\theta_1} \; . \tag{6.34}$$

A straightforward exhaustion consideration permits finding a subset Ω_1 of Ω_0, still satisfying (6.34) (up to a factor $\frac{1}{2}$) and such that moreover, for each $\Omega_2 \subset \Omega_1$

$$\left\| \int_{\Omega_2} \chi_{\widetilde{\Gamma}_y} dy \right\|_2 > \tfrac{1}{2}K^{-1/2}\varepsilon^{\frac{3}{2}+\theta_1}|\Omega_2| \tag{6.35}$$

holds.

Redefining

$$\beta(y, y') = \beta_j(y, y') = \begin{cases} 1 & \text{if } \widetilde{\Gamma}_y \text{ and } \widetilde{\Gamma}_{y'} \text{ intersect} \\ 0 & \text{otherwise} \end{cases} \tag{6.36}$$

estimate (6.10) obviously still holds and hence (6.34) implies

$$\iint\limits_{\Omega_1\times\Omega_1} \frac{\beta(y,y')}{\varepsilon+|y-y'|} dy\,dy' > K^{-1}\varepsilon^{2\theta_1}\ . \tag{6.37}$$

Hence, for some

$$K^{-1}\varepsilon^{2\theta_1} < \gamma < 1 \tag{6.38}$$

one will get

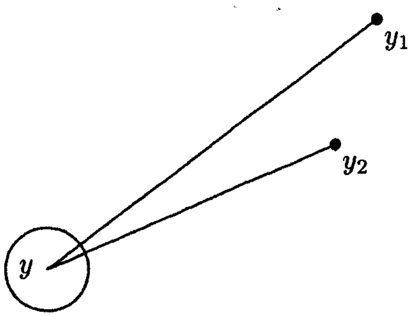
$$\begin{aligned} &\iint\limits_{\substack{\Omega_1\times\Omega_1\\ \gamma<|y-y'|<2\gamma}} \frac{\beta(y,y')}{\varepsilon+|y-y'|} dy\,dy' > K^{-1}\varepsilon^{2\theta_1}\left(\log\frac{1}{\varepsilon}\right)^{-1}\ ;\\ &\iint\limits_{\substack{\Omega_1\times\Omega_1\\ \gamma<|y-y'|<2\gamma}} \beta(y,y')dy\,dy' > K^{-1}\varepsilon^{2\theta_1}\left(\log\frac{1}{\varepsilon}\right)^{-1}\gamma\ . \end{aligned} \tag{6.39}$$

The $\log\frac{1}{\varepsilon}$-factors are irrelevant in what follows and we will drop them for simplicity.

Assume fixed $y^1, y^2 \in \Omega_1$ and a subset Ω_2 of Ω_1 of points y such that

$$\widetilde\Gamma_y \text{ intersects } \widetilde\Gamma_{y^1} \text{ and } \widetilde\Gamma_{y^2} \tag{6.40}$$

$$\text{angle } (y-y^1, y-y^2) > \frac{C}{K}\ , \tag{6.41}$$



From (6.30), one then gets for $i=1,2$, denoting λ_i the x_3-coordinate of a $\widetilde\Gamma_y - \widetilde\Gamma_{y^i}$ intersection point

$$\begin{cases} \left|a_{11}(y_1-y_1^i)\lambda_i + \eta_1(\lambda_i,y) - \eta_1(\lambda_i,y^i) - \omega_1'(y) + \omega_1'(y^i)\right| < 2\varepsilon\\ \left|a_{22}(y_2-y_2^i)\lambda_i + \eta_2(\lambda_i,y) - \eta_2(\lambda_i,y^i) - \omega_2'(y) + \omega_2'(y^i)\right| < 2\varepsilon \end{cases} \tag{6.42}$$

and making the subtraction for $i = 1, 2$ in (6.42)

$$\begin{cases} \big|a_{11}(y_1 - y_1^1)\lambda_1 - a_{11}(y_1 - y_1^2)\lambda_2 + \eta_1(\lambda_1, y) - \eta_1(\lambda_1, y^1) - \eta_1(\lambda_2, y) + \\ \eta_1(\lambda_2, y^2) + \omega_1'(y^1) - \omega_1'(y^2)\big| < 4\varepsilon \\ \\ \big|a_{22}(y_2 - y_2^1)\lambda_1 - a_{22}(y_2 - y_2^2)\lambda_2 + \eta_2(\lambda_1, y) - \eta_2(\lambda_1, y^1) - \eta_2(\lambda_2, y) + \\ \eta_2(\lambda_2, y^2) + \omega_2'(y^1) - \omega_2'(y^2)\big| < 4\varepsilon \end{cases} \tag{6.43}$$

Our aim is to consider

$$\begin{aligned} a_{ii}(y_i - y_i^1)\lambda_1 - a_{ii}(y_i - y_i^2)\lambda_2 + \eta_i(\lambda_1, y) \\ - \eta_i(\lambda_1, y^1) - \eta_i(\lambda_2, y) + \eta_i(\lambda_2, y^2) + \omega_i'(y^1) - \omega_i'(y^2) = 0 \end{aligned}$$

$$(i = 1, 2) \tag{6.44}$$

as implicit equations in λ_1, λ_2 which we seek to obtain as smooth functions of y, i.e.

$$\begin{cases} \lambda_1 = \lambda_1(y) \\ \lambda_2 = \lambda_2(y) \end{cases}$$

Here y is taken in a neighborhood of a point, of size δ, to be specified. This is achieved by the implicit function theorem. The solution (6.45) may not be unique but the number of solutions should be suitably bounded. The main point is to control the (λ_1, λ_2)-Jacobian. Observe that by (6.31) and the $\frac{1}{K}$-restriction on x_3, one has

$$\big|\partial_\lambda \eta_i(\lambda, y) - \partial_\lambda \eta_i(\lambda, z)\big| < \frac{1}{K} O\big(|y - z|\big) . \tag{6.46}$$

Hence

$$J_{\lambda_1, \lambda_2} = \begin{bmatrix} a_1(y_1 - y_1^1) + \frac{1}{K} O\big(|y - y1|\big) & a_{11}(y_1 - y_1^2) + \frac{1}{K} O\big(|y - y^2|\big) \\ \\ a_{22}(y_2 - y_2^1) + \frac{1}{K} O\big(|y - y^1|\big) & a_{22}(y_2 - y_2^2) + \frac{1}{K} O\big(|y - y^2|\big) \end{bmatrix}, \tag{6.47}$$

and thus

$$\det J_{\lambda_1, \lambda_2} > c\big|\det(y - y^1, y - y^2)\big| - \frac{1}{K} O\big(|y - y^1||y - y^2|\big) . \tag{6.48}$$

In view of hypothesis (6.41), taking the constant C appropriately, one finds a lower bound

$$\det J_{\lambda_1, \lambda_2} > \frac{1}{K}|y - y^1||y - y^2| . \tag{6.49}$$

If one assumes

$$|y - y^i| > \gamma_1 \tag{6.50}$$

it follows from (6.49) that

$$\det J_{\lambda_1,\lambda_2} > \frac{1}{K}\gamma_1^2 . \tag{6.51}$$

One may thus take

$$\delta \sim \frac{\gamma_1}{K} , \tag{6.52}$$

and, since $|\lambda_i| < \frac{1}{K}$, the number of solutions to (6.45) may be estimated by γ_1^{-2}. Here

$$\kappa = \varepsilon^{-\theta_2} \quad \text{and} \quad \gamma_1 > \varepsilon^{\theta_3} , \tag{6.53}$$

for certain θ_2, θ_3 depending on θ_1 where $\theta_2, \theta_3 \to 0$ for $\theta_1 \to 0$. Of course, once (6.45) obtained, (6.42) also yields

$$\begin{cases} \omega_1' = \omega_1'(y) \\ \omega_2' = \omega_2'(y) \end{cases} \tag{6.54}$$

as smooth functions of y on this δ-neighborhood. Thus our aim of obtaining a smooth translation function for a "large" subset Ω_3 of Ω_2 is achieved in this case. The main problem is that condition (6.41) may not be realizable, which is the reason for the complications in what follows.

Consider the set of triplets

$$\begin{aligned} T = \Big\{ &(y^1, y^2, y^3) \in \Omega_1^3 \mid |y^1 - y^i| \sim \gamma, \beta(y^1, y^i) = 1 \ (i = 2,3) \\ &\text{and angle } (y^1 - y^2, y^1 - y^3) > \frac{C}{K} \Big\} . \end{aligned} \tag{6.55}$$

Choosing $\theta_4 > 0$, if we assume

$$\text{meas}(T) > \varepsilon^{\theta_4} , \tag{6.56}$$

one may find points y^2, y^3 in Ω_1 to which correspond a set of points $y = y^1$ say Ω_2 of measure $|\Omega_2| > \varepsilon^{\theta_4}$, such that (6.40), (6.41) hold. In (6.50), one has $\gamma_1 = \gamma$. This brings us to the previous situation. Hence, suppose

$$\text{meas}(T) < \varepsilon^{\theta_4} . \tag{6.57}$$

Coming back to (6.39), one gets a subset Ω_4 of Ω_1 such that

$$|\Omega_4| > K^{-1}\varepsilon^{2\theta_1+}\gamma \tag{6.58}$$

and to each $y \in \Omega_4$ corresponds a set

$$\Omega(y) = \left\{y' \in \Omega_1 \;\middle|\; |y-y'| \sim \gamma \quad \text{and} \quad \beta(y,y') = 1\right\} \tag{6.59}$$

with

$$\big|\Omega(y)\big| > K^{-1}\varepsilon^{2\theta_1+}\gamma\ . \tag{6.60}$$

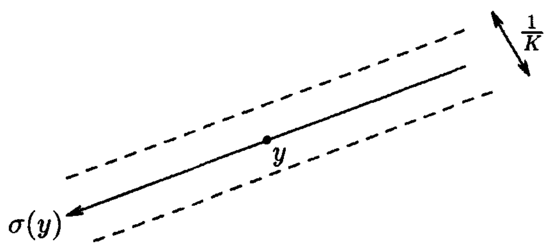
It is easily seen from (6.57) that if we let

$$\varepsilon^{\frac{1}{3}\theta_4} < K^{-1}\varepsilon^{2\theta_1+}\gamma \tag{6.61}$$

one has necessarily

$$\gamma > \varepsilon^{2\theta_1+} \tag{6.62}$$

and moreover, we may assume that for $y \in \Omega_4$, the set $\Omega(y)$ is contained in a $\frac{1}{K}$-neighborhood of a line-segment through y, in a direction $\sigma(y)$



In view of (6.60), (6.62), the relative density of $\Omega(y)$ in this strip is at least $\varepsilon^{5\theta_1}$.

Consider following set of triplets

$$T_1 = \Big\{(y^1,y^2,y^3) \in \Omega_4 \times \Omega_1 \times \Omega_1 \;\Big|\; y^i \in \Omega(y^1)\ (i=2,3)\ , \\ \text{angle}(y^2-y^1, y^3-y^1) > \frac{1}{K}\varepsilon^{5\theta_1}\Big\}\ . \tag{6.63}$$

It follows from (6.58), (6.60), (6.62) that

$$\text{meas}(T_1) > K^{-3}\varepsilon^{13\theta_1}\ . \tag{6.64}$$

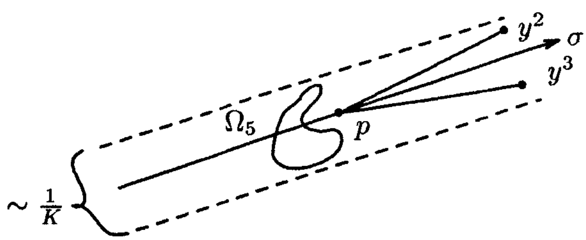
Hence, we may find points $y_2, y_3 \in \Omega_1$ and points $y \in \Omega_5 \subset \Omega_4 \cap B(p, \varepsilon^{\theta_5})$ where

$$\begin{cases} |\Omega_5| > K^{-3}\varepsilon^{13\theta_1+2\theta_5} \\ y^i \in \Omega(y) \quad \text{for} \quad i=2,3\ , \quad y \in \Omega_5 \\ \text{angle}(y^2-p, y^3-p) > \frac{1}{K}\varepsilon^{5\theta_1}\ . \end{cases} \tag{6.65}$$

Here θ_5 is choosen to satisfy

$$\varepsilon^{\theta_5} < \frac{1}{K}\varepsilon^{6\theta_1}\ . \tag{6.66}$$

Fixing some $y \in \Omega_5$, both y^2, y^3 lie in the $\frac{1}{K}$- strip centered at y in direction $\sigma(y)$. From the construction, it follows that this direction $\sigma(y)$ only deviates by $\sim \frac{1}{K}$ from the directions $[p, y_i]$, $i = 2, 3$. Thus, we may fix a direction σ for which we have the following situation



Let $\eta = (\eta_1, \eta_2)$ appearing in (6.30) and consider the real analytic function

$$\tau(s) = \det\left[D_y\partial^2_{x_3}\eta(0, p+s\sigma)\sigma\ ,\ \begin{pmatrix} a_{11}\sigma_1 \\ a_{22}\sigma_2 \end{pmatrix}\right]\ . \tag{6.67}$$

We distinguish 2 cases

$$\underline{\text{Case I}}: \qquad \max_{|t|\leq 1}|\tau(s)| \leq \varepsilon^{\theta_6} \tag{6.68}$$

$$\underline{\text{Case II}}: \qquad \max_{|t|\leq 1}|\tau(s)| > \varepsilon^{\theta_6}\ . \tag{6.69}$$

Observe that by the real analyticity, if (6.69) holds, then one also gets

$$\text{meas}\left[s \in [-1,1]\ \big|\ |\tau(s)| < \delta \cdot \varepsilon^{\theta_6}\right] < \delta^{c_1}\ , \tag{6.70}$$

for some constant c_1. From the way η is derived from φ appearing in (6.1), assumed real analytic in x, y on a neighborhood of the domain under consideration, one may assume c_1 uniform in p and σ. This is clear in the polynomial case and may be shown for real analytic functions along the lines of [B3]. We shall deal separately with these 2 cases.

Case I. By the hypothesis (6.35) and (6.65), we have that

$$\left\| \int_{\Omega_5} \chi_{\widetilde{\Gamma}_y} dy \right\|_2 > K^{-\frac{7}{2}} \varepsilon^{\frac{3}{2}+15\theta_1+2\theta_5} . \tag{6.71}$$

Repeating the considerations (6.37)-(6.39) yields a point $y^1 \in \Omega_5$ and a subset Ω_6 of Ω_5 of points y, such that

$$\begin{cases} |y-y^1| \sim \gamma_1 \quad \text{where} \quad \varepsilon^{\theta_5} > \gamma_1 > K^{-7}\varepsilon^{30\theta_1+4\theta_5} & (6.72) \\ \beta(y,y^1) = 1 \quad \text{for} \quad y \in \Omega_6 & (6.73) \\ \Omega_6 \subset B(q, \varepsilon^{\theta_7}) & (6.74) \\ |\Omega_6| > K^{-7}\varepsilon^{30\theta_1+4\theta_5+2\theta_7} . & (6.75) \end{cases}$$

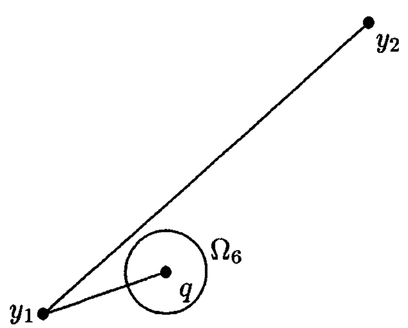
Here θ_7 is chosen such that

$$\varepsilon^{\theta_7} < \frac{1}{K}\varepsilon^{5\theta_1}\gamma_1 . \tag{6.76}$$

Hence, either for $i = 2$ or $i = 3$, one gets

$$\text{angle}(y^i - y^1, y - y^1) > \frac{1}{2K}\varepsilon^{5\theta_1} . \tag{6.77}$$

Assume this holds for $i = 2$



Since by (6.65), $y^2 \in \Omega(y^1)$ we have $\beta(y^1, y^2) = 1$ and may assume $\Gamma_{y^1} + \omega(y^1), \Gamma_{y^2} + \omega(y^2)$ intersect each other, say for $x_3 = t$. Proceeding as above (cf. 6.42), this yields the equations

$$\begin{cases} a_{11}(y_1^1 - y_1^2)t + \eta_1(t, y^1) - \eta_1(t, y^2) - \omega_1'(y^1) + \omega_1'(y^2) = 0 \\ a_{22}(y_2^1 - y_2^2)t + \eta_2(t, y^1) - \eta_2(t, y^2) - \omega_2'(y^1) + \omega_2'(y^2) = 0 . \end{cases} \tag{6.78}$$

We use (6.78) to substitute in (6.43), valid since $\widetilde{\Gamma}_y$ intersects $\widetilde{\Gamma}_{y^2}$ by (6.65) and $\widetilde{\Gamma}_{y^1}$ by (6.73), the expressions $\omega_i'(y^1) - \omega_i'(y^2)$.

We further denote

$$\mu_1 = \lambda_1 - t \ , \qquad \mu_2 = \lambda_2 - t \ , \tag{6.79}$$

$$\beta_i(\mu, z) = \eta_i(t + \mu, z) - \eta_i(t, z) \tag{6.80}$$

and get from (6.43), (6.78)

$$\begin{aligned} \big|a_{ii}(y_i - y_i^1)\mu_1 - a_{ii}(y_i - y_i^2)\mu_2 + \beta_i(\mu_1, y) - \beta_i(\mu_1, y^1) - \\ - \beta_i(\mu_2, y) + \beta_i(\mu_2, y^2)\big| < 4\varepsilon \quad (i = 1, 2) \end{aligned} \tag{6.81}$$

Our goal is to consider again

$$\begin{aligned} a_{ii}(y_i - y_i^1)\mu_1 - a_{ii}(y_i - y_i^2)\mu_2 + \beta_i(\mu_1, y) - \beta_i(\mu_1, y^1) - \\ - \beta_i(\mu_2, y) + \beta_i(\mu_2, y^2) = 0 \quad (i = 1, 2) \end{aligned} \tag{6.82}$$

locally as implicit equations in μ_1, μ_2 which we seek to obtain as smooth functions of y, i.e.

$$\begin{cases} \mu_1 = \mu_1(y) \\ \mu_2 = \mu_1(y) \end{cases} \tag{6.83}$$

where y is taken in a neighborhood of a point $\overline{y}$ where intersection with $\Gamma_{y^1} + \omega(y^1), \Gamma_{y^2} + \omega(y^2)$ occurs.

We may assume that

$$\left(\frac{\varepsilon^{5\theta_1}}{2K} <\right) \text{angle}(y^2 - y^1, y - y^1) < \frac{C}{K} \ . \tag{6.84}$$

Since otherwise (6.41) holds in which case the a_{ii}-terms in (6.82) permit the control of det J_{μ_1, μ_2}. This is the simple case we treated earlier.

Oberve that by (6.31)

$$\begin{cases} \big|\beta_i(\mu_1, y) - \beta_i(\mu_1, y^1)\big| < \frac{C}{K}|\mu_1||y - y^1| \\ \\ \big|\beta_i(\mu_2, y) - \beta_i(\mu_2, y^2)\big| < \frac{C}{K}|\mu_2||y - y^2| \ . \end{cases} \tag{6.85}$$

It thus follows from (6.82), (6.85) that

$$|y - y^2||\mu_2| \sim |y - y^1||\mu_1| \ . \tag{6.86}$$

Since $y^2 \in \Omega(y)$ by (6.65), we have

$$\text{angle}(y - y^2, \sigma) < \frac{\varepsilon^{-3\theta_1}}{K} \tag{6.88}$$

and hence, by (6.84), also

$$\text{angle}(y - y^1, \sigma) < \frac{\varepsilon^{-3\theta_1}}{K} \,. \tag{6.89}$$

Consider (6.82), we have

$$\begin{aligned} J_{\mu_1,\mu_2} = \big[&a_{ii}(y_i - y_i^1) + \partial_{x_3}\eta_i(t+\mu_1, y) - \partial_{x_3}\eta_i(t+\mu_1, y^1), \\ &a_{ii}(y_i - y_i^2) + \partial_{x_3}\eta_i(t+\mu_2, y) - \partial_{x_3}\eta_i(t+\mu_2, y^2)\big] \,. \end{aligned} \tag{6.90}$$

Further, for $i,j = 1,2$,we have by the mean value theorem

$$\partial_{x_3}\eta(\lambda, y) - \partial_{x_3}\eta(\lambda, y^j) = \int_0^1 D_y\partial_{x_3}\eta\big(\lambda, y^j + s(y - y^j)\big)(y - y^j)ds \,. \tag{6.91}$$

Again by (6.31), since $|\lambda| < \frac{1}{K}$

$$D_y\partial_{x_3}\eta(\lambda, z)(y - y^j) = \lambda D_y\partial_{x_3}^2\eta(0, z)(y - y^j) + O\left(\frac{1}{K^2}\right)|y - y^j| \,. \tag{6.92}$$

Since $\text{dist}(z, p + \mathbf{R}\sigma) < \frac{1}{K}$, there is a probability measure ρ_j on $[-1,1]$ such that

$$(6.91) = \lambda \int D_y\partial_{x_3}^2\eta(0, p + s\sigma)(y - y^j)\rho_j(ds) + O\left(\frac{1}{K^2}\right)|y - y^j| \,. \tag{6.93}$$

Substitution in (6.90) yields for some s_1, s_2

$$\begin{aligned} \det J_{\mu_1,\mu_2} = \det\big[&a_{ii}(y_- y_i^1) + (t+\mu_1)D_y\partial_{x_3}^2\eta_i(0, p + s_1\sigma)(y - y^1)\,, \\ &a_{ii}(y_i - y_i^2) + (t+\mu_2)D_y\partial_{x_3}^2\eta_i(0, p + s_2\sigma)(y - y^2)\big] \\ &+ O\left(\frac{1}{K^2}\right)|y - y^1||y - y^2| \,. \end{aligned} \tag{6.94}$$

The first term of (6.94) equals, because of (6.88), (6.89)

$$\begin{aligned} &a_{11}a_{22}\det[y - y^1, y - y^2] + \\ &+ O\left(\frac{1}{K}\right)\left|\det\left[D_y\partial_{x_3}^2\eta(0, p + s_j\sigma)(\sigma), \begin{pmatrix} a_{11}\sigma_1 \\ a_{22}\sigma_2 \end{pmatrix}\right]\right||y - y^1||y - y^2| \\ &+ O\left(\frac{\varepsilon^{-3\theta_1}}{K^2}\right)|y - y^1||y - y^2| \,. \end{aligned} \tag{6.95}$$

In view of (6.77) and the assumption (6.68), it follows from (6.94), (6.95) that

$$\det J_{\mu_1,\mu_2} > \left\{ c\frac{\varepsilon^{5\theta_1}}{K} - O\left(\frac{1}{K}\right)\varepsilon^{\theta_6} - O\left(\frac{\varepsilon^{-3\theta_1}}{K^2}\right)\right\} |y-y^1||y-y^2| \ . \tag{6.96}$$

We take

$$\theta_6 > 6\theta_1 \tag{6.97}$$

$$K > \varepsilon^{-9\theta_1} \ , \quad \text{i.e.} \quad \theta_2 > 9\theta_1 \ , \tag{6.98}$$

so that (6.96) yields by (6.72)

$$\begin{aligned} \det J_{\mu_1,\mu_2} &> c\varepsilon^{5\theta_1} K^{-1} |y-y^1||y-y^2| \\ &> c\varepsilon^{8\theta_1} K^{-1} \gamma_1 \ . \end{aligned} \tag{6.99}$$

This is the desired lower estimate, enabling the application of the implicit function theorem.

Case II. Choose any point $y^1 \in \Omega_5$ for which thus $\Omega(y^1)$ lies in the $\frac{1}{K}$ – (p,σ) strip as described above. Also, the density of $\Omega(y^1)$ in this strip is $> \varepsilon^{5\theta_1}$. It follows from (6.70) that there is a subset Ω_7 of $\Omega(y^1)$ satifying for $y \in \Omega_7$

$$\left| \det \left[D_y \partial^2_{x_3} \eta(0,y)\sigma \ , \ \begin{pmatrix} a_{11}\sigma_1 \\ a_{22}\sigma_2 \end{pmatrix} \right] \right| > \varepsilon^{\theta_6} \varepsilon^{C_1\theta_1} \tag{6.100}$$

where $C_1 = c_1^{-1}$ and

$$\operatorname{diam} \Omega_7 < \frac{1}{K} \ , \qquad |\Omega_7| > \frac{\varepsilon^{6\theta_1}}{K^2} \ . \tag{6.101}$$

In order to get (6.100), we assume

$$\frac{1}{K} \ll \varepsilon^{\theta_6 + C_1\theta_1} \ , \qquad \text{i.e.} \quad \theta_2 > C_1\theta_1 + \theta_6 \ . \tag{6.102}$$

Again using (6.35)

$$\left\| \int_{\Omega_7} \chi_{\widetilde{\Gamma}_y} dy \right\|_2 > K^{-\frac{5}{2}} \varepsilon^{\frac{3}{2} + 8\theta_1} \ , \tag{6.103}$$

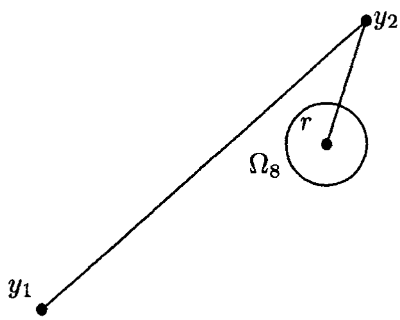
which permits the construction of $y^2 \in \Omega_7$ and $\Omega_8 \subset \Omega_7$ similarly to the way $\Omega_6 \subset \Omega_5$ has been obtained.

Thus for $y \in \Omega_8$

$$\begin{cases} |y-y^2| \sim \gamma_2 > K^{-5}\varepsilon^{16\theta_1} & (6.104) \\ \beta(y,y^2)=1 & (6.105) \\ \Omega_8 \subset B(r,\varepsilon^{\theta_8}) & (6.106) \\ |\Omega_8| > K^{-5}\varepsilon^{16\theta_1+2\theta_8} & (6.107) \end{cases}$$

where θ_8 is chosen such that

$$\varepsilon^{\theta_8} < \frac{1}{K}\cdot\gamma_2 \tag{6.108}$$



For $y \in \Omega_8$, $\widetilde{\Gamma}_y$ intersects $\widetilde{\Gamma}_{y^1}$ and $\widetilde{\Gamma}_{y^2}$. Since $y^2 \in \Omega_7 \subset \Omega(y^1)$, also $\widetilde{\Gamma}_{y^1}$ intersects $\widetilde{\Gamma}_{y^2}$. We are thus in the same situation as in Case I, having instead of (6.68) property (6.100) for $y \in \Omega_8$. It follows from (6.86) that

$$|\mu_1||y-y^1| < C|\mu_2||y-y^2| \tag{6.109}$$

and thus by (6.62)

$$|\mu_1| < \varepsilon^{-3\theta_1}|y-y^2||\mu_2| \ . \tag{6.110}$$

Write

$$\Omega_8 = \Omega_9 \cap \Omega_{10} \ , \tag{6.111}$$

where we define

$$y \in \Omega_9 \Longleftrightarrow \big|\mu_2(y)\big| > \varepsilon^{\theta_9} \ . \tag{6.112}$$

Hence the points $y \in \Omega_{10}$ have the property that

$$\operatorname{dist}(\widetilde{\Gamma}_y, P) < \varepsilon^{\theta_9} \ , \tag{6.113}$$

where P denotes the intersection point of $\Gamma_{y^1}+\omega(y^1), \Gamma_{y^2}+\omega(y^2)$. We treat these sets separately.

The set Ω_9. Introduce the new variable

$$\mu_1' = \frac{\mu_1}{\mu_2} \tag{6.114}$$

satisfying by (6.110)

$$|\mu_1'| < \varepsilon^{-3\theta_1}|y - y^2| \tag{6.115}$$

By (6.112), (6.81) yields after division by μ_2

$$\Big|a_{ii}(y_i - y_i^1)\mu_1' - a_{ii}(y_i - y_i^2) + \frac{1}{\mu_2}\left[\beta_i(\mu_1'\mu_2, y) - \beta_i(\mu_1'\mu_2, y^1)\right] - \frac{1}{\mu_2}\left[\beta_i(\mu_2, y) - \beta_i(\mu_2, y^2)\right]\Big| < \varepsilon^{1-\theta_9} \quad \text{for} \quad i = 1, 2\ . \tag{6.116}$$

Our aim now is to get from the implicit equations

$$a_{ii}(y_i - y_i^1)\mu_1' - \frac{1}{\mu_2}\left[\beta_i(\mu_2, y) - \beta_i(\mu_2, y^2)\right] + \\ + \frac{1}{\mu_2}\left[\beta_i(\mu_1'\mu_2, y) - \beta_i(\mu_1'\mu_2, y^1)\right] - a_{ii}(y_i - y_i^2) = 0$$

$$(i = 1, 2) \tag{6.117}$$

$$\begin{cases} \mu_1' = \mu_1'(y) \\ \mu_2 = \mu_2(y) \end{cases} \tag{6.118}$$

as a smooth solution. From (6.114) this then also yields (6.83).

Consider again J_{μ_1', μ_2}.

By (6.80)

$$\frac{1}{\mu_2}\left[\beta(\mu_2, y) - \beta(\mu_2, y^2)\right] = \int_0^1 \int_0^1 D_y \partial_{x_3}\eta\big(t + \tau\mu_2, y^2 + s(y - y^2)\big)(y - y^2)d\tau\, ds \tag{6.119}$$

thus

$$\partial_{\mu_2}\left\{\frac{1}{\mu_2}\left[\beta(\mu_2, y) - \beta(\mu_2, y^2)\right]\right\} \sim \\ \sim \int_0^1 D_y \partial_{x_3}^2 \eta\big(0, y^2 + s(y - y^2)\big)(y - y^2)ds + O\left(\frac{1}{K}\right)|y - y^2|\ . \tag{6.120}$$

As in Case I, we may assume (6.88), (6.89) valid. Hence, for some probility measure ρ_2 on $[-1,1]$

$$(6.120) \sim \left\{\int_0^1 D_y \partial^2_{x_3}(0, p+s\sigma)(\sigma)\rho_2(ds) + O\left(\frac{1}{K}\right)\right\} |y-y^2| . \quad (6.121)$$

Similarly

$$\frac{1}{\mu_2}\left[\beta_i(\mu_1'\mu_2, y) - \beta_i(\mu_1'\mu_2, y^1)\right] =$$

$$\mu_1' \int_0^1 \int_0^1 \left\langle \nabla_y \partial_{x_3}\eta_i\big(t+\tau\mu_1'\mu_2, y^1 + s(y-y^1)\big), y-y^1\right\rangle d\tau\, ds \quad (6.122)$$

and hence

$$\partial_{\mu_1'}\{\quad\} = O\Big(\frac{1}{K}\Big)|y-y^1| + O(|\mu_1'||\mu_2||y-y^1|) = O\Big(\frac{1}{K}\Big)|y-y^1| \quad (6.123)$$

$$\partial_{\mu_2}\{\quad\} = O\big(|\mu_1'|^2|y-y^1|\big) = O\big(|\mu_1'||y-y^2|\big) \le \varepsilon^{-3\theta_1}|y-y^2|^2 \quad (6.124)$$

in the last line using (6.86), (6.115).

From (6.117) and (6.121), (6.123), (6.124)

$$\det J_{\mu_1', \mu_2} = \det\Bigg[a_{ii}(y_i - y_i^1) + O\left(\frac{1}{K}\right)|y-y^1|,$$
$$D_y\partial^2_{x_3}(o, p+s\sigma)(\sigma)|y-y^2| + O\left(\frac{1}{K} + \varepsilon^{-3\theta_1}\gamma_2\right)|y-y^2|\Bigg] . \quad (6.125)$$

It follows from (6.100) that the main contribution

$$\big|\det\big[a_{ii}(y_i-y_i^1), D_y\partial^2_{x_3}(0, p+s\sigma)(\sigma)|y-y^2|\big]\big| >$$
$$\varepsilon^{\theta_6 + c_1\theta_1}|y-y^1||y-y^2| . \quad (6.126)$$

The error terms contribute for

$$O\left(\frac{1}{K} + \varepsilon^{-3\theta_1}\gamma_2\right)|y-y^1||y-y^2| = O\left(\varepsilon^{-3\theta_1}\frac{1}{K}\right)|y-y^1||y-y^2| . \quad (6.127)$$

Since, by (6.101), $\gamma_2 < \frac{1}{K}$.

It follows that under assumption (6.102), (6.126) yields a lower bound on $\det J_{\mu_1', \mu_2}$. Thus, by (6.104)

$$|\det J_{\mu_1', \mu_2}| > \varepsilon^{\theta_6 + C_1\theta_1}\gamma_2 . \quad (6.128)$$

The implicit function theorem allows on a "large" subset of Ω_8 (or Ω_1) to replace the translation function by a smooth function (controlled by $|\det J|^{-1}$), but the ε-neighborhood of the curves has to be replaced by the larger $\frac{\varepsilon^{1-\theta_9}}{|\det J|}$-neighborhood, in view of (6.116).

The set Ω_{10}. Using (6.113), we will make a direct estimate on

$$\left\| \int_{\Omega_{10}} \chi_{\widetilde{\Gamma}_y} dy \right\|_q . \tag{6.129}$$

Fix $\varepsilon^{\theta_{10}}$ and consider a $B(P, \varepsilon^{\theta_{10}})$-neighborhood of P. By Hölder's inequality, one has since $q < 2$

$$\left\| \varepsilon^{-2} \int_{\Omega_{10}} \chi_{\widetilde{\Gamma}_y \cap B(P, \varepsilon^{\theta_{10}})} dy \right\|_q \le \| \quad \|_2^{\frac{2}{p}} \| \quad \|_1^{\frac{2}{q}-1}$$
$$\le \varepsilon^{-\frac{1}{p}} \cdot \varepsilon^{\theta_{10}(\frac{2}{q}-1)} |\Omega_{10}|^{\frac{1}{q}} \tag{6.130}$$

$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, using the estimate (6.15).

Next, consider what happens outside $B(P, \varepsilon^{\theta_{10}})$.

Assume $\Gamma_{y^1}, \Gamma_{y^2}$ translated such that P lies in the intersection of both curves. If $P = (\overline{x}_1, \overline{x}_2, \overline{x}_3)$, it follows from (6.3) that for $i = 1, 2$ the translate $\Gamma_{y_i} + \omega(y^i)$ satisfies the equation

$$\begin{cases} x_1^i - \overline{x}_1 + a_{11} y_1^i (x_3 - \overline{x}_3) + \partial_{y_1} \psi(x_3, y^i) - \partial_{y_1} \psi(\overline{x}_3, y^i) = 0 \\ x_2^i - \overline{x}_2 + a_{22} y_2^i (x_3 - \overline{x}_3) + \partial_{y_2} \psi(x_3, y^i) - \partial_{y_2} \psi(\overline{x}_3, y^i) = 0 . \end{cases} \tag{6.131}$$

Subtraction for $i = 1, 2$ gives easily for $(x_1^i, x_2^i, x_3) \in \Gamma_{y^i} + \omega(y^i)$

$$|x_1^1 - x_1^2| + |x_2^1 - x_2^2| > c|y^1 - y^2||x_3 - \overline{x}_3| + o(|y^1 - y^2||x_3 - \overline{x}_3|)$$
$$= c|y^1 - y^2||x_3 - \overline{x}_3| . \tag{6.132}$$

Hence

$$\text{dist}\left((\Gamma_{y^1} + \omega(y^1)) \backslash B(P, \varepsilon^{\theta_{10}}), (\Gamma_{y^2} + \omega(y^2)) \backslash B(P, \varepsilon^{\theta_{10}})\right) >$$
$$\varepsilon^{\theta_{10}} |y^1 - y^2| . \tag{6.133}$$

It follows that if (6.113) holds, then

$$\text{dist}\left(\widetilde{\Gamma}_{y^1} \backslash B(P, \varepsilon^{\theta_{10}}), \widetilde{\Gamma}_{y^2} \backslash B(P, \varepsilon^{\theta_{10}})\right) > \varepsilon^{\theta_{10}} |y^1 - y^2| - \varepsilon^{\theta_9} \tag{6.134}$$

and therefore

$$(\widetilde{\Gamma}_{y^1} \cap \widetilde{\Gamma}_{y^2}) \backslash B(P, \varepsilon^{\theta_{10}}) = \emptyset \quad \text{if} \quad y^1, y^2 \in \Omega_{10} \quad \text{and} \quad |y^1 - y^2| > \varepsilon^{\theta_9 - \theta_{10}} . \tag{6.135}$$

Consequently, by (6.10)

$$\left\| \int_{\Omega_{10}} \chi_{\widetilde{\Gamma}_y \setminus B(P,\varepsilon^{\theta_{10}})} dy \right\|_2^2 = \iint\limits_{\substack{\Omega_{10}\times\Omega_{10} \\ |y^1-y^2|<\varepsilon^{\theta_9-\theta_{10}}}} \frac{\varepsilon^3}{\varepsilon - |y^1-y^2|} dy^1\, dy^2$$

$$< \varepsilon^{3+\theta_9-\theta_{10}} |\Omega_{10}| \tag{6.136}$$

and

$$\left\| \varepsilon^{-2} \int_{\Omega_{10}} \chi_{\widetilde{\Gamma}_y \setminus B(P,\varepsilon^{\theta_{10}})} dy \right\|_2 < \varepsilon^{-\frac{1}{2}+\frac{\theta_9-\theta_{10}}{2}} |\Omega_{10}|^{\frac{1}{2}} \tag{6.137}$$

$$\left\| \varepsilon^{-2} \int_{\Omega_{10}} \chi_{\widetilde{\Gamma}_y \setminus B(P,\varepsilon^{\theta_{10}})} dy \right\|_q < \varepsilon^{-\frac{1}{p}+\frac{\theta_9-\theta_{10}}{p}} |\Omega_{10}|^{\frac{1}{q}} \; . \tag{6.138}$$

Combining (6.130), (6.138) gives

$$\left\| \varepsilon^{-2} \int_{\Omega_{10}} \chi_{\widetilde{\Gamma}_y} dy \right\|_q < (\varepsilon^{\theta_{10}(1-\frac{2}{p})} + \varepsilon^{\frac{\theta_9-\theta_{10}}{p}}) \varepsilon^{-\frac{1}{p}} |\Omega_{10}|^{\frac{1}{q}} \; . \tag{6.139}$$

Choose

$$\theta_{10} = \frac{\theta_9}{p-1} \tag{6.140}$$

yielding then

$$\left\| \varepsilon^{-2} \int_{\Omega_{10}} \chi_{\widetilde{\Gamma}_y} dy \right\|_q < \varepsilon^{\frac{p-2}{p(p-1)}\theta_9} \varepsilon^{-\frac{1}{p}} |\Omega_{10}|^{\frac{1}{q}} \; . \tag{6.141}$$

Summary.

From the preceding, fixing j, one obtains (6.25), where the sets Ω_α either satisfy (6.141), i.e.

$$\left\| \varepsilon^{-2} \int_{\Omega_\alpha} \chi_{\widetilde{\Gamma}_y} dy \right\|_q < \varepsilon^{\frac{p-2}{p(p-1)}\theta_9} \varepsilon^{-\frac{1}{p}} |\Omega_\alpha|^{\frac{1}{q}} \tag{6.142}$$

or for $y \in \Omega_\alpha$, one has

$$\widetilde{\Gamma}_y \subset \varepsilon^{1-\theta_{11}}\text{-neighborhood of } \Gamma_y + \omega''(y)$$

where ω'' is a translation function which is smooth with derivatives controlled by the lower bound $\varepsilon^{\theta_{12}}$ on the Jacobians appearing in the process. The number θ_{11} may then be taken

$$\theta_{11} = \theta_9 + \theta_{12} \ . \tag{6.143}$$

All the sets Ω_α introduced also satisfy

$$|\Omega_\alpha| > \varepsilon^{\theta_{13}} \tag{6.144}$$

where the $\theta_j = \theta_j(\theta_1) \overset{\theta_1 \to 0}{\longrightarrow} 0$.

We also summarize the conditions on the θ_j along the construction. The number θ_1 appearing in (6.28) comes in the initial hypothesis (6.34)

$$K = \varepsilon^{-\theta_2} \ ; \qquad \text{see (6.53)} \ .$$

From (6.61), (6.38), the condition

$$2\theta_2 + 5\theta_1 < \frac{1}{3}\theta_4 \ . \tag{6.145}$$

From (6.66)

$$\theta_5 > \theta_2 + 6\theta_1 \ . \tag{6.146}$$

From (6.72), (6.76)

$$\theta_7 > 35\theta_1 + 8\theta_2 + 4\theta_5 \ . \tag{6.147}$$

From (6.97), (6.98)

$$\theta_6 > 6\theta_1 \ , \qquad \theta_2 > 9\theta_1 \ . \tag{6.148}$$

From (6.102)

$$\theta_2 > C_1\theta_1 + \theta_6 \ . \tag{6.149}$$

From (6.104), (6.108)

$$\theta_8 > 6\theta_2 + 16\theta_1 \ . \tag{6.150}$$

This yields $\theta_1, \theta_6, \theta_2, \theta_4, \theta_5, \theta_7, \theta_8$ as linear functions of θ_1. It also follows from the construction that θ_{12}, θ_{13} may be taken to be linear functions of θ_1. The θ_9 is an independent parameter appearing in (6.142) and (6.143) (with different effect). From (6.144), the number of Ω_α's is bounded by $\sim \varepsilon^{-\theta_{13}}$.

Those for which (6.142) holds have thus in (6.28) a total contribution which is crudely estimated by

$$K \cdot \varepsilon^{-\theta_{13}} \cdot \varepsilon^{2-\frac{1}{p}+\frac{p-2}{p(p-1)}\theta_9} < \varepsilon^{2-\frac{1}{p}+\theta_{14}} , \tag{6.151}$$

provided

$$\frac{p-2}{p(p-1)}\theta_9 > \theta_2 + \theta_{13} + \theta_{14} . \tag{6.152}$$

We next analyze the case with smooth translation function. As made clear from the example in section 2, we may not expect to gain something here over the $\varepsilon^{2-\frac{1}{p}}$-estimate without an additional assumption on the phase function φ. Denote $\widetilde{\widetilde{\Gamma_y}}$ an $\varepsilon^{1-\theta_{11}}$-neighborhood of $\Gamma_y + \omega''(y)$ (cf. above).

We have to estimate for $f \in L^p_+$, $\|f\|_p \le 1$

$$\varepsilon^{-2} \int \left\langle f, \chi_{\tilde{\tilde{\Gamma}}_y} \right\rangle dy . \tag{6.153}$$

Averaging first f over cubes of size $\varepsilon^{1-\theta_{11}}$, it amounts to consider $g \ge 0$, $\|g\|_p \le 1$,

$$|\nabla g| < \frac{1}{\varepsilon} \tag{6.154}$$

and the expression

$$\varepsilon^{-2\theta_{11}} \int_0^{1/K} \int g\big(a_{11}x_3y_1 + \eta_1(x_3,y) - \omega_1''(y), a_{22}x_3y_2 + \eta_2(x_3,y) - \\ - \omega_2''(y)\big) dx_3dy_1dy_2 \tag{6.155}$$

where ω'' is smooth with

$$|\partial^{(\alpha)}\omega''| \lesssim \varepsilon^{-\theta_{12}} . \tag{6.156}$$

From the way (6.30) is deduced from (6.3), this yields

$$\varepsilon^{-2\theta_{11}} \int \int_{I_j} g(a_{11}x_3y_1 + \partial_{y_1}\psi(x_3,y) - \omega_1(y), a_{22}x_3y_2 + \partial_{y_2}\psi(x_3,y) \\ - \omega_2(y)\big) dx_3dy_1dy_2 . \tag{6.157}$$

The relation between ω and ω'' is given by (6.32), writing ω_i'' instead of ω_i' and hence also

$$|\partial^{(\alpha)}\omega| \lesssim \varepsilon^{-\theta_{12}} . \tag{6.158}$$

At this point, it is natural to consider the coordinate transformation T : $(y_1, y_2, x_3) \mapsto (x_1, x_2, x_3)$ given by

$$x_i = a_{ii}x_3 y_i + \partial_{y_i}\psi(x_3, y) - \omega_i(y) \qquad (i = 1, 2) \tag{6.159}$$

and for which
$\det DT =$

$$= \begin{vmatrix} a_{11}x_3 + \partial^2_{y_1,y_1}\psi(x_3,y) - \partial_{y_1}\omega_1(y) & \partial^2_{y_1,y_2}\psi(x_3,y) - \partial_{y_2}\omega_1(y) \\ \partial^2_{y_1,y_2}\psi(x_3,y) - \partial_{y_1}\omega_2(y) & a_{22}x_3 + \partial^2_{y_2,y_2}\psi(x_3,y) - \partial_{y_2}\omega_2(y) \end{vmatrix} . \tag{6.160}$$

Analyzing this expression yields

$$\det DT(y, x_3) = [\det D_y^2\varphi] + \sum_{i,j=1}^{2} \omega_{ij}(y)[\partial^2_{y_i y_j}\varphi] + \det J\omega \tag{6.161}$$

where ω_{ij} and $\det J\omega$ only depend on y.

Consider the following hypothesis

$$\begin{cases} \text{The Hessian determinant } \det\left(\frac{\partial^2\varphi}{\partial y^2}\right)(x_3, 0) \text{ is not} \\ \text{a linear combination of the second-order} \\ y\text{-derivatives } \frac{\partial^2\varphi}{\partial y_i \partial y_j}(x_3, 0), \text{ as functions of } x_3 \end{cases} \tag{6.162}$$

Since φ was assumed real analytic, (6.162) implies a non-vanishing Wronskian determinant at $x_3 = 0$ and hence on a neighborhood of $(x_3 = 0, y = 0)$. Thus (6.162) remains valid if y is taken in a neighborhood of 0. In fact, there is a positive number $c_2 > 0$ such that

$$\int |\det DT|^{-c_2} dx_3 dy < c_2^{-1} . \tag{6.163}$$

This constant c_2 only depends on φ.

Write (6.157) as

$$\varepsilon^{-2\theta_{11}} \int_U (g \circ T) dx_3 dy_1 dy_2 . \tag{6.164}$$

Split U as $U = U_1 + U_2$ where

$$\left.\begin{array}{lll} |\det DT| > \varepsilon^{\theta_{15}} & \text{on} & U_1 \\ |\det DT| \le \varepsilon^{\theta_{15}} & \text{on} & U_2 \end{array}\right\} \tag{6.165}$$

Hence, from (6.163)

$$\operatorname{meas} U_2 < c_2^{-1} \varepsilon^{c_2 \theta_{15}} \ . \tag{6.166}$$

The region U_1 may be broken up into domains $U_{1,j}$ where T is invertible. The number of such domains depends on the derivative estimate for T and the lower bound on $|\det DT|$. In fact, from (6.158), one gets a bound on their number by

$$\varepsilon^{-\theta_{16}} \ , \quad \text{where } \theta_{16} \sim \theta_{15} + \theta_{12} \ . \tag{6.167}$$

On each of these $U_{1,j}$, one has for a bounded function $h \ge 0$

$$\begin{aligned} \int_{U_{1,j}} (h \circ T) dx_3 dy_1 dy_2 &= \int_{T(U_{1,j})} \frac{h}{|(\det T) \circ T^{-1}|} dx_1 dx_2 dx_3 \\ &< \varepsilon^{-\theta_{15}} \int h \ . \end{aligned} \tag{6.168}$$

Hence, from (6.166), (6.168), (6.167)

$$\int_U (h \circ T) dx \le c_2^{-1} \varepsilon^{c_2 \theta_{15}} \|h\|_\infty + \varepsilon^{-\theta_{15} - \theta_{16}} \int h \tag{6.169}$$

$$\le c_2^{-1} \varepsilon^{c_2 \theta_{15}} \|h\|_\infty + \varepsilon^{-C\theta_{12}} \varepsilon^{-C\theta_{15}} \|h\|_1 \ . \tag{6.170}$$

Here θ_{15} is a parameter. For an appropriate choice, we find

$$\int_U (h \circ T) dx \le c_2 \varepsilon^{-C\theta_{12}} \|h\|_1^{c_3} \|h\|_\infty^{1-c_3} \ , \tag{6.171}$$

for some positive number c_2 only depending on φ.

This yields an exponent $p = p(\varphi) < \infty$ satisfying

$$\int_U |h \circ T| dx < \varepsilon^{-C\theta_{12}} \|h\|_p \ . \tag{6.172}$$

In particular, from (6.164), (6.157), (6.155) and (6.153) are bounded by

$$\varepsilon^{-2\theta_{11} - C\theta_{12}} \|g\|_p \le \varepsilon^{-2\theta_{11} - C\theta_{12}} \tag{6.178}$$

if we let $p = p(\varphi)$.

The (6.143)-contribution is thus $< \varepsilon^{-2\theta_9 - C\theta_{12} - \theta_{13}} < \varepsilon^{-2\theta_9 - C\theta_1}$, from what preceeds. Hence, from (6.151), there is the total estimate

$$\varepsilon^{2-\frac{1}{p}+\theta_{14}} + \varepsilon^{2-2\theta_9-C\theta_1} \tag{6.179}$$

on the first term of (6.28).

From condition (6.152),

$$\varepsilon^{2-\frac{1}{p}+\frac{p-2}{p(p-1)}\theta_9 - C\theta_1} + \varepsilon^{2-\frac{1}{p}+(\frac{1}{p}-2\theta_9-C\theta_1)} . \tag{6.180}$$

If we let θ_1 be sufficiently small, we find for some $\theta_{17} > 0$ the bound on (6.28), (6.23)

$$\varepsilon^{2-\frac{1}{p}+\theta_{17}} + \varepsilon^{2-\frac{1}{p}+\frac{2}{p}\theta_1} . \tag{6.181}$$

Consequently, there is an inequality

$$\|\mathcal{M}_\varepsilon\|_{p\to 1} < \varepsilon^{-\frac{1}{p}+\theta_{18}} \tag{6.182}$$

for some $\theta_{18} > 0$. This is the required property $\gamma(p) < \frac{1}{p}$ of (5.34) in order to get (4.2). Thus there is the following theorem.

THEOREM. *Let φ of the form (1.12) be real analytic (on a neighborhood of 0) and such that*
(6.162) det $\left(\frac{\partial^2\varphi}{\partial y^2}\right)(x_3, 0)$ *is not a linear combination of the second derivatives* $\frac{\partial^2\varphi}{\partial y_i \partial y_j}(x_3, 0)$.
Then the operators T_N defined by (1.1), i.e.

$$T_N f(x) = \int e^{iN\varphi(x,y)} a(x,y) f(y) dy \tag{6.183}$$

where a is supported on a suitable neighborhood of 0, satisfy for some $q < 4$, the bound

$$\|T_N f\|_q < C N^{-3/q} \|f\|_\infty . \tag{6.184}$$

Remarks.

(1) The relevance of condition (6.162) is clear from the example in section 2. One has indeed in this case

$$\frac{\partial^2 \varphi}{\partial y^2}(x_3, 0) = \begin{pmatrix} x_3^2 & x_3 \\ x_3 & 0 \end{pmatrix} \tag{6.185}$$

and its determinant is $-x_3^2 = -\frac{\partial^2 \varphi}{\partial y_1^2}(x_3, 0)$.

(2) It follows from section 3 that in general we may not expect to have (6.184) for all $q > 3$. For this, it suffices that

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \varphi}{\partial x_3^2}\right)\Big|_{x_3=0=y} \text{ is not a multiple of } \frac{\partial^2}{\partial y^2}\left(\frac{\partial \varphi}{\partial x_3}\right)\Big|_{x_3=0=y} . \tag{6.186}$$

(3) We used the hypothesis of real analyticity of φ in a few places in the previous argument. It is possible to avoid this. It is clear how an adequate strengthening of (6.162) for C^∞-functions may be formulated with the same effect. If we do not assume φ real anlaytic, (6.70) is not valid anymore. For C^∞-functions, one has to do a further partitioning of the y-domain to get the required information on $\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \varphi}{\partial x_3^2}\right)$, which leads to additional technicalities.

7. Further Comments

(i) Factorization.

Consider the operator (1.1) and assume we have shown an inequality

$$\|T_N f\|_q = CN^{-d/q}\|f\|_\infty \tag{7.1}$$

($q > 2$). By general factorization theory, one may then find a probability measure $\mu(dy)$ such that for $r > q$

$$\|T_N f\|_q \leq CN^{-d/q}\|f\|_{L^r(d\mu)} \,. \tag{7.2}$$

It is a natural to ask when μ may be replaced by the standard measure dy. If for instance T_N corresponds to the restriction to a sphere, one may use a standard averaging argument over the orthogonal group to get Lebesgue measure. This case is special and a procedure to deal with a general phase function, even assuming linearity in x, seems unclear. However, if $\varphi(x,y)$ is quadratic in y and has the form (for $d=3$)

$$\varphi(x,y) = x_1y_1 + x_2y_2 + \psi_1(x_3)y_1^2 + \psi_2(x_3)y_2^2 \tag{7.3}$$

there is a way of exploiting translation operators. Define for $z=(z_1,z_2)$, $|z|$ small, the operator

$$\tau_z f(y) = f(y_1+z_1, y_2+z_2) \tag{7.4}$$

and write

$$\big\|T_N(\tau_z f)\big\|_q \le CN^{-3/q}\|\tau_z f\|_{L^r(d\mu)}\,. \tag{7.5}$$

Replacing y by $y-z$, one finds

$$\big\|T_N(\tau_z f)\big\|_q = \Big\|\int e^{iN\varphi(x,y-z)}a(x,y-z)f(y)dy\Big\|_q \tag{7.6}$$

where by (7.3)

$\varphi(x,y-z) =$

function of $x + \big(x_1 - 2z_1\psi_1(x_3)\big)y_1 + \big(x_2 - 2z_2\psi_2(x_3)\big)y_2 + \psi_1(x_3)y_1^2 + \psi_2(x_3)y_2^2\,.$ (7.7)

Substituting (7.7) in (7.6), one may of course ignore the x-terms. Make the following change of variable in x

$$\begin{cases} x_1' = x_1 - 2z_1\psi_1(x_3) \\ x_2' = x_2 - 2z_2\psi_2(x_3) \end{cases} \tag{7.8}$$

which is measure preserving. That brings (7.6) in the form

$$\Big\|\int e^{iN\varphi(x',y)}a\big(x_1' + 2z_1\psi_1(x_3), x_2' + 2z_2\psi_2(x_3), x_3, y-z\big)f(y)dy\Big\|_q\,. \tag{7.9}$$

Replacing the left member of (7.5) by (7.9) and applying a standard averaging argument in z, one finds finally

$$\Big\|\int e^{iN\varphi(x,y)}\overline{a}(x,y)f(y)dy\Big\|_q \le CN^{-3/q}\|f\|_r \tag{7.10}$$

for some other localizing function $\overline{a}$ (which is of course irrelevant).

(ii) Kakeya and Nikodym maximal inequalities in $\mathbf{R}^3$.

Using the notations of [B1] for the Kakeya maximal function f_δ^* and Nikodym maximal function f_δ^{**} (of excentricity δ), the conjectured bounds in $\mathbf{R}^3$ are given by

$$\|f_\delta^*\|_p \quad \text{and} \quad \|f_\delta^{**}\|_p \ll \left(\frac{1}{\delta}\right)^{\frac{3}{p}-1+\varepsilon} \|f\|_p \tag{7.11}$$

where $\varepsilon > 0$ is arbitrary and $p \leq 3$.

This fact was verified in [B1] if $p \leq \frac{7}{3}$. As it follows from [B1], the knowledge of (7.11) for any $p > 2$ has an application to the Bochner-Riesz problem described in the introduction, in the sense that m_λ is shown to be a bounded multiplier on $L^p(\mathbf{R}^3)$ for $\frac{3}{2+\lambda} < p < \frac{3}{1-\lambda}$ and where λ may take certain values $< \frac{1}{4}$.

Our purpose is to use the ideas of section 6 of the paper to get very simple proofs of (7.11) for certain $p > 2$. We do not intend here to try to optimize the method. (The argument in [B1] seems more performing anyway.) Although the arguments of Kakeya and Nikodym are in many respects analogous, it is preferable to give them separately.

(I) Estimates on f_δ^*. Standard techniques cf. [St2] reduce the question to showing a minoration

$$|A| \gg \delta^{3-p+\varepsilon}\sigma^p \tag{7.12}$$

assuming $A \subset B(0,1)$ having a property that for a subset $\Omega \subset S^2$, $|\Omega| > \frac{1}{2}$, each direction $\xi \in \Omega$ may be translated into a line $L_\xi \| \xi$ such that

$$|A \cap L_\xi^\delta| > \sigma\delta^2 \ . \tag{7.13}$$

Here L^δ stands for a δ-neighborhood of L.

Let ξ run in a δ-net $\mathcal{E} \subset \Omega$. It follows from (7.13) that

$$\left\langle \chi_A, \sum_{\xi\in\mathcal{E}} \chi_{L_\xi^\delta} \right\rangle \gtrsim \sigma \tag{7.14}$$

and thus

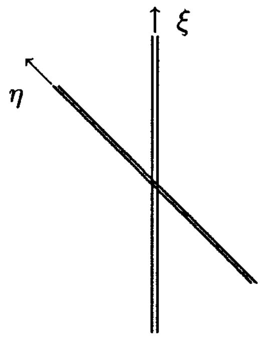
$$|A|^{1/2} \left\| \sum_{\xi\in\mathcal{E}} \chi_{L_\xi^\delta} \right\|_2 \geq \sigma \ . \tag{7.15}$$

Proceeding as in the previous section, one has

$$\Big\| \sum_{\xi\in\mathcal{E}} \chi_{L^\delta_\xi} \Big\|_2^2 = \sum_{\xi,\eta\in\mathcal{E}} |L^\delta_\xi \cap L^\delta_\eta| \tag{7.16}$$

where

$$|L^\delta_\xi \cap L^\delta_\eta| < \frac{\delta^3 \beta(\xi,\eta)}{|\xi-\eta|+\delta} \tag{7.17}$$



and

$$\beta(\xi,\eta) = 1 \iff L^\delta_\xi \cap L^\delta_\eta \neq \emptyset\ . \tag{7.18}$$

Substitution of (7.17) in (7.16) yields the bound

$$\delta^3 \sum_{\xi,\eta} \frac{\beta(\xi,\eta)}{\delta+|\xi-\eta|} < \delta^{-1} \tag{7.19}$$

and by (7.15)

$$|A| > \delta\sigma^2\ . \tag{7.20}$$

This is (7.12) for $p=2$.

Fix now a parameter $0<\lambda<1$ which afterwards will be chosen as a function of σ. If we have

$$\Big\| \sum_{\xi\in\mathcal{E}} \chi_{L^\delta_\xi} \Big\|_2 < \frac{\lambda}{\delta^{1/2}} \tag{7.21}$$

it follows from (7.15) that

$$|A| > \frac{1}{\lambda^2}\delta\sigma^2\ . \tag{7.22}$$

Alternatively, let

$$\Big\| \sum_{\xi \in \mathcal{E}} \chi_{L_\xi^\delta} \Big\|_2 > \lambda \delta^{-1/2} . \tag{7.23}$$

A standard exhaustive argument permits then to find a subset $\mathcal{E}_1 \subset \mathcal{E}$ such that

$$\Big\| \sum_{\xi \in \mathcal{E}_1} \chi_{L_\xi^\delta} \Big\|_2 > \lambda \delta^{-1/2} ; \qquad |\mathcal{E}_1| > \lambda^2 \delta^{-2} \tag{7.24}$$

and for any $\mathcal{E}_2 \subset \mathcal{E}_1$

$$\Big\| \sum_{\xi \in \mathcal{E}_2} \chi_{L_\xi^\delta} \Big\|_2 > \lambda \delta^{3/2} |\mathcal{E}_2| \tag{7.25}$$

(we omit constants).

Taking for $\mathcal{E}_2 = \mathcal{E}_1$ gives as above

$$\begin{aligned} \lambda^2 \delta^{-1} &< \delta^3 \sum_{\xi,\eta \in \mathcal{E}_1} \frac{\beta(\xi,\eta)}{\delta + |\xi - \eta|} \\ &< \delta^3 \Big[\sum_{\xi,\eta} \frac{1}{(\delta + |\xi-\eta|)^2} \Big]^{1/2} \Big[\sum_{\xi,\eta \in \mathcal{E}_1} \beta(\xi,\eta) \Big]^{1/2} \\ &< \left(\log \frac{1}{\delta} \right)^{1/2} \delta \Big[\sum_{\xi,\eta \in \mathcal{E}_1} \beta(\xi,\eta) \Big]^{1/2} . \end{aligned} \tag{7.26}$$

Thus

$$\sum_{\xi,\eta \in \mathcal{E}_1} \beta(\xi,\eta) > \lambda^4 \left(\log \frac{1}{\delta} \right)^{-1} \delta^{-4} \tag{7.27}$$

and therefore one may find $\xi_1 \in \mathcal{E}_1$ and $\mathcal{E}_2 \subset \mathcal{E}_1$ satisfying

$$\beta(\xi,\xi_1) = 1 \quad \text{for} \quad \xi \in \mathcal{E}_2 \tag{7.28}$$

$$|\mathcal{E}_2| > \lambda^4 \left(\log \frac{1}{\delta} \right)^{-1} \delta^{-2} . \tag{7.29}$$

Using next (7.25), it follows similarly that

$$\sum_{\xi,\eta \in \mathcal{E}_2} \beta(\xi,\eta) > \lambda^4 \left(\log \frac{1}{\delta} \right)^{-1} \delta^4 |\mathcal{E}_2|^4 \tag{7.30}$$

$$> \lambda^{20} \left(\log \frac{1}{\delta} \right)^{-5} \delta^{-4} . \tag{7.31}$$

Hence, there is a point $\xi_2 \in \mathcal{E}_2$ and $\mathcal{E}_3 \subset \mathcal{E}_2$ satisfying

$$|\xi_1 - \xi_2| > \lambda^{10} \left(\log \frac{1}{\delta}\right)^{-3} \tag{7.32}$$

$$\beta(\xi, \xi_2) = 1 \qquad \text{for} \qquad \xi \in \mathcal{E}_3 \tag{7.33}$$

$$|\mathcal{E}_3| > \lambda^{20} \left(\log \frac{1}{\delta}\right)^{-5} \delta^{-2} \tag{7.34}$$

Thus, from (7.28), (7.33)

$$L_\xi^\delta \cap L_{\xi_1}^\delta \neq \emptyset \ , \ L_\xi^\delta \cap L_{\xi_2}^\delta \neq \emptyset \ . \tag{7.35}$$

It is clear from (7.34) that one may assume for $\xi \in \mathcal{E}_3$

$$\alpha = \text{angle}\big(\xi, \text{plane}(L_{\xi_1}, L_{\xi_2})\big) > \lambda^{20} \left(\log \frac{1}{\delta}\right)^{-5} . \tag{7.36}$$

Denote P the intersection of L_{ξ_1}, L_{ξ_2}. A simple geometrical analysis of 3 almost concurrent lines shows that for $\xi \in \mathcal{E}_3$ satisfying the above properties, one has for the intersection point Q_ξ of L_ξ and $[L_{\xi_1}, L_{\xi_2}]$

$$\text{dist}(P, Q_\xi) < \frac{\delta}{\alpha|\xi_1 - \xi_2|} \ . \tag{7.37}$$

Thus, by (7.32), (7.36)

$$\text{dist}(P, Q_\xi) < \lambda^{-30} \cdot \left(\log \frac{1}{\delta}\right)^{8} \cdot \delta \ . \tag{7.38}$$

We may therefore find one more set $\mathcal{E}_4 \subset \mathcal{E}_3$ and some point Q fulfilling the conditions

$$Q \in L_\xi^\delta \quad \text{if} \quad \xi \in \mathcal{E}_4 \tag{7.39}$$

$$|\mathcal{E}_4| > \lambda^{80} \cdot \left(\log \frac{1}{\delta}\right)^{-21} \cdot \delta^{-2} \ . \tag{7.40}$$

This geometrical configuration has moreover the property that

$$|A \cap L_\xi^\delta| > \sigma \delta^2 \tag{7.41}$$

for $\xi \in \mathcal{E}_4$, since $\mathcal{E}_4 \subset \Omega$. Letting $Q = 0$, integrating in polar coordinates yields

$$|A| \geq \int_{|r|>\frac{\sigma}{2}} \chi_A(r\xi) r^2\, dr\, d\xi > c\sigma^3 \cdot \delta^2 |\mathcal{E}_4| \,. \tag{7.42}$$

Thus, from (7.40), the following minoration

$$|A| > \lambda^{80} \left(\log \frac{1}{\delta}\right)^{-21} \cdot \sigma^3 \tag{7.43}$$

is obtained, complementary to (7.22). Choosing λ optimally gives

$$|A| \ll \delta^{1-\frac{1}{41}-\varepsilon} \sigma^{2+\frac{1}{21}} \tag{7.44}$$

which is (7.12) with $p = 2 + \frac{1}{41}$.

<u>II.Estimates on f_δ^{**}.</u> In this case we have to prove (7.12) assuming (7.13) holds for all $\xi \in \Omega$, where L_ξ is now a line through ξ and $\Omega \subset B(0,1)$ has measure $> \frac{1}{2}$.

Partition the unit sphere S^2 in caps of diameter $\frac{1}{10}$, one may select one of them, say C centered at e_3 and a set $\Omega_1 \subset \Omega$, $|\Omega_1| > \frac{1}{10^3}$, so that

$$\xi \in \Omega_1 \Rightarrow \text{the direction of } L_\xi \text{ belongs to } C \,. \tag{7.45}$$

Consider next an intersection Ω_2 of Ω_1 and a translate of the e_1, e_2-plane, such that $|\Omega_2| > \frac{1}{10^3}$. The collection of tubes under consideration is obtained by taking a δ-net $\mathcal{E}$ in Ω_2, thus $|\mathcal{E}| \sim \delta^{-2}$ and $(L_\xi^\delta)_{\xi \in \mathcal{E}}$. It is clear from construction that for $\xi, \eta \in \mathcal{E}$, $|\xi - \eta| > 10\delta$

$$L_\xi^\delta \cap L_\eta^\delta \neq \emptyset \Rightarrow \text{angle}(L_\xi, L_\eta) \gtrsim |\xi - \eta| \,. \tag{7.56}$$

This is the property needed to get (7.17). One then reasons exactly along the lines above and has either (7.22) or a configuration with concurrent lines, which leads to (7.43). the same conclusion (7.44) follows.

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