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NONSTATIONARY RANDOM VIBRATION ANALYSIS OF LINEAR ELASTIC STRUCTURES WITH FINITE ELEMENT METHOD*

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ABSTRACT

At present, the finite element method is an efficient method for analyzing structural dynamic problems. When the physical quantities such as displacements and stresses are resolved in the spectra and the dynamic matrices are obtained in spectral resolving form, the relative equations cannot be solved by the vibration mode resolving method as usual. For solving such problems, a general method is put forward in this paper. The excitations considered with respect to nonstationary processes are as follows:

$$\boldsymbol{P}(t)=\{P_i(t)\},P_i(t)=a_i(t)P_i^0(t),$$

$a_i(t)$ is a time function already known. We make Fourier transformation for the discretized equations obtained by finite element method, and by utilizing the behaviour of orthogonal increment of spectral quantities in random process[1], some formulas of relations about the spectra of excitation and response are derived. The cross power spectral denisty matrices of responses can be found by these formulas, then the structrual safety analysis can be made. When $a_i(t)=1$ $(i=1,2,\cdots,n)$, the method stated in this paper will be reduced to that which is used in the special case of stationary process.

I. Single-Freedom-Degree Case

The basic equation is

$$M\ddot{U}+C\dot{U}+KU=P \tag{1.1}$$

In the above equation, U is displacement of the particle, M and C are mass of particle and coefficient of damping respectively, K is the rigidity coefficient, the dot symbol represents differentiating with respect to time, and P is an excitation of nonstationary process, expressed as follows:

$$P(t)=a(t)P^o(t) \tag{1.2}$$

Making spectral resolving for $P(t)$, we have

$$P(t)=\int_{-\infty}^{\infty}\tilde{P}(\omega)e^{i\omega t}d\omega \tag{1.3}$$

Assume that Fourier transformations of $a(t)$ and $P^o(t)$ are $\tilde{a}(\omega)$ and $\tilde{P}^o(\omega)$ respectively; from the character of Fourier Transformation, we have

$$\tilde{P}(\omega)=\tilde{a}(\omega)*\tilde{P}^o(\omega) \tag{1.4}$$

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$\tilde{P}(\omega)$ is the convolution of $\tilde{a}(\omega)$ and $\tilde{P}^o(\omega)$. Then, we have:

$$P(t)=\iint\tilde{P}^o(\omega_1)\tilde{a}(\omega_2-\omega_1)e^{i\omega_2 t}\,d\omega_1 d\omega_2 \tag{1.5a}$$

The complex conjugate variable of $P(t)$ is

$$P^*(t)=\iint\tilde{P}^{o*}(\omega_3)\tilde{a}^*(\omega_4-\omega_3)e^{-i\omega_4 t}d\omega_3 d\omega_4 \tag{1.5b}$$

For the sake of simplicity, in the above two equations and equations below the integral limits $-\infty$ and ∞ are omitted. Product of eqs.(1.5 a) and (1.5 b) is:

$$P(t)P^*(t)=\iiiint\tilde{P}^o(\omega_1)\tilde{P}^{o*}(\omega_3)\tilde{a}(\omega_2-\omega_1)\tilde{a}^*(\omega_4-\omega_3)\cdot e^{i(\omega_2-\omega_4)t}d\omega_1 d\omega_2 d\omega_3 d\omega_4 \tag{1.6}$$

Note that expectation of the product of $\tilde{P}^o(\omega_1)$ and $\tilde{P}^{o*}(\omega_3)$ may be written as:

$$E[\tilde{P}^o(\omega_1)\tilde{P}^{o*}(\omega_3)]=S_P(\omega_1)\delta(\omega_1-\omega_3) \tag{1.7}$$

δ expresses Dirac function and $S_P(\omega_1)$ is power spectral density function with respect to $P^o(t)$. The expectation of eq.(1.6) may be written as follows:

$$\begin{aligned}E[P(t)P^*(t)]=E|P(t)|^2&=\iiiint E[\tilde{P}^o(\omega_1)\tilde{P}^{o*}(\omega_3)]\\&\cdot\tilde{a}(\omega_2-\omega_1)\tilde{a}^*(\omega_4-\omega_3)e^{i(\omega_2-\omega_4)t}d\omega_1 d\omega_2 d\omega_3 d\omega_4\\&=\iiiint S_P(\omega_1)\delta(\omega_1-\omega_3)\tilde{a}(\omega_2-\omega_1)\tilde{a}^*(\omega_4-\omega_3)\\&\cdot e^{i(\omega_2-\omega_4)t}d\omega_1 d\omega_2 d\omega_3 d\omega_4\\&=\iiint S_P(\omega_1)\tilde{a}(\omega_2-\omega_1)\tilde{a}^*(\omega_4-\omega_1)e^{i(\omega_2-\omega_4)t}d\omega_1 d\omega_2 d\omega_4\\&=\int S_P(\omega_1)\left[\int\tilde{a}(\omega_2-\omega_1)e^{i\omega_2 t}d\omega_2\right]\left[\int\tilde{a}^*(\omega_4-\omega_1)e^{-i\omega_4 t}d\omega_4\right]d\omega_1\end{aligned} \tag{1.8}$$

Mean square value $E|P(t)|^2$ may usually be formulated as follows:

$$E|P(t)|^2=\int S_P(t,\omega)d\omega \tag{1.9a}$$

$$S_P(t,\omega)=|a(t,\omega)|^2 S_P(\omega) \tag{1.9b}$$

If we define that

$$a(t,\omega)=\int\tilde{a}(\omega'-\omega)e^{i\omega' t}d\omega' \tag{1.10}$$

by comparing eq.(1.8) with (1.9 a,b), we see that eq.(1.8) is equivalent to(1.9 a,b). In practice, we have

$$a(t,\omega)=\int\tilde{a}(\omega'-\omega)e^{i(\omega'-\omega)t}\cdot e^{i\omega t}d(\omega'-\omega)=a(t)e^{i\omega t} \tag{1.11}$$

Substituting the above equation into eq.(1.9 b), we obtain:

$$S_P(t,\omega)=a^2(t)S_P(\omega) \tag{1.12}$$

Similar to derivation of eq.(1.8), we may find the correlation function of $P(t_1)$ and $P^*(t_2)$ as follows:

$$\begin{aligned}R_P(t_1,t_2)&=E[P(t_1)P^*(t_2)]\\&=\int S_P(\omega_1)a(t_1)a(t_2)e^{i\omega_1(t_1-t_2)}d\omega_1\\&=a(t_1)a(t_2)\int S_P(\omega)e^{i\omega(t_1-t_2)}d\omega\end{aligned} \tag{1.13}$$

This result can also be derived from eq.(1.2) directly,

$$\begin{aligned}E[P(t_1)P^*(t_2)]&=a(t_1)a(t_2)E[P^o(t_1)P^{o*}(t_2)]\\&=a(t_1)a(t_2)\int S_P(\omega)e^{i\omega(t_1-t_2)}d\omega\end{aligned} \tag{1.14}$$

Now, we may consider the solution of eq.(1.1). The frequency response function of eq.(1.1) is

$$H(\omega)=\frac{1}{-M\omega^2+Ci\omega+K} \tag{1.15}$$

$$\tilde{U}(\omega)=H(\omega)\tilde{P}(\omega) \tag{1.16}$$

Then we have

$$U(t)=\int\tilde{U}(\omega)e^{i\omega t}d\omega=\int H(\omega)\tilde{P}(\omega)e^{i\omega t}d\omega$$

$$=\iint H(\omega_2)\tilde{a}(\omega_2-\omega_1)\tilde{P}^o(\omega_1)e^{i\omega_2 t}d\omega_1 d\omega_2 \tag{1.17a}$$

$$U^*(t)=\iint H^*(\omega_4)\tilde{a}^*(\omega_4-\omega_3)\tilde{P}^{o*}(\omega_3)e^{-i\omega_4 t}d\omega_3 d\omega_4 \tag{1.17b}$$

The expectation of the product of the above two equations is

$$E[U(t)U^*(t)]=\iiiint E[\tilde{P}^o(\omega_1)\tilde{P}^{o*}(\omega_3)]H(\omega_2)H^*(\omega_4)$$

$$\cdot\tilde{a}(\omega_2-\omega_1)\tilde{a}^*(\omega_4-\omega_3)e^{i(\omega_2-\omega_4)t}d\omega_1 d\omega_2 d\omega_3 d\omega_4$$

$$=\int S_P(\omega_1)\left[\int H(\omega_2)\tilde{a}(\omega_2-\omega_1)e^{i\omega_2 t}d\omega_2\right]$$

$$\cdot\left[\int H^*(\omega_4)\tilde{a}^*(\omega_4-\omega_1)e^{-i\omega_4 t}d\omega_4\right]d\omega_1 \tag{1.18}$$

The above equation may be rewritten as

$$E[U(t)U^*(t)]=\int S_U(t,\omega)d\omega$$

$$=\int|b(t,\omega)|^2 S_P(\omega)d\omega \tag{1.19}$$

where

$$S_U(t,\omega)=|b(t,\omega)|^2 S_P(\omega) \tag{1.20a}$$

$$b(t,\omega)=\int H(\omega')\tilde{a}(\omega'-\omega)e^{i\omega' t}d\omega' \tag{1.20b}$$

Usually $a(t)$ and $\tilde{a}(\omega)$ are all known in practical problems. If only $a(t)$ is known, $\tilde{a}(\omega'-\omega)$ may be obtained by the following equation:

$$\tilde{a}(\omega'-\omega)=\frac{1}{2\pi}\int a(\tau)e^{-i(\omega'-\omega)\tau}d\tau \tag{1.21}$$

From eqs.(1.21),(1.20 b) and (1.20 a), $b(t,\omega)$ and response spectral density $S_U(t,\omega)$ may be obtained. Starting from eqs. (1.17 a,b), similar to the above derivation, correlation function $E[U(t_1)U^*(t_2)]$ is found as

$$R_U(t_1,t_2)=E[U(t_1)U^*(t_2)]$$

$$=\int[b(t_1,\omega)b^*(t_2,\omega)]S_P(\omega)d\omega \tag{1.22}$$

According to the formulas commonly used in random process:

$$R_{U^{(n)}U^{(m)}}(t_1,t_2)=E\left[\frac{d^nU(t_1)}{dt_1^n}\ \frac{d^mU(t_2)}{dt_2^m}\right]$$

$$=\frac{\partial^{n+m}R_U(t_1,t_2)}{\partial t_1^n\partial t_2^m} \tag{1.23}$$

$$R_{U^{(n)}U^{(m)}}(t_1,t_2)=\int\frac{d^nb(t_1,\omega)}{dt_1^n}\cdot\frac{d^mb^*(t_2,\omega)}{dt_2^m}S_P(\omega)d\omega \tag{1.24}$$

the correlation function of any degree derivatives can be obtained.

If time function $a(\tau)=1$, then eqs.(1.21) and (1.20 b) reduce to the following two equations respectively:

$$\bar{a}(\omega'-\omega)=\delta(\omega'-\omega) \tag{1.25a}$$

$$b(t,\omega)=H(\omega)e^{i\omega t} \tag{1.25b}$$

and eqs.(1.19) and (1.22) reduce to

$$E[U(t)U^*(t)]=\int H(\omega)H^*(\omega)S_P(\omega)d\omega \tag{1.26a}$$

$$R_U(t_1,t_2)=E[U(t_1)U^*(t_2)]$$
$$=\int H(\omega)H^*(\omega)S_P(\omega)e^{i\omega(t_1-t_2)}d\omega \tag{1.26b}$$

respectively. These are the well-known equations in textbooks on stationary random process.

II. Multi-Freedom-Degree Case

In this article, the results obtained above will be generalized to multi-freedom-degree case. According to dynamic finite-element-method in spectral resolving form, the basic equation may be written as

$$\mathbf{K}(\omega)\tilde{\mathbf{U}}(\omega)=\tilde{\mathbf{P}}(\omega) \tag{2.1}$$

$\tilde{\mathbf{U}}(\omega)$ is nodal displacement matrix, $\tilde{\mathbf{P}}(\omega)$ is reduced nodal excitation matrix, in each of both matrices there are n independent components. $\mathbf{K}(\omega)$ is an $n\times n$ dynamic rigidity matrix obtained by discretization with finite-element-method. When a special value is assigned to ω, eq.(2.1) may be solved with a method similar to that in static case. Solving eq.(2.1), we obtain

$$\tilde{\mathbf{U}}(\omega)=\mathbf{K}^{-1}(\omega)\tilde{\mathbf{P}}(\omega) \tag{2.2}$$

where $\tilde{P}(\omega)$ is excitation in spectral form with respect to nonstationary process $\mathbf{P}(t)$. Since $\mathbf{P}(t)$ may be expressed as

$$\mathbf{P}(t)=a(t)\mathbf{P}^o(t) \tag{2.3}$$

where $a(t)$ is a function known and $\mathbf{P}^o(t)$ is a stationary random vector process, the response $U(t)$ will also be a nonstationary random process. According to finite-element-method, there exist internal force-displacement relations:

$$\tilde{\mathbf{T}}(\omega)=\mathbf{E}(\omega)\tilde{\mathbf{U}}(\omega) \tag{2.4}$$

by which we may obtain $\tilde{\mathbf{T}}(\omega)$ and also $\tilde{\mathbf{U}}(\omega)$, $\tilde{\mathbf{T}}(\omega)$. In structural vibration problems, it is usually supposed that the mean values of excitations are zero, i.e.,

$$\left.\begin{aligned}&E[\mathbf{P}(t)]=0,\ E[\mathbf{P}^o(t)]=0\\&E[\tilde{\mathbf{P}}(\omega)]=0,\ E[\tilde{\mathbf{P}}^o(\omega)]=0\end{aligned}\right\} \tag{2.5}$$

$\tilde{\mathbf{P}}(\omega)$ and $\tilde{\mathbf{P}}^o(\omega)$ are the spectral solutions of $P(t)$ and $\mathbf{P}^o(t)$ respectively. The important statistical quantities in random process are correlation matrix $\mathbf{R}_P(t_1,t_2)$ and spectral matrix $\mathbf{S}_P(t,\omega)$. The former may be written as

$$\mathbf{R}_P(t_1,t_2)=\begin{bmatrix} R_{P(11)}(t_1,t_2) & R_{P(12)}(t_1,t_2) & \cdots & R_{P(1n)}(t_1,t_2) \\ R_{P(21)}(t_1,t_2) & R_{P(22)}(t_1,t_2) & \cdots & R_{P(2n)}(t_1,t_2) \\ \vdots & \vdots & & \vdots \\ R_{P(n1)}(t_1,t_2) & R_{P(n2)}(t_1,t_2) & \cdots & R_{P(nn)}(t_1,t_2) \end{bmatrix} \quad (2.6)$$

where

$$\mathbf{R}_P(t_1,t_2)=E[\mathbf{P}(t_1)\mathbf{P}^t(t_2)] \quad (2.7)$$

In the above equation the index t on the right upper corner means taking its transposed matrix, and we have

$$\mathbf{S}_P(t,\omega)=\begin{vmatrix} S_{P(11)}(t,\omega) & S_{P(12)}(t,\omega) & \cdots & S_{P(1n)}(t,\omega) \\ S_{P(21)}(t,\omega) & S_{P(22)}(t,\omega) & \cdots & S_{P(2n)}(t,\omega) \\ \vdots & \vdots & & \vdots \\ S_{P(n1)}(t,\omega) & S_{P(n2)}(t,\omega) & \cdots & S_{P(nn)}(t,\omega) \end{vmatrix} \quad (2.8)$$

Between the above two matrices there exists the following relation:

$$\mathbf{R}_P(t,t)=\int \mathbf{S}_P(t,\omega)d\omega \quad (2.9)$$

Similar to eq. (1.7) in single-degree case, we have

$$E[\tilde{\mathbf{P}}^o(\omega)\tilde{\mathbf{P}}^{ot*}(\omega')]=\mathbf{S}_P(\omega)\delta(\omega-\omega') \quad (2.10)$$

$\mathbf{S}_P(\omega)$ is the cross power spectral density matrix with respect to $\mathbf{P}^o(t)$ and we also have

$$\tilde{\mathbf{P}}(\omega) = \tilde{a}(\omega) * \tilde{\mathbf{P}}^o(\omega) \quad (2.11)$$

$$\mathbf{P}(t) = \iint \tilde{\mathbf{P}}^o(\omega_1)\tilde{a}(\omega_2-\omega_1)e^{i\omega_2 t}d\omega_1 d\omega_2 \quad (2.12a)$$

$$\mathbf{P}^{t*}(t) = \iint \tilde{\mathbf{P}}^{ot*}(\omega_3)\tilde{a}(\omega_4-\omega_3)e^{-i\omega_4 t}d\omega_3 d\omega_4 \quad (2.12b)$$

$$\begin{aligned}\mathbf{R}_P(t,t)&=E[\mathbf{P}(t)\mathbf{P}^t(t)]=E[\mathbf{P}(t)\mathbf{P}^{t*}(t)]\\ &=\iiiint E[\tilde{\mathbf{P}}^o(\omega_1)\tilde{\mathbf{P}}^{ot*}(\omega_3)]\tilde{a}(\omega_2-\omega_1)\tilde{a}^*(\omega_4-\omega_3)\\ &\quad \cdot e^{i(\omega_2-\omega_4)t}d\omega_1 d\omega_2 d\omega_3 d\omega_4\\ &=\iiiint \mathbf{S}_P(\omega_1)\delta(\omega_1-\omega_3)\tilde{a}(\omega_2-\omega_1)\tilde{a}^*(\omega_4-\omega_3)\\ &\quad \cdot e^{i(\omega_2-\omega_4)t}d\omega_1 d\omega_2 d\omega_3 d\omega_4\\ &=\iiint \mathbf{S}_P(\omega_1)\tilde{a}(\omega_2-\omega_1)\tilde{a}^*(\omega_4-\omega_1)e^{i\omega_2 t}e^{-i\omega_4 t}d\omega_1 d\omega_2 d\omega_4\\ &=\int \mathbf{S}_P(\omega)a(t,\omega)a^*(t,\omega)d\omega \end{aligned} \quad (2.13)$$

$$a(t,\omega)=\int \tilde{a}(\omega'-\omega)e^{i\omega' t}d\omega'=a(t)e^{i\omega t} \quad (2.14)$$

$$\mathbf{R}_P(t,t)=\int \mathbf{S}_P(t,\omega)d\omega \quad (2.15)$$

$$\mathbf{S}_P(t,\omega)=a^2(t)\mathbf{S}_P(\omega) \quad (2.16)$$

$$\begin{aligned}\mathbf{R}_P(t_1,t_2)&=E[\mathbf{P}(t_1)\mathbf{P}^{t*}(t_2)]=\int \mathbf{S}_P(\omega)\tilde{a}(t_1,\omega)\tilde{a}^*(t_2,\omega)d\omega\\ &=a(t_1)a(t_2)\int \mathbf{S}_P(\omega)e^{i\omega(t_1-t_2)}d\omega\end{aligned} \quad (2.17)$$

Now, we find the spectral density matrix $S_U(t,\ \omega)$ with respect to response $U(t)$ Resolving $U(t)$ into spectra and considering eq.(2.2), we obtain

$$U(t)=\int \tilde{U}(\omega)e^{i\omega t}d\omega=\int K^{-1}(\omega)\tilde{P}(\omega)e^{i\omega t}\,d\omega$$

$$=\iint K^{-1}(\omega_2)\tilde{a}(\omega_2-\omega_1)\tilde{P}^{o}(\omega_1)e^{i\omega_2 t}\,d\omega_1\,d\omega_2 \tag{2.18a}$$

$$U'^{*}(t)=\iint \tilde{P}^{o\prime *}(\omega_3)\tilde{a}^{*}(\omega_4-\omega_3)K^{-1\prime *}(\omega_4)e^{-i\omega_4 t}d\omega_3 d\omega_4 \tag{2.18b}$$

By finding expectation of the product of the above two equations and considering eq.(2.10), we have

$$\begin{aligned}R_U(t,t)&=E[U(t)U'^{*}(t)]\\&=\iiiint K^{-1}(\omega_2)\tilde{a}(\omega_2-\omega_1)E[\tilde{P}^{o}(\omega_1)\tilde{P}^{o\prime *}(\omega_3)]\tilde{a}^{*}(\omega_4-\omega_3)\\&\quad\cdot K^{-1\prime *}(\omega_4)e^{i\omega_2 t}e^{-i\omega_4 t}d\omega_1 d\omega_2 d\omega_3 d\omega_4\\&=\iiiint K^{-1}(\omega_2)\tilde{a}(\omega_2-\omega_1)S_P(\omega_1)\delta(\omega_1-\omega_3)\tilde{a}^{*}(\omega_4-\omega_3)\\&\quad\cdot K^{-1\prime *}(\omega_4)e^{i\omega_2 t}e^{-i\omega_4 t}d\omega_1 d\omega_2 d\omega_3 d\omega_4\\&=\iiint K^{-1}(\omega_2)\tilde{a}(\omega_2-\omega_1)S_P(\omega_1)\tilde{a}^{*}(\omega_4-\omega_1)\\&\quad\cdot K^{-1\prime *}(\omega_4)e^{i\omega_2 t}e^{-i\omega_4 t}d\omega_1 d\omega_2 d\omega_4\\&=\int\Big[\int K^{-1}(\omega_2)\tilde{a}(\omega_2-\omega_1)e^{i\omega_2 t}\,d\omega_2\Big]S_P(\omega_1)\\&\quad\cdot\Big[\int K^{-1\prime *}(\omega_4)\,\tilde{a}^{*}(\omega_4-\omega_1)e^{-i\omega_4 t}\,d\omega_4\Big]d\omega_1\end{aligned} \tag{2.19}$$

Eq. (2.29) may be rewritten as follows:

$$R_U(t,\ t)=\int b(t,\omega)S_P(\omega)b'^{*}(t,\omega)d\omega \tag{2.20}$$

where

$$b(t,\ \omega)=\int K^{-1}(\omega')\tilde{a}(\omega'-\omega)e^{i\omega' t}d\omega' \tag{2.21}$$

Since $R_U(t,t)$ also may be resolved into spectral form:

$$R_U(t,\ t)=\int S_U(t,\ \omega)d\omega \tag{2.22}$$

by comparing eq.(2.23) with (2.20),(2.21), we obtain

$$S_U(t,\omega)=b(t,\omega)S_P(\omega)b'^{*}(t,\omega) \tag{2.23}$$

Similar to derivation in the first article, we obtain

$$\begin{aligned}R_U(t_1,t_2)&=E[U(t_1)U'(t_2)]\\&=\int b(t_1,\omega)S_P(\omega)b'^{*}(t_2,\ \omega)d\omega\end{aligned} \tag{2.24}$$

In structural vibration analysis, the cross correlation matrix with respect to internal force T is more important. From eqs.(2.4), (2.18 a), we obtain

$$\begin{aligned}T(t)&=\int \tilde{T}(\omega)e^{i\omega t}d\omega\\&=\iint E(\omega_2)K^{-1}(\omega_2)\tilde{a}(\omega_2-\omega_1)\tilde{P}^{o}(\omega_1)e^{i\omega_2 t}d\omega_1 d\omega_2\end{aligned} \tag{2.25}$$

Replace eq. (2.24) by the following equation:

$$R_T(t_1,t_2)=\int c(t_1,\ \omega)S_P(\omega)c'^{*}(t_2,\omega)d\omega \tag{2.26a}$$

where

$$c(t,\omega)=\int E(\omega')K^{-1}(\omega')\tilde{a}(\omega'-\omega)e^{i\omega' t}d\omega' \tag{2.26b}$$

In the same manner, with respect to $\dot{T}$, we have

$$\dot{T}(t)=\iint i\omega_2 E(\omega_2)K^{-1}(\omega_2)\tilde{a}(\omega_2-\omega_1)\tilde{P}^{o}(\omega_1)e^{i\omega_2 t}d\omega_1 d\omega_2 \tag{2.27}$$

$$R_{\dot{T}}(t_1,t_2)=\int d(t_1,\omega)S_P(\omega)d'^{*}(t_2,\omega)d\omega \tag{2.28a}$$

$$d(t,\omega)=\int i\omega' E(\omega')K^{-1}(\omega')\tilde{a}(\omega'-\omega)e^{i\omega' t}d\omega' \tag{2.28b}$$

$$R_{T\dot{T}}(t_1,t_2)=\int c(t_1,\omega)S_P(\omega)d'^{*}(t_2,\omega)d\omega \tag{2.29}$$

Since

$$R_T(t,t)=\int S_T(t,\omega)d\omega \tag{2.30a}$$

we get:

$$S_T(t,\omega)=c(t,\omega)S_P(\omega)c'^{*}(t,\omega) \tag{2.30b}$$

and we have

$$R_{\dot{T}}(t,t)=\int S_{\dot{T}}(t,\omega)d\omega \tag{2.31a}$$

$$S_{\dot{T}}(t,\omega)=d(t,\omega)S_P(\omega)d'^{*}(t,\omega) \tag{2.31b}$$

$$R_{T\dot{T}}(t,t)=\int S_{T\dot{T}}(t,\omega)d\omega \tag{2.32a}$$

$$S_{T\dot{T}}(t,\omega)=c(t,\omega)S_P(\omega)d'^{*}(t,\omega) \tag{2.32b}$$

When the variation of function $a(t)$ is slower than that of $e^{i\omega t}$ with respect to t, we have the following approximate relations:

$$S_{\dot{T}}(t,\omega)=\omega^2 S_T(t,\omega) \tag{2.33}$$

$$S_{T\dot{T}}(t,\omega)=-i\omega S_T(t,\omega) \tag{2.34}$$

In the case of stationary process, i.e., suppose $a(t)=1$, we have

$$b(t,\omega)=K^{-1}(\omega)e^{i\omega t} \tag{2.35a}$$

$$c(t,\omega)=E(\omega)K^{-1}(\omega)e^{i\omega t} \tag{2.35b}$$

$$d(t,\omega)=i\omega E(\omega)K^{-1}(\omega)e^{i\omega t} \tag{2.35c}$$

$$S_U(t,\omega)=S_U(\omega)=K^{-1}(\omega)S_P(\omega)K^{-1\prime *}(\omega) \tag{2.36}$$

$$S_T(t,\omega)=S_T(\omega)=E(\omega)K^{-1}(\omega)S_P(\omega)K^{-1\prime *}(\omega)E'^{*}(\omega) \tag{2.37}$$

$$S_{\dot{T}}(t,\omega)=S_{\dot{T}}(\omega)=\omega^2 S_T(\omega) \tag{2.38}$$

$$S_{T\dot{T}}(t,\omega)=S_{T\dot{T}}(\omega)=-i\omega S_T(\omega) \tag{2.39}$$

In structural analysis, perhaps the diagonal element of $S_T(t,\omega)$ and $S_{\dot{T}}(t,\omega)$ are more important, which are written as follows:

$$S_{T(kk)}(t,\omega)=\sum_{i=1}^{n}\sum_{j=1}^{n}c_{(ki)}(t,\omega)S_{P(ij)}(\omega)c'^{*}_{(jk)}(t,\omega) \tag{2.40}$$

$$S_{\dot{T}(kk)}(t,\omega)=\sum_{i=1}^{n}\sum_{j=1}^{n}d_{(ki)}(t,\omega)S_{P(ij)}(\omega)d'^{*}_{(jk)}(t,\omega) \approx \omega^2 S_{T(kk)}(t,\omega) \tag{2.41}$$

and we have

$$R_{T(k)}(t,\ t)=\int S_{T(kk)}(t,\omega)d\omega \tag{2.12}$$

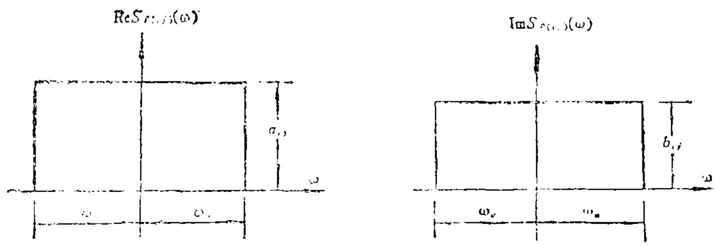
$$R_{\tilde{T}(k)}(t,\ t)=\int S_{\tilde{T}(kk)}(t,\omega)d\omega \tag{2.13}$$

III. Computing Method and Safety Analysis

In practical problems, suppose that $P(t)$ and $S_P(\omega)$ are obtained by statistical method, we desire to get the form of $S_P(\omega)$ as simple as possible, by limiting the value of ω to a certain interval and the relative integrals will also be integrated numerically in a certain interval. The maximum absolute value of ω is denoted by ω_u. Suppose when $|\omega|\leqslant\omega_u$ we have

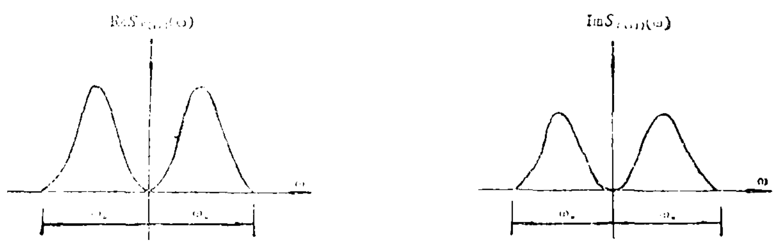
$$S_{P(ij)}(\omega)=a_{ij}(\omega)+ib_{ij}(\omega) \tag{3.1}$$

Otherwise, $S_{P(ij)}(\omega)=0$. There are two typical forms of spectral functions shown in Figs.1 and 2 respectively.



(a) Real spectrum (b) Imaginary spectrum

Fig. 1.



(b) Real spectrum (b) Imaginary spectrum

Fig. 2.

From eq.(1.21) we see that, when $a(t)=1$, $\bar{a}(\omega)=\delta(\omega)$ when $a(t)=\delta(t)$, $\bar{a}(\omega)=\frac{1}{2\pi}$. The true form of function $\bar{a}(\omega)$ should be one between the above two cases. If the time the vibration takes is between $-T$ and T, $\bar{a}(\omega)$ may be approximately expressed in the form shown in Fig.3.

For structural dynamic problems, we must find the forms of $E(\omega)$ and $K(\omega)$ first by finite element method. When ω is a certain given value, these matrices possess [illegible]. By eq.(2.26 [illegible]), for certain values of both t and ω, $c(t,\omega)$

may be found by numerical integration, and from eqs.(2.30 a,b) $S_T(t,\omega)$ can be obtained. In the same manner, we obtain $S_{\dot{T}}(t,\omega)$ and $R_{\dot{T}}(t,\omega)$. If the form of $\tilde{a}(\omega)$ may be expressed as Fig.3, by substituting $\omega'-\omega=\omega''$ into eq.(2.26 b) and some calculation, we have

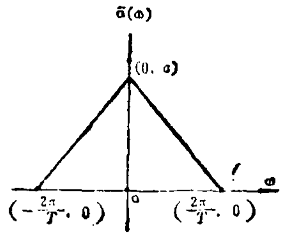


Fig.3.

$$c(t,\omega)=e^{i\omega t}\int_{-2\pi/T}^{2\pi/T}E(\omega''+\omega)K^{-1}(\omega''+\omega)\tilde{a}(\omega'')e^{i\omega'' t}\,d\omega'' \tag{3.2}$$

when $t=0$, it reduces to

$$c(0,\omega)=\int_{-2\pi/T}^{2\pi/T}E(\omega''+\omega)K^{-1}(\omega''+\omega)\tilde{a}(\omega'')d\omega'' \tag{3.3}$$

When correlation functions $R_{T(nn)}(t,t)$ and $R_{\dot{T}(nn)}(t,t)$ are obtained, we can find the degree of safety of structures according to the following method[2].

Suppose that $\lambda_{(nn)}$ is the strength of the n-th member and the number, which the internal force surpasses $\lambda_{(nn)}$ per unit time, is denoted by $\nu^+_{\lambda(nn)}(t)$, we have

$$\nu^+_{\lambda(nn)}(t)=\frac{1}{2\pi}\sqrt{\frac{R_{\dot{T}(nn)}(t,t)}{R_{T(nn)}(t,t)}}\exp\left\{-\frac{\lambda^2_{(nn)}}{2R_{T(nn)}(t,t)}\right\} \tag{3.4}$$

Then the dynamic reliability of the n-th member will be

$$P_{(nn)}(\lambda,-\lambda)=\exp\left[-2\int_{-T}^{T}\nu^+_{\lambda(nn)}(t)dt\right] \tag{3.5}$$

Appendix A

A simple Example of Calculation

From eq.(1.20 a) we see that $|H(\omega)|^2$ in the stationary case will be replaced by $|b(t,\omega)|^2$ in the nonstationary case. In the following, we consider the relation $|b(t,\omega)|^2$ and $|H(\omega)|^2$ in the single-freedom-degree case. We have

$$H(\omega)=\frac{1}{\omega_0^2-\omega^2+2i\beta\omega_0\omega} \tag{A-1}$$

Assume the shape of $\tilde{a}(\omega)$ is shown as Fig.3 and

$$a=\frac{T}{2\pi} \tag{A-2}$$

then we have

$$\begin{aligned}b(t,\omega)&=e^{i\omega t}\int_{-2\pi/T}^{2\pi/T}H(\omega''+\omega)\tilde{a}(\omega'')e^{i\omega'' t}d\omega''\\&=ae^{i\omega t}\int_{-2\pi/T}^{2\pi/T}H(\omega''+\omega)\left(1-\frac{\omega}{2\pi/T}\right)e^{i\omega'' t}d\omega''\end{aligned} \tag{A-3}$$

where

$$H(\omega)=\frac{\omega_0^2-\omega^2-2i\beta\omega_0\omega}{(\omega_0^2-\omega^2)^2+4\beta^2\omega_0^2\omega^2} \tag{A-4}$$

For practical computation, Filton's integrating method will be used[3], which may be illustrated by the following numerical example.

$$\frac{2\pi}{T}=0.2,\quad \omega_0=4,\quad \beta=0.025,$$

$$H(\omega)=\frac{(16-\omega^2)-0.2\,i\omega}{(16-\omega^2)^2+0.04\,\omega^2}$$

The integral interval (-0.2, 0.2) is divided into 4 parts, i.e., $n=2$, according to Filton's integrating method, we obtain:

$$e^{-i\omega t}\,b(t,\omega)=ah\left[H(\omega)\cdot\beta+\frac{\gamma}{2}H(\omega-0.1)e^{-iht}+\frac{\gamma}{2}H(\omega+0.1)e^{iht}\right] \quad (A\text{-}5)$$

where

$$\left.\begin{aligned}\beta&=2\theta^{-3}[\theta(1+\cos^2\theta)-2\sin\theta\cos\theta]\\ \gamma&=4\theta^{-3}(\sin\theta-\theta\cos\theta)\\ \theta&=ht=0.1\,t\end{aligned}\right\} \quad (A\text{-}6)$$

Detailed calculation is omitted and the results are listed in the following Table:

TABLE A-1 Values of $|H(\omega)|^2$ and $|b(t,\omega)|^2$

ω	$\lvert H(\omega)\rvert^2$	$\lvert b(0,\omega)\rvert^2$	$\lvert b(15,\omega)\rvert^2$	$\lvert b(30,\omega)\rvert^2$
3.5	0.0689	0.0718	0.0156	0.0000057
3.6	0.1020	0.1095	0.0250	0.0000294
3.7	0.1700	0.1877	0.0491	0.0001359
3.8	0.3321	0.3811	0.1575	0.001147
3.9	0.8114	0.6559	0.4602	0.011263
4.0	1.5625	0.6946	0.7379	0.027428
4.1	0.7527	0.6295	0.4346	0.010752
4.2	0.2946	0.3427	0.1200	0.001079
4.3	0.1441	0.1614	0.0442	0.000131
4.4	0.0829	0.0893	0.0207	0.0000261
4.5	0.0530	0.0558	0.0122	0.0000073

By comparing the values of $|H(\omega)|^2$ with $|b(0,\omega)|^2$, we may form the following preliminary conclusions. When $t=0$, values of $|b(0,\omega)|^2$ have the same order of magnitude with respect to $|H(\omega)|^2$, the curve of $|b(0,\omega)|^2$ is flatter than $|H(\omega)|^2$, and at the point ω for from ω_0, the values of $|b(0,\omega)|^2$ and $|H(\omega)|^2$ are approximately equal to each other. When t increases, $|b(t,\omega)|^2$ has a quickly descending tendency. Because the behaviour of oscillation exists in the nonstationary case, the values of $|b(t,\omega)|^2$ increase in a short interval, such as $|b(15,4)|^2>|b(0,4)|^2$, but when t still increases, $|b(t,\omega)|^2$ approaches zero.

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