Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 20 (3—4), (1969), pp. 431—442.

M-MIXING SYSTEMS. I

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To Professor G. ALEXITS *on his 70th birthday*

Introduction

The properties of mixing sequences of random variables were investigated by a number of authors. Their aim was to generalize theorems (first of all limit theorems) valid for independent random variables to a class of weakly dependent random variables. In order to obtain such theorems slightly different concepts of mixing were introduced. However the essential idea of these definitions is a condition saying that "the future is independent from the long past". More precisely let $\xi_1, \xi_2, \ldots$ be a sequence of random variables and let $\mathscr{B}_m^n$ $(m \leqq n)$ denote the smallest σ-algebra with respect to which the random variables $\xi_m, \xi_{m+1}, \ldots, \xi_n$ are measurable. Then a mixing condition says that the elements of $\mathscr{B}_1^k$ are nearly independent from the elements of $\mathscr{B}_{k+l}^\infty$ if l is large enough, i.e. we assume

$$|\mathbf{P}(AB)-\mathbf{P}(A)\mathbf{P}(B)| \leqq f(l) \tag{1}$$

where $A \in \mathscr{B}_1^k$, $B \in \mathscr{B}_{k+l}^\infty$ and $f(l)$ is a function converging to 0 with a certain rate.

A quite different way to define a concept of weak dependence is due to G. ALEXITS (see [1] and [2]).

He introduced the following:

DEFINITION. A sequence $\xi_1, \xi_2, \ldots$ of random variables is called an equinormed strongly *multiplicative* system (ESMS) if

$$\begin{gathered}\mathbf{E}(\xi_i) = 0, \quad \mathbf{E}(\xi_i^2) = 1 \qquad (i = 1, 2, \ldots)\\ \mathbf{E}(\xi_{i_1}^{r_1}\xi_{i_2}^{r_2}\ldots\xi_{i_n}^{r_n}) = \mathbf{E}(\xi_{i_1}^{r_1})\mathbf{E}(\xi_{i_2}^{r_2})\ldots\mathbf{E}(\xi_{i_n}^{r_n}) \qquad (i_1<i_2<\ldots<i_n;\ n = 1, 2, \ldots)\end{gathered} \tag{2}$$

where r_j $(j=1, 2, \ldots, n)$ can be equal to 1 or 2. (The existence of the mentioned expectations is assumed.)

Alexits himself and others obtained results showing that in some sense this condition is able to substitute the condition of independence.

In the present paper we try to give a common genaralization of the ESMS and the systems with mixing property.

Namely we introduce the following

DEFINITION. A sequence $\xi_1, \xi_2, \ldots$ of random variables is called *M-mixing* if there is a function $f(l)$ $(l=1, 2, \ldots)$ convering to 0 such that

$$|\mathbf{E}(\xi_{i_1}^{r_1}\xi_{i_2}^{r_2}\ldots\xi_{i_n}^{r_n}\xi_{j_1}^{s_1}\xi_{j_2}^{s_2}\ldots\xi_{j_m}^{s_m})-\mathbf{E}(\xi_{i_1}^{r_1}\xi_{i_2}^{r_2}\ldots\xi_{i_n}^{r_n})\mathbf{E}(\xi_{j_1}^{s_1}\xi_{j_2}^{s_2}\ldots\xi_{j_m}^{s_m})| \leqq f(j_1-i_n)$$

where $i_1<i_2<\ldots<i_n<j_1<j_2<\ldots<j_m$; $n=1, 2, \ldots$; $m=1, 2, \ldots$ and r_l and s_k $(l=1, 2, \ldots, n;\ k=1, 2, \ldots, m)$ can be equal to 1 or 2.

The fact that an ESMS is an M-mixing sequence is obvious.

The following theorem of IBRAGIMOV ([3], Theorem 17. 2. 2) shows the connection between the mixing sequences and the M-mixing systems.

THEOREM OF IBRAGIMOV. *Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables obeying condition* (1). *Further let ξ and η be random variables measurable with respect to $\mathcal{B}_1^k$ and $\mathcal{B}_{k+l}^\infty$ respectively, for which there exist positive numbers δ, c_1, c_2 such that*

$$\mathbf{E}(|\xi|^{2+\delta}) < c_1 \quad \text{and} \quad \mathbf{E}(|\eta|^{2+\delta}) < c_2.$$

Then we have

$$|\mathbf{E}(\xi\eta) - \mathbf{E}(\xi)\mathbf{E}(\eta)| \leqq \left(4 + 3\left(c_1^{\frac{1}{1+\delta}} c_2^{\frac{1+\delta}{2+\delta}} + c_1^{\frac{1+\delta}{2+\delta}} c_2^{\frac{1}{1+\delta}}\right)\right)(f(l))^{\frac{\delta}{2+\delta}}.$$

In § 1 we give some known theorems for mixing sequences, in § 2 the known results of ESMS are repeated. The aim of these two paragraphs is just to give a comparison to the new results.

In § 3 a convergence theorem (and a strong law of large numbers) are formulated for M-mixing sequences. § 4 contains the proofs.

The paper "M-mixing systems. II" will contain a central limit theorem and a law of iterated logarithm for M-mixing sequences.

§ 1. Mixing systems

The investigation of mixing systems started by the paper of ROSENBLATT ([4]). He proved a central limit theorem for such sequences. A detailed treatment can be found in [3]. (See also [5].) In [3] it is assumed that the sequence of random variables is not only mixing but it is a stationary sequence too. This second restriction generally can be dropped (or replaced by a weaker one) without any difficulty.

A typical result of this type is the following

THEOREM MI—1 ([3], Theorem 18. 5. 3). *Let $\xi_1, \xi_2, \ldots$ be a stationary sequence obeying condition* (1). *Assume that there exists a $\delta > 0$ such that*

$$\mathbf{E}(\xi_j^{2+\delta}) < \infty$$

and

$$\sum_{n=1}^{\infty} (f(n))^{\frac{\delta}{2+\delta}} < \infty.$$

Then

$$\sigma^2 = \mathbf{E}(\xi_1^2) + 2\sum_{j=2}^{\infty} \mathbf{E}(\xi_i \xi_j) < \infty$$

and

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\sum_{j=1}^{n} \xi_j}{\sigma\sqrt{n}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$$

provided that $\sigma > 0$.

A strong law of large numbers for mixing systems is given in [6]. (See also [7] Theorem 8. 2. 1.) In this paper the concept of mixing is defined in a different way, namely the condition says that (only) the present is independent from the long past. More precisely it is assumed that

$$|\mathbf{P}(AB)-\mathbf{P}(A)\mathbf{P}(B)| \leqq f(l)$$

where $A \in B_1^k$, $B \in \mathscr{B}_{k+l}^{k+l}$ and $f(l)$ is a function converging to 0 ($\mathscr{B}_a^b$ is defined in the Introduction). A sequence of this type is called $*$-mixing.

For this type of mixing systems the following theorem is proved.

THEOREM MI—2 ([6]). *Let $\xi_1, \xi_2, \ldots$ be a $*$-mixing sequence such that $\mathbf{E}(\xi_n)=0$, $\mathbf{E}(\xi_n^2)<\infty$ $(n=1, 2, \ldots)$. Suppose that $\mathbf{E}(|\xi_n|)\leqq K$ $(n=1, 2, \ldots;\ K>0$ is constant) and*

$$\sum_{n=1}^{\infty} \frac{\mathbf{E}(\xi_n^2)}{n^2} < \infty. \tag{3}$$

Then

$$\mathbf{P}\left(\frac{\xi_1+\xi_2+\ldots+\xi_n}{n} \to 0\right) = 1.$$

§ 2. Multiplicative systems

The fundamental theorem of ESMS was obtained by ALEXITS and TANDORI.

THEOREM MU—1 ([1], [2]). *Let $\xi_1, \xi_2, \ldots$ be a uniformly bounded* ESMS, *further let $c_1, c_2, \ldots$ be a sequence of real numbers for which*

$$\sum_{k=1}^{\infty} c_k^2 < \infty.$$

Then

$$\sum_{k=1}^{\infty} c_k \xi_k$$

is convergent almost everywhere.

A central limit theorem and a law of iterated logarithm for ESMS were obtained by the author [8] (see also [9] and [10]).

THEOREM MU—2 ([8]). *If $\xi_1, \xi_2, \ldots$ is a uniformly bounded* ESMS, *then*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\xi_1+\xi_2+\ldots+\xi_n}{\sqrt{n}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt.$$

THEOREM MU—3 ([8] and [7] Theorem 3. 3. 3). *If $\xi_1, \xi_2, \ldots$ is a uniformly bounded* ESMS, *then*

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\xi_1+\xi_2+\ldots+\xi_n}{\sqrt{n \log \log n}} \leqq 7\right) = 1.$$

A generalization of the concept of ESMS was studied by the author, namely the systems were investigated in which condition (2) holds if $n \leqq 4$. For this type of systems we obtained

THEOREM MU—4 ([7], Theorem 3. 3. 4). *Let $\xi_1, \xi_2, \dots$ be a sequence of random variables for which*

$$\mathbf{E}(\xi_i^6) \leqq K \qquad (i = 1, 2, \dots)$$

$$\mathbf{E}(\xi_i^2 \xi_j \xi_k) = \mathbf{E}(\xi_i^2 \xi_j) = \mathbf{E}(\xi_i \xi_j \xi_k \xi_l) = \mathbf{E}(\xi_i \xi_j \xi_k) = \mathbf{E}(\xi_i \xi_j) = \mathbf{E}(\xi_i) = 0$$

where the indices i, j, k, l are different and K is a positive constant. Further let $c_1, c_2, \dots$ be a sequence of real numbers and suppose that there exists an integer r (depending on $\{c_k\}$) such that

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty$$

where [1]

$$l(x) = l_1(x) = \begin{cases} \log x & \text{if} \quad x \geqq 2 \\ 1 & \text{if} \quad 0 < x < 2 \end{cases}$$

and $l_r(x)$ is the r-th iterated of $l(x)$ i.e. $l_r(x) = l(l_{r-1}(x))$. Then the series

$$\sum_{k=1}^{\infty} c_k \xi_k$$

is convergent almost everywhere.

§ 3. A convergence theorem

Now we formulate our main

THEOREM MM—1.[2] *Let $\xi_1, \xi_2, \dots$ be a sequence of random variables obeying the following conditions*

(i) $$\mathbf{E}(\xi_i) = 0, \quad \mathbf{E}(\xi_i^4) \leqq K \qquad (i = 1, 2, \dots)$$

(ii) $$|\mathbf{E}(\xi_i \xi_j)| \leqq f(j-i) \qquad (i < j)$$

(iii) $$|\mathbf{E}(\xi_i \xi_j \xi_k \xi_l)| \leqq \min\big(f(l-k), f(j-i)\big) \qquad (i < j < k < l)$$

(iv) $$|\mathbf{E}(\xi_i^2 \xi_j \xi_k)| \leqq f(k-j) \qquad (i < j < k)$$

(v) $$|\mathbf{E}(\xi_i \xi_j^2 \xi_k)| \leqq \min\big(f(k-j), f(j-i)\big) \qquad (i < j < k)$$

(vi) $$|\mathbf{E}(\xi_i \xi_j \xi_k^2)| \leqq f(j-i) \qquad (i < j < k)$$

(vii) $$\mathbf{E}(\xi_i^2 \xi_j^2) \leqq 1 + f(j-i) \qquad (i < j)$$

where K is a positive constant greater than 1, and $f(k)$ is a decreasing function defined on the integers for which there exists a positive constant d such that

(4) $$f(k) \leqq e^{-dk}.$$

[1] Here and in what follows $\log x$ means the logarithm with base 2.
[2] This Theorem clearly contains Theorem MU—4.

Further let $c_1, c_2, \ldots$ *be a sequence of real numbers and suppose that there exists an integer r (depending on* $\{c_k\}$*) such that*

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty \tag{5}$$

where

$$l(x) = l_1(x) = \begin{cases} \log x & \text{if} \quad x \geqq 2 \\ 1 & \text{if} \quad 0 < x < 2 \end{cases}$$

and $l_r(x)$ *is the r-th iterated of* $l(x)$ *i.e.* $l_r(x) = l(l_{r-1}(x))$. *Then the series*

$$\sum_{k=1}^{\infty} c_k \xi_k$$

is convergent almost everywhere.

Making use of the Kronecker lemma (see e.g. [7] Theorem 1. 2. 2) Theorem MM—1 implies the following strong law of large numbers.

THEOREM MM—2. *Let* $\xi_1, \xi_2, \ldots$ *be a sequence of random variables obeying the conditions of Theorem* MM—1, *further let* $c_1, c_2, \ldots$ *be a sequence of real numbers and suppose that there exists an integer r such that*

$$\sum_{k=1}^{\infty} \frac{c_k^2}{k^2} l_r^2(k) < \infty \tag{6}$$

(where $l_r(k)$ *is defined in Theorem* MM—1*).*

Then

$$\mathbf{P}\left(\frac{c_1\xi_1 + c_2\xi_2 + \ldots + c_n\xi_n}{n} \to 0\right) = 1.$$

It can be seen that in some sense this theorem is stronger than Theorem MI—2 but in some sense this is the weaker one. Namely in this theorem there is a more strict restriction about the meaning of the „long past" (condition (4)) but there is no restriction about the whole long past only about two or three members of it and now we do not take into consideration all events of the long past, just the moments (see Theorem of Ibragimov). Since in condition (6) the integer r can be as large as we wish, this condition is not much stronger than condition (3).

The proof of Theorem MM—1 is based on an inequality analogous to the Rademacher—Mensov inequality (see e.g. [7] Theorem 3. 1. 1).

THEOREM MM—3. *Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be a sequence of random variables, obeying the conditions* (i)—(vii) *of Theorem* MM—1, *where (now)* $f(k)$ *is a decreasing function for which*

$$\sum_{k=1}^{n} kf(k) \leqq \frac{1}{8}.$$

Then

$$\mathbf{E}\left(\max_{1 \leqq k \leqq n}\left(\sum_{j=1}^{k} c_j\xi_j\right)^4\right) \leqq 24K(\log 4n)^4 \left(\sum_{j=1}^{n} c_j^2\right)^2$$

where $\{c_j\}_{j=1}^n$ *is an arbitrary sequence of real numbers.*

§ 4. Proofs

First of all we give four lemas.

LEMMA 1. *Under the conditions of Theorem* MM—3 *we have*

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{E}^2(\xi_i^3\xi_j) \leqq \frac{8}{3}K \quad (i=1,2,\dots,n).$$

PROOF. Let $\mathbf{E}(\xi_i^3\xi_j)=d_{ij}$ and consider the inequality

$$0\leqq \mathbf{E}\left[\left(\xi_i^2-\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}\xi_i\xi_j\right)^2\right] = \mathbf{E}(\xi_i^4)+\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}^2\mathbf{E}[(\xi_i^2\xi_j^2)]-2\sum_{j=1}^{n} d_{ij}^2+$$

$$+2\sum_{\substack{1\leqq k<l\leqq n\\ k\neq i;\, l\neq i}} d_{il}d_{ik}\mathbf{E}(\xi_i^2\xi_k\xi_l) \leqq K+\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}^2\big(1+f(j-i)\big)-2\sum_{j=1}^{n} d_{ij}^2+$$

$$+\sum_{\substack{1\leqq k<l\leqq n\\ k\neq i;\, l\neq i}} (d_{il}^2+d_{ik}^2)\mathbf{E}(\xi_i^2\xi_k\xi_l) \leqq K+\frac{9}{8}\sum_{j=1}^{n} d_{ij}^2-2\sum_{j=1}^{n} d_{ij}^2+$$

$$+\sum_{\substack{1\leqq k<l\leqq n\\ k\neq i;\, l\neq i}} d_{il}^2\max\big(f(k-l),f(k-i)\big)+\sum_{\substack{1\leqq k<l\leqq n\\ k\neq i;\, l\neq i}} d_{ik}^2\max\big(f(k-l),f(k-i)\big)\leqq$$

$$\leqq K+\frac{9}{8}\sum_{j=1}^{n} d_{ij}^2-2\sum_{j=1}^{n} d_{ij}^2+\frac{4}{8}\sum_{j=1}^{n} d_{ij}^2 = K-\frac{3}{8}\sum_{j=1}^{n} d_{ij}^2$$

which implies our lemma.

LEMMA 2. *Under the conditions of Theorem* MM—3 *we have*

$$\mathbf{E}\left[\left(\sum_{j=1}^{n} c_j\xi_j\right)^4\right] \leqq 24K\left(\sum_{j=1}^{n} c_j^2\right)^2.$$

PROOF. We have

$$\mathbf{E}\left[\left(\sum_{j=1}^{n} c_j\xi_j\right)^4\right] = \sum_{j=1}^{n} c_j^4\mathbf{E}(\xi_j^4)+6\sum_{1\leqq i<j\leqq n} c_i^2c_j^2\mathbf{E}(\xi_i^2\xi_j^2)+4\sum_{1\leqq i<j\leqq n} c_i^3c_j\mathbf{E}(\xi_i^3\xi_j)+$$

$$+4\sum_{1\leqq i<j\leqq n} c_ic_j^3\mathbf{E}(\xi_i\xi_j^3)+12\sum_{1\leqq i<j<k\leqq n} c_i^2c_jc_k\mathbf{E}(\xi_i^2\xi_j\xi_k)+$$

$$+12\sum_{1\leqq i<j<k\leqq n} c_ic_j^2c_k\mathbf{E}(\xi_i\xi_j^2\xi_k)+12\sum_{1\leqq i<j<k\leqq n} c_ic_jc_k^2\mathbf{E}(\xi_i\xi_j\xi_k^2)+$$

$$+24\sum_{1\leqq i<j<k<l\leqq n} c_ic_jc_kc_l\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)\leqq$$

$$\leqq K\sum_{j=1}^{n} c_j^4+6K\sum_{1\leqq i<j\leqq n} c_i^2c_j^2+8\sum_{i=1}^{n} c_i^3\sqrt{\sum_{j=1}^{n} c_j^2\sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{E}^2(\xi_i^3\xi_j)}+$$

$$+6\sum_{1\le i<j<k\le n} c_i^2(c_j^2+c_k^2)f(k-j)+6\sum_{1\le i<j<k\le n} c_j^2(c_i^2+c_k^2)|\mathbf{E}(\xi_i\xi_j^2\xi_k)|+$$

$$+6\sum_{1\le i<j<k\le n}(c_i^2+c_j^2)c_k^2 f(j-i)+6\sum_{1\le i<j<k<l\le n}(c_i^2+c_j^2)(c_k^2+c_l^2)|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|\leqq$$

$$\leqq 3K\left(\sum_{j=1}^n c_j^2\right)^2+14K\left(\sum_{i=1}^n c_i^2\right)^2+\sum_{1\le i<j\le n} c_i^2c_j^2+\sum_{1\le i<k\le n} c_i^2 c_k^2+$$

$$+6\sum_{1\le i<j<k\le n} c_i^2c_j^2 f(k-j)+6\sum_{1\le i<j<k\le n} c_j^2c_k^2 f(j-i)+\sum_{1\le i<k\le n} c_i^2c_k^2+$$

$$+\sum_{1\le j<k\le n} c_j^2c_k^2+6\sum_{1\le i<j<k<l\le n}(c_i^2+c_j^2)(c_k^2+c_l^2)|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|\leqq$$

$$\leqq 20K\left(\sum_{j=1}^n c_j^2\right)^2+6\sum_{1\le i<j<k<l\le n} c_i^2c_k^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|+6\sum_{1\le i<j<k<l\le n} c_i^2c_l^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|+$$

$$+6\sum_{1\le i<j<k<l\le n} c_j^2c_k^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|+6\sum_{1\le i<j<k<l\le n} c_j^2c_l^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|.$$

The last four members of the last sum can be estimated in the same way. As an example the estimation of

$$I_1=\sum_{1\le i<j<k<l\le n} c_i^2c_k^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|$$

will be given. We take into consideration two possible cases.

Case 1. $j-i\leqq l-k$,

Case 2. $j-i>l-k$.

The members of I_1 for which the restriction of Case 1 holds can be estimated by

$$\sum_{1\le i<k\le n} c_i^2c_k^2\sum_{l=k+1}^{\infty}\sum_{j=i+1}^{i+l-k} f(l-k)=\sum_{1\le i<k\le n} c_i^2c_k^2\sum_{l=k+1}^{\infty}(l-k)f(l-k)\leqq$$

$$\leqq\frac{1}{8}\sum_{1\le i<k\le n} c_i^2c_k^2.$$

The sum of the members of I_1 obeying the restriction of Case 2 can be estimated in the same way, so we have

$$I_1\leqq\frac{1}{4}\sum_{1\le i<k\le n} c_i^2c_k^2.$$

Hence we obtained

$$\mathbf{E}\left[\left(\sum_{j=1}^n c_j\xi_j\right)^4\right]\leqq 24K\left(\sum_{j=1}^n c_j^2\right)^2$$

which is the statement of our Lemma.

LEMMA 3. *If* $\xi_1, \xi_2, \ldots$ *is a sequence of random variables obeying the conditions* (i)—(vii) *of Theorem* MM—1 *with a function* $f_1(k)$ *and if* $m_1 < m_2 < \ldots$ *is a sequence of integers then the sequence*

$$\psi_k = \begin{cases} \dfrac{1}{\alpha_k} \sum\limits_{j=m_k+1}^{m_{k+1}} c_j \xi_j & \text{if} \quad \alpha_k > 0 \\ 0 & \text{if} \quad \alpha_k = 0 \end{cases}$$

where $\alpha_k = \left[\sum\limits_{j=m_k+1}^{m_{k+1}} c_j^2 \right]^{1/2}$ *is obeying the condition* (i)—(vii) *with a function* $f_2(k)$ *for which*

$$f_2(k) \leqq 8 \sum_{l=k}^{\infty} l f_1(l).$$

PROOF. Condition (i) is proved by Lemma 1. The others can be proved in the same way. So as an example we prove that

$$|\mathbf{E}(\psi_i \psi_j \psi_k \psi_l)| \leqq 8 \sum_{j=l-k}^{\infty} j f_1(j)$$

provided that $i < j < k < l$.

We have

$$|\mathbf{E}(\psi_i \psi_j \psi_k \psi_l)| = \left| \mathbf{E} \left[\sum_{\varkappa=m_i+1}^{m_{i+1}} \frac{c_\varkappa}{\alpha_i} \xi_\varkappa \sum_{\lambda=m_j+1}^{m_{j+1}} \frac{c_\lambda}{\alpha_j} \xi_\lambda \sum_{\mu=m_k+1}^{m_{k+1}} \frac{c_\mu}{\alpha_k} \xi_\mu \sum_{\nu=m_l+1}^{m_{l+1}} \frac{c_\nu}{\alpha_l} \xi_\nu \right] \right| \leqq$$

$$\leqq \sum_\varkappa \sum_\lambda \sum_\mu \sum_\nu \left(\frac{c_\varkappa^2}{\alpha_i^2} + \frac{c_\lambda^2}{\alpha_j^2} \right) \left(\frac{c_\mu^2}{\alpha_k^2} + \frac{c_\nu^2}{\alpha_l^2} \right) |\mathbf{E}(\xi_\varkappa \xi_\lambda \xi_\mu \xi_\nu)| \leqq$$

$$\leqq \sum_\varkappa \sum_\mu \frac{c_\varkappa^2}{\alpha_i^2} \frac{c_\mu^2}{\alpha_k^2} \Big[\sum_\nu f_1(\nu-\mu)(\nu-\mu) + \sum_{\{\lambda: \lambda-\varkappa>\nu-\mu\}} f_1(\lambda-\varkappa)(\lambda-\varkappa) \Big] +$$

$$+ \sum_\varkappa \sum_\nu \frac{c_\varkappa^2}{\alpha_i^2} \frac{c_\nu^2}{\alpha_l^2} \Big[\sum_\mu f_1(\nu-\mu)(\nu-\mu) + \sum_{\{\varkappa: \lambda-\varkappa>\nu-\mu\}} f_1(\lambda-\varkappa)(\lambda-\varkappa) \Big] +$$

$$+ \sum_\lambda \sum_\mu \frac{c_\lambda^2}{\alpha_j^2} \frac{c_\mu^2}{\alpha_k^2} \Big[\sum_\nu f_1(\nu-\mu)(\nu-\mu) + \sum_{\{\lambda: \lambda-\varkappa>\nu-\mu\}} f_1(\lambda-\varkappa)(\lambda-\varkappa) \Big] +$$

$$+ \sum_\lambda \sum_\nu \frac{c_\lambda^2}{\alpha_j^2} \frac{c_\nu^2}{\alpha_l^2} \Big[\sum_\mu f_1(\nu-\mu)(\nu-\mu) + \sum_{\{\varkappa: \lambda-\varkappa>\nu-\mu\}} f_1(\lambda-\varkappa)(\lambda-\varkappa) \Big] \leqq$$

$$\leqq 8 \sum_{j=m_l+1-m_{k+1}}^{\infty} j f_1(j) \leqq 8 \sum_{j=l-k}^{\infty} j f_1(j).$$

LEMMA 4. *If* $c_1, c_2, \ldots$ *is a sequence of real numbers for which*

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty$$

then there exists a sequence $n_1 < n_2 < \dots$ *of integers for which*

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right) l_{r-1}^2(k) < \infty$$

and

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right)^2 l^4(n_{k+1} - n_k) < \infty.$$

See the proof in [7] Lemma 3. 3. 3.

PROOF OF THEOREM MM—3. First of all we assume that $n = 2^\nu$ $(\nu = 1, 2, \dots)$ and introduce the following notations

$$\sigma_j = c_1 \xi_1 + c_2 \xi_2 + \dots + c_j \xi_j,$$

$$\psi_{\alpha\beta} = c_{\alpha+1} \xi_{\alpha+1} + c_{\alpha+2} \xi_{\alpha+2} + \dots + c_\beta \xi_\beta$$

where

$$\alpha = \mu 2^k; \quad \beta = \beta(\alpha) = (\mu+1) 2^k; \qquad \mu = 0, 1, 2, \dots, 2^{\nu-k} - 1; \; k = 0, 1, 2, \dots, \nu.$$

Consider the random variable σ_j as the sum of some $\psi_{\alpha\beta}$. Let

$$\sigma_j = \sum_i \psi_{\alpha_i \beta_i}$$

where $\beta_1 - \alpha_1 > \beta_2 - \alpha_2 > \dots$. Clearly the number of the members of the sum $\sum_i \psi_{\alpha_i \beta_i}$ is less than ν. Therefore by the Cauchy inequality we have

$$\sigma_j^4 = \left(\sum_i \psi_{\alpha_i \beta_i}\right)^4 \leqq \nu^2 \left(\sum_i \psi_{\alpha_i \beta_i}^2\right)^2 \leqq \nu^3 \sum_i \psi_{\alpha_i \beta_i}^4$$

which implies

$$\int_\Omega \max_{1 \leqq j \leqq 2^\nu} \sigma_j^4 \, d\mathbf{P} \leqq \nu^3 \sum_{\alpha, \beta} \int_\Omega \psi_{\alpha\beta}^4 \, d\mathbf{P} \leqq \nu^3 \sum_{k=0}^{\nu} \sum_{\mu=0}^{2^{\nu-k}-1} \int_\Omega \left(\sum_{i=\mu 2^k + 1}^{(\mu+1)2^k} c_i \xi_i \right)^4 d\mathbf{P}$$

where α and $\beta = \beta(\alpha)$ run through all their possible values.

Making use of Lemma 1 we have

$$\mathbf{E}\left(\max_{1 \leqq j \leqq 2^\nu} \sigma_j^4 \right) \leqq (\nu+1)^4 24K \left(\sum_{j=1}^{2^\nu} c_j^2 \right)^2.$$

This inequality proves our statement in the case $n = 2^\nu$. If $2^\nu \leqq n < 2^{\nu+1}$ then

$$\mathbf{E}\left(\max_{1 \leqq j \leqq n} \sigma_j^4 \right) \leqq 24K(\nu+2)^4 \left(\sum_{j=1}^{n} c_j^2 \right)^2 \leqq 24K(\log 4n)^4 \left(\sum_{j=1}^{n} c_j^2 \right)^2$$

which completes our proof.

PROOF OF THEOREM MM—1. First of all choose an integer $S \geqq 2$ such that

$$\sum_{k=s}^{\infty} k e^{-dk} \leqq \frac{1}{2}.$$

Now we prove that

$$\sum_{k=1}^{\infty} c_{ks+t}\xi_{ks+t} \quad (t = 0, 1, 2, \dots, s-1)$$

is convergent almost everywhere, which implies our Theorem.

Put

$$c_{ks+t} = \gamma_k, \quad \xi_{ks+t} = \eta_k.$$

As a first step we prove the almost everywhere convergence of the series $\sum_{k=1}^{\infty} \gamma_k \eta_k$ under the condition

$$\sum_{k=1}^{\infty} c_k^2 l^2(k) < \infty.$$

Set

$$\vartheta_n = \sum_{k=n}^{\infty} \gamma_k \eta_k.$$

Then

$$\mathbf{E}(\vartheta_n^2) \leqq \frac{3}{2} \sum_{k=n}^{\infty} \gamma_k^2 \leqq \frac{3}{2} \frac{A}{l^2(n)} \quad \left(\text{where } A = \sum_{k=1}^{\infty} \gamma_k^2 l^2(k)\right)$$

and

$$\sum_{n=1}^{\infty} \mathbf{E}(\vartheta_{2^n}^2) \leqq \frac{3A}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By Beppo Levi theorem this fact implies

$$\vartheta_{2^n} \to 0.$$

By Theorem MM—3 we have

$$\mathbf{E}\left(\max_{2^n \leqq k < 2^{n+1}} \left(\sum_{j=2^n}^{k} \gamma_j \eta_j \right)^4 \right) \leqq 24Kl^4(2^{n+2}) \left(\sum_{j=2^n+1}^{2^{n+1}} \gamma_j^2 \right)^2 = 24K(n+2)^4 \left(\sum_{j=2^n+1}^{2^{n+1}} \gamma_j^2 \right)^2$$

which is less than

$$24K \left(\sum_{j=2^n+1}^{2^{n+1}} \gamma_j^2 \log^2 j \right)^2$$

if n is large enough. Hence

$$\sum_{n=1}^{\infty} \mathbf{E}\left(\max_{2^n \leqq k < 2^{n+1}} \left| \sum_{j=2^n}^{k} \gamma_j \eta_j \right|^4 \right) < \infty$$

and

$$\max_{2^n \leqq k < 2^{n+1}} \left| \sum_{j=2^n}^{k} \gamma_j \eta_j \right|^4 \to 0$$

which proves the convergence of the series

$$\sum_{k=1}^{\infty} \gamma_k \eta_k$$

in the case when

$$\sum_{k=1}^{\infty} \gamma_k^2 \log^2 k < \infty.$$

Now we prove our Theorem by induction. Suppose (as the condition of our induction) that if $\{a_k\}$ is a sequence of real numbers for which

$$\sum_{k=1}^{\infty} a_k^2 l_{r-1}^2(k) < \infty$$

and ψ_k is a sequence of random variables obeying the conditions of Theorem MM—1. Then

$$\sum_{k=1}^{\infty} a_k \psi_k$$

is convergent.

Now let $\{b_k\}$ be a sequence of real numbers for which

$$\sum_{k=1}^{\infty} b_k^2 l_r^2(k) < \infty$$

and denote by $\{n_k\}$ a sequence of integers for which

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} b_j^2 \right) l_{r-1}^2(k) < \infty$$

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} b_j^2 \right)^2 l^4(n_{k+1} - n_k) < \infty.$$

By Lemma 2 the sequence

$$\psi_k = \begin{cases} \dfrac{1}{\alpha_k} \displaystyle\sum_{j=n_k+1}^{n_{k+1}} b_j \eta_j & \text{if} \quad \alpha_k > 0 \\ 0 & \text{if} \quad \alpha_k = 0 \end{cases}$$

$\left(\text{where } \alpha_k = \left[\sum_{j=n_k+1}^{n_{k+1}} b_j^2 \right]^{1/2} \right)$ is obeying the conditions of Theorem MM—1. This fact implies — by the condition of our induction — that

$$\sum_{k=1}^{\infty} \alpha_k \psi_k$$

is convergent almost everywhere.

In order to prove our theorem it is enough to show that

$$\sum_{k=1}^{\infty} \mathbf{E} \left[\max_{n_k+1 \leq t < n_{k+1}} \left(\sum_{j=n_k+1}^{t} b_j \eta_j \right)^4 \right] < \infty.$$

However this fact follows immediately from Theorem MM—3.

(Received 4 March 1969)

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References

[1] G. ALEXITS, *Convergence problems of orthogonal series* (Pergamon Press and Akadémiai Kiadó, 1960).

[2] G. ALEXITS—K. TANDORI, Über das Konvergenzhalten einer Klasse von Orthogonalreihen, *Annales Univ. Sci. Budapest, Sectio Math.*, **3** (1961), pp. 15—19.

[3] I. A. IBRAGIMOV—JU. V. LINNIK, *Independent and stationarily connected variables* (Russian) ("Nauka", Moscow, 1965).

[4] M. ROSENBLATT, A central limit theorem and a strong mixing condition, *Proc. Nat. Acad. Sci.*, **42** (1956), pp. 43—47.

[5] R. I. SERFLING, Contributions to central limit theory for dependent random variables, *Annals of Math. Stat.*, **39** (1968), pp. 1158—1175.

[6] J. R. BLUM—D. L. HANSON—L. KOOPMAN, On the strong law of large numbers for a class of stochastic processes, *Zeitschr. f. Wahrscheinlichkeitstheorie*, **2** (1963), pp. 1—11.

[7] P. RÉVÉSZ, *The laws of large numbers* (Akadémiai Kiadó, 1967, Academic Press, 1968).

[8] P. RÉVÉSZ, Some remarks on strongly multiplicative systems, *Acta Math. Acad. Sci. Hung.*, **16** (1956), pp. 441—446.

[9] F. MÓRICZ, Inequalities and theorems concerning strongly multiplacitive systems, *Acta Sci. Math. Szeged*, **29** (1968), pp. 115—136.

[10] M. CSÖRGŐ, On a law of iterated logarithm for strongly multiplicative system. In print.