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ON THE BUSEMANN-PETTY PROBLEM FOR PERTURBATIONS OF THE BALL

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1. Introduction

Let K, K' be two centrally symmetric convex bodies in $\mathbb{R}^n$, with centre at 0. Let V_r denote the r-dimensional volume function. We consider the following problem of H. Busemann and C.M. Petty (see [BP], [Bu1]):

Does the property

$$V_{n-1}(K \cap L) < V_{n-1}(K' \cap L) \tag{1.1}$$

for each $(n-1)$-dimensional subspace L of $\mathbb{R}^n$ imply that

$$V_n(K) < V_n(K') \ .$$

For $n = 2$, the question has an affirmative answer, as shown by Busemann in [Bu2]. Larman and Rogers [LR] constructed counterexamples in dimension $n \geq 12$, where K' is the euclidean ball $B_n = \{x \in \mathbb{R}^n \mid |x| = \left(\sum_1^n x_i^2\right)^{1/2} \leq 1\}$. Observe that if $K = B_n$ (or an ellipsoid, by the affine invariance of the problem) the answer is again affirmative. Indeed, let $\| \ \|$ be the norm on $\mathbb{R}^n$ induced by K', i.e.

$$\|x\| = \min\{\lambda \in \mathbb{R} \mid x \in \lambda K'\} \ . \tag{1.2}$$

Then

$$\frac{V_n(K')}{V_n(B_n)} = \int_{S^{n-1}} \|x\|^{-n} \sigma_{n-1}(dx) \tag{1.3}$$

where σ_{n-1} is the normalized invariant measure on the sphere $S_{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$. Similarly

$$\frac{V_{n-1}(K' \cap L)}{V_{n-1}(B_{n-1})} = \int_{S^{n-1} \cap L} \|x\|^{-n+1} \sigma_{n-2}(dx) \tag{1.4}$$

identifying $S^{n-1} \cap L$ and S^{n-2}.

Assuming $V_n(K') \leq V_n(B_n)$, it follows from (1.3) that

$$\int_{G_{n,n-1}} \left(\int_{S^{n-1}\cap L} \|x\|^{-n} \sigma_{n-2}(dx) \right) \mathbb{P}(dL) = \int_{S^{n-1}} \|x\|^{-n} \sigma_{n-1}(dx) \leq 1 \tag{1.5}$$

where $\mathbb{R}$ is the normalized invariant measure on the Grassmanian $G_{n,n-1}$. Thus, for some hyperspace L

$$\int_{S^{n-1}\cap L} \|x\|^{-n} \sigma_{n-1}(dx) \leq 1 \tag{1.6}$$

obviously implying

$$\int_{S^{n-1}\cap L} \|x\|^{-n+1} \sigma_{n-2}(dx) \leq 1 \tag{1.7}$$

and thus

$$V_{n-1}(K' \cap L) \leq V_{n-1}(B_n \cap L) \tag{1.8}$$

by (1.4). This proves the previous claim.

In fact, in our entire discussion we will consider only the case where $K' = B_n$. K. Ball (see [Ba1,2]) obtained counterexamples to the Busemann-Petty problem in dimension $n \geq 10$, considering the cube $Q_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ and using his estimate

$$V_{n-1}(Q_n \cap L) \leq \sqrt{2} \tag{1.9}$$

for every hyperplane L.

Lately[1], A. Giannopoulos [G] lowered the dimension to $n \geq 7$, constructing counterexamples of the form

$$K = A_n(a,b) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_1^{n-1} x_i^2 \leq a^2 \ , \ |x_n| \leq b \right\} \tag{1.10}$$

(cylinders) for certain choices of the parameters a, b.

The aim of this paper is to do a more systematic investigation of what happens when K is a small perturbation of B_n, i.e.

$$\delta(K, B_n) < \delta_0 \tag{1.11}$$

(1) Exposed by S. Pichorides, Orsay 1/90.

where $\delta(A, B)$ stands for the usual Hausdorff-distance between two sets A and B. The main idea is of course to study the variations of 1^{st}, 2^{nd}, etc. order of the volume ratio formula (1.3). This leads us to explicit problems in function theory (related to Radon transforms) and perhaps a better understanding of the role of the dimension.

Our main results are summarized in the following two theorems:

THEOREM 1. *Let K be a convex symmetric body in R^3, $V_3(K) = V_3(B_3)$ and $\delta(K, B_3) < \delta_0$ for some $\delta_0 > 0$ small enough. Then, for some 2-dimensional subspace L, $V_2(K \cap L) \geq V_2(B_2)$.*

THEOREM 2. *Theorem 1 does not hold in dimension $n \geq 7$ in the sense that there are small perturbations K of B_n, $V_n(K) = V_n(B_n)$ and $V_{n-1}(K \cap L) < V_{n-1}(B_{n-1})$, for each hyperplane L.*

Theorem 2 yields another construction of counterexamples in dimension $n \geq 7$, related in spirit to the original Larman-Rogers method [LR] (in particular, probabilistic techniques are used). The variations of (1.3) are however easier to deal with than the formula itself.

Theorem 1 excludes in dimension 3 counterexamples within a neighborhood of the euclidean ball. The method of proof is again based on analyzing the variations of (1.3) which is done by expanding the perturbation in spherical harmonics.

Adapting the proof for dimension 4 seems difficult. The present method fails in an essential way. This paper leaves unsettled dimensions 4,5,6 for the local problem (in the sense of (1.11)). The author feels the methods discussed below deserve further attention.

Letters $c > 0$, $d < \infty$ stand for constants which are at most dependent on dimension.

2. Approximation of the volume-ratio formula

Let K be a convex symmetric body in $\mathbb{R}^n$ and $\| \quad \|$ its induced norm. Then

$$\frac{V_n(K)}{V_n(B_n)} = \left(\int_{S^{n-1}} \|x\|^{-n} \right)^{1/n} . \tag{2.1}$$

Put $r(x) = \|x\|^{-1}$ and consider a perturbation

$$r(x) = 1 + t\varphi(x) \qquad (0 < t < 1 \quad \text{and} \quad |\varphi| \leq 1) .$$

Put

$$I(t) = \left(\int_{S^{n-1}} (1 + t\varphi(x))^n \right)^{1/n} .$$

Thus, taking derivatives

$$I'(t) = I(t)^{1-n} \int (1 + t\varphi)^{n-1} \varphi$$

$$I''(t) =$$

$$= (1-n) I(t)^{-n} I'(t) \int (1+t\varphi)^{n-1}\varphi + (n-1) I^{1-n} \int (1+t\varphi)^{n-2}\varphi^2$$

$$= (1-n) I(t)^{1-2n} \left(\int (1+t\varphi)^{n-1}\varphi \right)^2 + (n-1) I^{1-n} \int (1+t\varphi)^{n-2}\varphi^2$$

and also

$$|I'''(t)| \leq c \int |\varphi|^3 .$$

Hence, by Taylor's theorem

$$\left| I(t) - 1 - \left(\int_{S^{n-1}} \varphi \right) t - \frac{n-1}{2} \left[\int_{S^{n-1}} \varphi^2 - \left(\int_{S^{n-1}} \varphi \right)^2 \right] t^2 \right| \leq$$

$$\leq ct^3 \int_{S^{n-1}} |\varphi|^3 . \tag{2.2}$$

This formula will be used in the proofs of both Theorem 1 and Theorem 2. We start with Theorem 2.

3. Proof of Theorem 2

It clearly suffices to generate a perturbation K of B_n s.t.

$$\left(\frac{\operatorname{Vol} K}{\operatorname{Vol} B_n} \right)^{1/n} > \sup_L \left(\frac{\operatorname{Vol} K \cap L}{\operatorname{Vol} B_{n-1}} \right)^{\frac{1}{n-1}} . \tag{3.1}$$

If K is given as in section 2 above, it follows from (2.2) that (3.1) will hold if

$$\left(\int_{S^{n-1}} \varphi \right) t + \frac{n-1}{2} \left[\int_{S^{n-1}} \varphi^2 - \left(\int_{S^{n-1}} \varphi \right)^2 \right] t^2 \geq$$

$$\left(\int_{S^{n-1}\cap L}\varphi\right)t+\frac{n-2}{2}\left[\int_{S^{n-1}\cap L}\varphi^2-\left(\int_{S^{n-1}\cap L}\varphi\right)^2\right]t^2+Ct^3 \tag{3.2}$$

assuming $|\varphi|\leq 1$.

In addition, in order to ensure K to be convex symmetric, we impose a second derivative bound

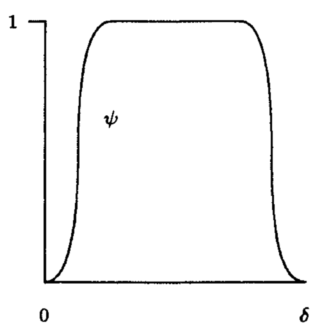
$$t\|D^2\varphi\|_\infty < c \tag{3.3}$$

and the condition

$$\varphi(x)=\varphi(-x) \quad \text{for} \quad x\in S^{n-1}\,. \tag{3.4}$$

Clearly (3.3) (for a suitable constant c) will imply indeed that $t=1+t\varphi$ generates a convex body.

Choose $\delta>0$ and partition the boundary of the cube $\left[-\frac{1}{2},\frac{1}{2}\right]^n$ in cells of size δ which are projected on S^{n-1}. The function φ is defined on those cells and transposed on S^{n-1} by the radial map. Consider on each cell a function of shape



Thus $\|\mathcal{D}^2\psi\|<C\delta^{-2}$ and ψ takes value 1 except on a set of relative measure $o(1)$. Define on the cell Q_α

$$\varphi\big|_{Q_\alpha}=\varepsilon_\alpha\psi_\alpha \qquad (\psi_\alpha \text{ reproducing } \psi)$$

and where $\varepsilon_\alpha = \pm 1$ are signs which will be randomly chosen, with this restriction that

$$\varphi|Q_\alpha = \alpha|\widetilde{Q}_\alpha \tag{3.5}$$

where $\widetilde{Q}_\alpha$ is antipodal to Q_α.

The function φ on S^{n-1} obtained this way has clearly the following properties

$$\left.\begin{array}{c} \varphi \text{ is symmetric} \\ |\varphi| \leq 1 \\ \int |\varphi|^2 = 1 - o(1) \\ \|D^2\varphi\| \leq C\delta^{-2} \end{array}\right\} . \tag{3.6}$$

Moreover, elementary probabilistic considerations yield that for random choice of the signs (subject to condition (3.5), irrelevant for this matter), one gets

$$\left|\int \varphi\right| \leq C\delta^{-\frac{n-1}{2}}\delta^{n-1} = C\delta^{\frac{n-1}{2}} \tag{3.7}$$

$$\left|\int_{S\cap L} \varphi\right| \leq C\delta^{\frac{n-2}{2}}\left(\log\frac{1}{\delta}\right)^{1/2} \text{, for each hyperplane } L . \tag{3.8}$$

To obtain (3.9) consider a net in the Grassmanian $G_{n,n-1}$ and use the (elementary) measure concentration properties for linear combinations of Rademacher functions on $\{1,-1\}^N$. The reader will easily work out the details.

Conditions (3.3) and (3.6) force

$$t \sim \delta^2 . \tag{3.9}$$

Clearly (3.2) is implied by

$$\frac{n-1}{2}\int_{S^{n-1}} \varphi^2 - \frac{n-2}{2}\int_{S^{n-1}\cap L} \varphi^2 \gg$$

$$\gg \frac{1}{t}\left(\left|\int_{S^{N-1}} \varphi\right| + \left|\int_{S^{n-1}\cap L} \varphi\right|\right) + \left(\int_{S^{n-1}} \varphi\right)^2 + t . \tag{3.10}$$

The left member of (3.10) dominates, by (3.6)

$$\frac{n-1}{2} - \frac{n-2}{2} - o(1) > \frac{1}{3} .$$

By (3.7),(3.8),(3.9), the right member is bounded by

$$C\delta^{-2}\left\{\delta^{\frac{n-1}{2}} + \delta^{\frac{n-2}{2}}\left(\log\frac{1}{\delta}\right)^{1/2}\right\} \overset{\delta\to 0}{\longrightarrow} 0$$

provided $\frac{n-2}{2} > 2$.

Remarks.

1. It will be clear in the next section (when studying the corresponding Radon transform) that the probabilistic method of achieving (3.8) yields an essentially optimal result (up to the logarithmic factor).
2. The method described above yields approximations of B_3 in C^α-topology ($\alpha < \frac{1}{2}$) which are (not necessarily convex) counterexamples. Observe that forcing convexity for the body K requires a C^2-perturbation, although in general the gauge-function of a convex body does not admit C^2-bounds. This fact is one of the reasons for the dimension gap comparing Theorems 1 and 2.

4. Proof of Theorem 1

Let K be given by $r = 1 + \varphi$, where

$$\|\varphi\|_\infty = \delta < \delta_0 . \tag{4.1}$$

One has to show that

$$\left(\int_{S^2}(1+\varphi)^n\right)^{1/n} < \sup_L \left(\int_{S^2\cap L}(1+\varphi)^{n-1}\right)^{\frac{1}{n-1}} . \tag{4.2}$$

Writing $\varphi = \varphi - \int_S \varphi + \int_S \varphi$ and rescaling, one may clearly assume

$$\int_S \varphi = 0 . \tag{4.3}$$

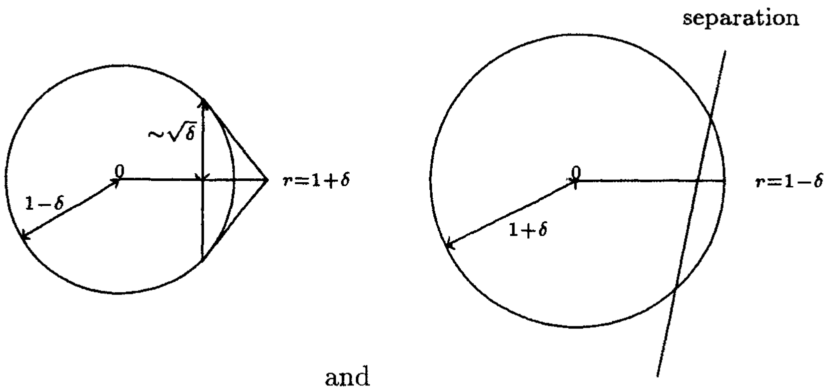
Again by (2.2), (4.2) is implied by the following inequality (let $t = 1$ in (2.2))

$$\frac{n-1}{2}\int_S \varphi^2 + C\delta^3 < \sup_L \left\{ \left(\int_{S\cap L} \varphi\right) + \tfrac{1}{2}\left[\int_{S\cap L}\varphi^2 - \left(\int_{S\cap L}\varphi\right)^2\right]\right\}. \tag{4.4}$$

Since the second term on the right of (4.4) is positive, it suffices to show

$$\int_S \varphi^2 + \delta^3 \ll \sup_L \int_{S\cap L} \varphi\,. \tag{4.5}$$

Observe that because K is convex and $\|r-1\|_\infty = \delta$, one has $|r-1| > \frac{\delta}{2}$ on a $\sim \sqrt{\delta}$-neighborhood of some point in S. There are in fact two cases



Hence, clearly

$$\int_{S^2} \varphi^2 = \int_{S^2} |1-r|^2 > c\delta^2\left(\sqrt{\delta}\right)^2 = c\delta^3 \tag{4.6}$$

and (4.5) may be written

$$\int_S \varphi^2 \ll \sup_L \int_{S\cap L} \varphi\,. \tag{4.5'}$$

We need one more geometric fact

$$\left|r(\xi) - r(\xi')\right| = \left|\varphi(\xi) - \varphi(\xi')\right| \le C\sqrt{\delta}|\xi - \xi'| \tag{4.7}$$

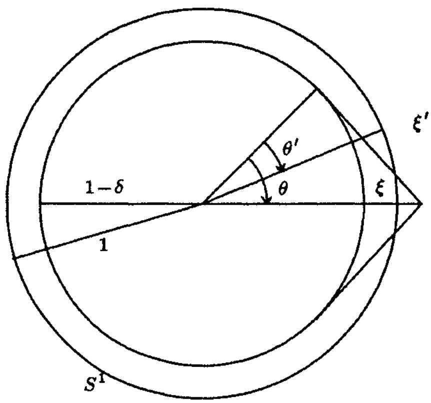
or equivalently

$$\|D\varphi\| < C\sqrt{\delta}\ . \tag{4.8}$$

The proof of (4.7) follows from simple geometric considerations. Obviously the problem is 2-dimensional. Assume $r(\xi) > r(\xi')$. The fact that

$$K \supset \operatorname{conv}\big((1-\delta)B_3, r(\xi)\xi\big)$$

is used to get a lower bound on $r(\xi')$



With θ, θ' as above, one has

$$\begin{aligned} r(\xi) &= (1-\delta)(1+tg^2\varphi)^{1/2} \\ r(\xi') &\geq (1-\delta)(1+tg^2\theta')^{1/2} \quad \text{if} \quad 0 < \theta' \leq \theta \\ &\geq 1-\delta \quad \text{if} \quad \theta' \leq 0\ . \end{aligned}$$

Since $r(\xi) < 1+\delta$

$$\theta < C\sqrt{\delta}\ .$$

- **Case (i):** $\theta' > 0$.
  Thus

$$\begin{aligned} \big|r(\xi) - r(\xi')\big| &= r(\xi) - r(\xi') \\ &\leq (1-\delta|\big[(1+tg^2\theta)^{1/2} - (1+tg^2\theta')^{1/2}\big] \\ &\leq \frac{1}{\cos\theta} - \frac{1}{\cos\theta'} \\ &\leq C|\theta||\theta-\theta'| \leq \sqrt{\delta}|\xi-\xi'|\ . \end{aligned}$$

- **Case (ii):** $\theta' < 0$.

  Then

$$\begin{aligned}\left|r(\xi) - r(\xi')\right| &\leq (1-\delta)\left[(1+tg^2\theta)^{1/2} - 1\right] \\ &\leq C|\theta|^2 \\ &\leq C\sqrt{\delta}|\xi - \xi'| \,.\end{aligned}$$

This proves (4.8).

Let

$$\varphi = \sum_{\substack{k>0 \\ k \text{ even}}} Y_k$$

be the expansion of φ is spherical harmonics (since φ is even, only even degree harmonics appear).

Define the Radon-transform

$$\tilde{\varphi}(\xi) = \int_{S\cap L_\xi} \varphi$$

for $\xi \in S^2$, where L_ξ has the obvious meaning.

Since this operation commutes with the orthogonal transformations

$$\tilde{\varphi} = \sum_{\substack{k>0 \\ K \text{ even}}} \lambda_k Y_k \tag{4.9}$$

where, from the Funk-Hecke formula

$$\begin{aligned}\lambda_k &\sim \int_{-1}^{1} P_k(t)(1-t^2)^{\frac{n-3}{2}} \delta_0(dt) \\ &= P_k(0)\end{aligned}$$

where δ_0 = Dirac measure at 0 and $P_k(t)$ is a Legendre polynomial (see [M] for these matters).

For $n = 3$

$$P_k(t) \sim \frac{\left(-\frac{1}{2}\right)^k}{\Gamma(k+1)} \left(\frac{d}{dt}\right)^k (1-t^2)^k$$

(Rodrigues' formula) and it follows that

$$\begin{aligned}P_k(0) &= 0 \quad \text{if } k \text{ odd} \\ &\sim \left(-\frac{1}{2}\right)^k \frac{1}{k!} \binom{k}{\frac{k}{2}} \left(\frac{d}{dt}\right) (-t^2)^{k/2} \quad \text{if } k \text{ even}\,.\end{aligned}$$

Hence, for k even

$$|\lambda_k| \sim 2^{-k} \binom{k}{\frac{k}{2}} \sim \frac{1}{\sqrt{k}} . \tag{4.10}$$

Since $\int \varphi = 0$, also $\int \widetilde{\varphi} = 0$. Therefore

$$\sup_L \int_{S \cap L} \varphi = \|(\widetilde{\varphi})^+\|_\infty \geq \int_S |\widetilde{\varphi}| \tag{4.11}$$

(4.5′) amounts thus to get

$$\|\widetilde{\varphi}\|_1 \gg \sum \|Y_k\|_2^2 . \tag{4.12}$$

We need two further observations.

(A) – One has

$$\|\widetilde{\varphi}\|_2^2 \leq \|\widetilde{\varphi}\|_1 \|\widetilde{\varphi}\|_\infty . \tag{4.13}$$

Obviously, from (4.7)

$$\left|\widetilde{\varphi}(\xi) - \widetilde{\varphi}(\xi')\right| \leq C\sqrt{\delta}|\xi - \xi'| .$$

Thus if $\left|\widetilde{\varphi}(\xi_0)\right| = \tau = \|\widetilde{\varphi}\|_\infty$, then $|\widetilde{\varphi}| > \frac{\tau}{2}$ on the neighborhood of ξ_0 of radius $\sim \delta^{-1/2}\tau$. Hence

$$\|\widetilde{\varphi}\|_2^2 \geq c\tau^2 \cdot \delta^{-1} \cdot \tau^2 = \delta^{-1}\|\widetilde{\varphi}\|_\infty^4 \tag{4.14}$$

and from (4.13),(4.14)

$$\begin{aligned} \|\widetilde{\varphi}\|_2^2 &\leq c\delta^{1/4}\|\widetilde{\varphi}\|_1\|\widetilde{\varphi}\|_2^{1/2} \\ \|\widetilde{\varphi}\|_1 &\geq c\delta^{-1/4}\|\widetilde{\varphi}\|_2^{3/2} . \end{aligned} \tag{4.15}$$

(B) – The function φ has a harmonic extension to B_3 again given by $\sum Y_k$, where Y_k are considered as functions on B_3. From the general potential theory

$$\|\partial_n \varphi\|_{L^2(S)} \leq \|\nabla_{\text{tangential}}\, \varphi\|_{L^2(S)} .$$

One has

$$\partial_n \varphi = \sum k Y_k$$

and it follows from (4.8) that

$$\sum k^2 \|Y_k\|_2^2 \leq c\delta \ . \tag{4.16}$$

Reduce by (4.15) inequality (4.12) to

$$\|\widetilde{\varphi}\|_2^{3/2} \gg \delta^{1/4} \sum \|Y_k\|_2^2 \ . \tag{4.17}$$

By (4.9), (4.10)

$$\|\widetilde{\varphi}\|_2^2 \sim \sum \frac{1}{k} \|Y_k\|_2^2 \ .$$

Choose k_0 and estimate by (4.16)

$$\sum \|Y_k\|_2^2 \leq \sum_{k<k_0} \|Y_k\|_2^2 + \delta k_0^{-2}$$

and replace (4.17) by

$$\Big(\sum_{k<k_0} \|Y_k\|_2^2\Big)^{3/4} \gg \delta^{1/4} k_0^{3/4} \Big(\sum_{k<k_0} \|Y_k\|_2^2 + \delta k_0^{-2}\Big) \ . \tag{4.18}$$

Choose k_0 the smallest integer such that

$$\sum_{k<k_0} \|Y_k\|_2^2 > \delta k_0^{-2} \tag{4.19}$$

which is possible since

$$\sum \|Y_k\|_2^2 \leq \delta^2 \ .$$

It clearly follows from (4.16) that for this choice of k_0

$$\sum_{k<k_0} \|Y_k\|_2^2 \sim \delta k_0^{-2} \tag{4.20}$$

and

$$\sum_{k<k_0} \|Y_k\|_2^2 > c \sum_{k \geq k_0} \|Y_k\|_2^2$$

hence, by (4.6)

$$\sum_{k<k_0} \|Y_k\|_2^2 > c\|\varphi\|_2^2 > c\delta^3 . \tag{4.21}$$

Thus (4.18) becomes for this choice of k_0

$$\Big(\sum_{k<k_0} \|Y_k\|_2^2\Big)^{3/4} \gg \delta^{\frac{1}{4}+\frac{3}{8}} \Big(\sum_{k<k_0} \|Y_k\|^2\Big)^{5/8}$$

or

$$\Big(\sum_{k<k_0} \|Y_k\|_2^2\Big)^{1/8} \gg \delta^{\frac{1}{4}+\frac{3}{8}} . \tag{4.22}$$

By (4.21), this clearly holds for $\delta \to 0$, concluding the proof.

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