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ON THE SUMMABILITY OF THE FOURIER SERIES OF L^2 INTEGRABLE FUNCTIONS. IV

By

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To G. ALEXITS *on his seventieth birthday*

1. Let π_n denote the space of the complex trigonometric polynomials

$$f(x) = \sum_{-n}^{n} c_\nu e^{i\nu x} = \frac{a_0}{2} + \sum_{1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

of order n and let C_n be defined by

$$C_n = \max_{f \in \pi_n} \left| \int_0^{2\pi} \max_{0 \leqq k \leqq n} s_k(x; f)\, dx \right| \Big/ \left\{ \int_0^{2\pi} |f(x)|^2\, dx \right\}^{1/2} \tag{1. 1}$$

where $s_k(x; f)$ is the partial sum of $f(x)$ of order k.

The theorem of Carleson, namely that every $L^2(0, 2\pi)$ integrable function can be expanded into an almost everywhere convergent Fourier series is equivalent to the statement that the sequence $\{C_n\}_{n=0}^{\infty}$ is bounded [2].

The more difficult question of the behaviour of the quantities

$$A_n = \max_{f \in \pi_n} \int_0^{2\pi} \max_{0 \leqq k \leqq n} |s_k(x; f)|^2\, dx \Big/ \int_0^{2\pi} |f(x)|^2\, dx \qquad (n = 1, 2, \dots) \tag{1. 2}$$

was settled recently by R. A. HUNT [1]: he proved that the infinite sequence $A_1, A_2, \dots$ is bounded.

Another question is the following. Let the complex function $\varphi(x) \in L^2(-\pi, \pi)$ be of unit norm: $\|\varphi\| = \int_{-\pi}^{\pi} |\varphi(x)|^2 dx = 1$ and let $\{\varkappa_r\}_{r=-\infty}^{\infty}$ be a sequence of non-negative numbers not exceeding π. Let us consider the quantity

$$E(\varphi; \{\varkappa_r\}) = \sum_{r=-\infty}^{\infty} \left| \int_{-\varkappa_r}^{\varkappa_r} \varphi(x) e^{-irx}\, dx \right|^2. \tag{1. 3}$$

Do the $E(\varphi; \{\varkappa_r\})$'s have a finite bound if φ ranges over all $L^2(-\pi, \pi)$ integrable functions of unit norm and $\{\varkappa_r\}$ is a fixed sequence? (Of course, if $\varkappa_r = \pi$ for every r, then by Parseval's formula $E(\varphi; \{\varkappa_r\}) = 2\pi$). This would certainly be true if the quantity

$$E(\varphi) = \sum_{r=-\infty}^{\infty} \max_{0 \leqq \varkappa_r \leqq \pi} \left| \int_{-\varkappa_r}^{\varkappa_r} \varphi(x) e^{-irx}\, dx \right|^2 \tag{1. 4}$$

had a finite bound the $\varphi(x)$'s ranging over all $L^2(-\pi, \pi)$ integrable functions of unit norm.

The purpose of this paper is to show the equivalence of Hunt's result and of the problem mentioned in the last paragraph: *from $A_n < A$ it follows that*

$$\sup_{\|\varphi\|=1} E(\varphi) < \infty; \tag{1. 5}$$

conversely (1. 5) *implies Hunt's result.*

2. Let $x_1, x_2, \ldots, x_m$ be a strictly increasing sequence of real numbers and $k_1, k_2, \ldots, k_m$ non-negative integers not exceeding n. Let further $\mathbf{x}$ and $\mathbf{k}$ denote the m-vectors $\{x_1, \ldots, x_m\}$ and $\{k_1, \ldots, k_m\}$, respectively. Finally we shall use the notation

$$|||f||| = \left|\left|\left|\frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right|\right|\right| = \left\{\frac{|a_0|^2}{2} + \sum_{\nu=1}^{n} \left(|a_\nu|^2 + |b_\nu|^2\right)\right\}^{1/2}.$$

In part I of this paper [3] I have introduced the quantities

$$\Lambda(\mathbf{x}, \mathbf{k}) = \max_{f \in \pi_n} \sum_{r=1}^{m} |s_{k_r}(x_r; f)|^2 \Big/ |||f|||^2 \tag{2. 1}$$

and have shown that they are equal to the greatest eigenvalue of the matrix

$$[D_{\min(k_p, k_q)}(x_p - x_q)]_{p,q=1}^{m} \tag{2. 2}$$

where $D_l(x) = 1/2 + \cos x + \ldots + \cos lx$ is Dirichlet's kernel.

We shall call a function $f^* = a_0^*/2 + \Sigma(a_\nu^* \cos \nu x + b_\nu^* \sin \nu x)$ an extremal function of the maximum problem (2. 1) if one has

$$\Lambda(\mathbf{x}, \mathbf{k}) = \sum_{r=1}^{m} |s_{k_r}(x_r; f^*)|^2 \Big/ |||f^*|||^2. \tag{2. 3}$$

The existence of these extremal functions is obvious and we state

LEMMA 1. *There exists an extremal function of* (2. 1) *with real Fourier coefficients* a_ν^*, b_ν^*.

Indeed by introducing the notations $a_0/\sqrt{2} = \zeta_0$, $a_\nu = \zeta_\nu$, $b_\nu = \zeta_{-\nu}$ $(\nu = 1, 2, \ldots, n)$ we see that the numerator of the right-hand side of (2. 1) is an Hermitian form of the quantities $\zeta_{-n}, \zeta_{-n+1}, \ldots, \zeta_n$ with real coefficients. $\Lambda(\mathbf{x}, \mathbf{k})$ is the greatest eigenvalue of the corresponding symmetric matrix, and the quantities

$$\zeta_\nu = \zeta_\nu^* = \begin{cases} a_\nu^* & (\nu = 1, 2, \ldots, n), \\ a_0^*/\sqrt{2}, & \\ b_{-\nu}^* & (\nu = -1, -2, \ldots, -n) \end{cases}$$

are the components of the real eigenvector corresponding to the eigenvalue $\Lambda(\mathbf{x}, \mathbf{k})$

We quote another result of part I:

(2. 4) $$A_n(\mathbf{x}) = \max_{f\in\pi_n}\frac{1}{m}\sum_{r=1}^{m}\max_{l=0,1,\dots,n}|s_l(x_r;f)|^2\Big/|||f|||^2 = \frac{1}{m}\max_{\substack{k_r=0,1,\dots,n\\ r=1,2,\dots,m}} A(\mathbf{x};\mathbf{k}).$$

In the special case when $x_r = x_r^* = 2\pi r/m$ $(r=1, 2, \dots, m)$ the quantity $A_n(\mathbf{x})$ was denoted by $A_n^{(m)}$ and it was shown in part I that

(2. 5) $$2A_n^{(m)} \geqq A_n.$$

A counterpart of this inequality is

LEMMA 2.

(2. 6) $$2A_n^{(m)} \leqq \left(1+4\pi\frac{n}{m}\right)A_n.$$

Indeed let $k_1^*, k_2^*, \dots, k_m^*$ be an extremal sequence corresponding to the maximum problem (2. 3), i.e.

$$A_n^{(m)} = \frac{1}{m}A_n(k_1^*, k_2^*, \dots, k_m^*)$$

and a_ν^*, b_ν^* should denote, as before, the Fourier coefficients of the real extremal function f^* in (2. 4), only that now $k_1 = k_1^*, \dots, k_m = k_m^*$. So, if

(2. 7) $$|||f^*||| = \frac{1}{\pi}, \quad \text{i.e.} \quad \int_{-\pi}^{\pi}\{f^*(x)\}^2\,dx = 1,$$

we have

(2. 8) $$A_n^{(m)} = \frac{\pi}{m}\sum_{r=1}^{m} s_{k_r^*}^2\left(\frac{2\pi}{m}r, f^*\right).$$

On the other hand, by (1. 2), if $x_r^* = 2\pi r/m$,

$$A_n \geqq \int_{-\pi}^{\pi}\max_{0\leqq k\leqq n}|s_k(x;f^*)|^2\,dx \geqq \sum_{r=1}^{m}\int_{x_{r-1}^*}^{x_r^*} s_{k_r^*}^2(x;f^*)\,dx$$

and if

$$\min_{x_{r-1}^*\leqq x\leqq x_r^*} s_{k_r^*}^2(x;f^*) = s_{k_r}^2(x_r^*-\eta_r;f^*)$$

then *a fortiori*

(2. 9) $$A_n \geqq \sum_{r=1}^{m}\int_{x_{r-1}^*}^{x_r^*} s_{k_r^*}^2(x_r^*-\eta_r;f^*)\,dx = \frac{2\pi}{m}\sum_r s_{k_r^*}^2(x_r^*-\eta_r;f^*).$$

Let us now introduce the continuous function

$$g(\xi) = \frac{2\pi}{m}\sum_r s_{k_r^*}^2(x_r^*-\xi\eta_r, f^*).$$

By (2. 5) and (2. 7) $g(0) \geqq A_n$ and by (2. 9) $g(1) \leqq A_n$. Hence, there exists a ϑ, $0 \leqq \vartheta \leqq 1$, such that $g(\vartheta) = A_n$ and by (2. 7)

$$2A_n^{(m)} - A_n = \frac{2\pi}{m} \sum_r \{s_{k_r^*}^2(x_r^*; f^*) - s_{k_r^*}^2(x_r^* - \vartheta\eta_r; f^*\} =$$

$$= \frac{2\pi}{m} \sum_r \int\limits_{x_r^* - \vartheta\eta_r}^{x_r^*} \frac{d}{dx} s_{k_r^*}^2(x; f^*)\, dx = \frac{4\pi}{m} \sum_r \int\limits_{x_r^* - \vartheta\eta_r}^{x_r^*} s_{k_r^*}(x; f^*) \frac{d}{dx} s_{k_r^*}(x; f^*)\, dx.$$

Taking into regard that $(d/dx) s_{k_r^*}(x_r; f^*) = s_{k_r^*}(x_r, f^{*\prime})$ where

$$f^{*\prime} = \sum \nu (b_\nu^* \cos \nu x - a_\nu^* \sin \nu x) \tag{2. 10}$$

and using in turn Schwarz's inequality and $x_{r-1}^* \leqq x_r^* - \vartheta\eta_r$ we have

$$2A_n^{(m)} - A_n < \frac{4\pi}{m} \sum_r \left\{ \int\limits_{x_{r-1}^*}^{x_r^*} s_{k_r^*}^2(x; f^*)\, dx \right\}^{1/2} \left\{ \int\limits_{x_{r-1}^*}^{x_r^*} s_{k_r^*}^2(x; f^{*\prime})\, dx \right\}^{1/2}.$$

Again, by Cauchy's inequality

$$2A_n^{(m)} - A_n \leqq \frac{4\pi}{m} \left\{ \sum_r \int\limits_{x_{r-1}^*}^{x_r^*} s_{k_r^*}^2(x; f^*)\, dx \sum_r \int\limits_{x_{r-1}^*}^{x_r^*} s_{k_r^*}^2(x; f^{*\prime})\, dx \right\}^{1/2} \leqq$$

$$\leqq \frac{4\pi}{m} \left\{ \int\limits_0^{2\pi} \max_{0 \leqq k \leqq n} s_k^2(x; f^*)\, dx \cdot \int\limits_0^{2\pi} \max_{0 \leqq k \leqq n} s_k^2(x; f^{*\prime})\, dx \right\}^{1/2} \leqq$$

$$\leqq \frac{4\pi}{m} \left\{ A_n \int\limits_0^{2\pi} [f^*(x)]^2\, dx \cdot A_n \int\limits_0^{2\pi} [f^{*\prime}(x)]^2\, dx \right\}^{1/2}$$

by the definition (1. 2) of A_n. Finally by (2. 8) and (2. 10)

$$2A_n^{(m)} - A_n \leqq \frac{4\pi}{m} A_n \cdot n \tag{2. 11}$$

since by (2. 7)

$$\int\limits_0^{2\pi} [f^{*\prime}(x)]^2\, dx = \frac{1}{\pi} \sum_1^n \nu^2 (a_\nu^{*2} + b_\nu^{*2}) \leqq n^2.$$

We remark that from (2. 5) and (2. 6) we have

$$1 \leqq \frac{2A_n^{(2n)}}{A_n} \leqq 1 + 2\pi \tag{2. 12}$$

i.e. A_n and $A_n^{(2n)}$ have the same order of magnitude.

Let now in (2. 4) x_r be equal to $\pi r/m$. We shall denote the corresponding quantity $A_n(\mathbf{x})$ by $a_n^{(m)}$ and prove

LEMMA 3. $a_n^{(m)} \leqq 2A_n^{(2m)} \leqq 2a_n^{(m)}$.

It is trivial that $a_n^{(m)}$ may be defined in the more general way (cf. § 2 of Part I)

$$a_n^{(m)} = \max_{|||f|||=1} \frac{1}{m} \sum_{r=1}^{m} \max_{k=0,1,\dots,n} \left| s_k\left(\frac{\pi r}{m} + C; f\right)\right|^2 \tag{2.13}$$

where C is any real constant.

So we have, if $|||f|||=1$

$$\max_f \sum_{r=1}^{m} \max_k \left| s_k\left(\frac{\pi r}{m}; f\right)\right|^2 \leqq \max_f \sum_{r=1}^{2m} \max_k \left| s_k\left(\frac{\pi r}{m}; f\right)\right|^2 \leqq$$

$$\leqq \max_f \sum_{r=1}^{m} \max_k \left| s_k\left(\frac{\pi r}{m}; f\right)\right|^2 + \max_f \sum_{r=m+1}^{2m} \max_k \left| s_k\left(\frac{\pi r}{m}; f\right)\right|^2$$

and by (2. 13) the two terms on the right-hand side are equal. Dividing by m yields Lemma 3.

3. We now deal with the following maximum problem. Let $k_1, \dots, k_m$ be non-negative numbers, not necessarily integers, $\mathbf{z}$ an m-vector with complex components $z_1, \dots, z_m$ and let the vectors $\mathbf{x}$ and $\mathbf{k}$ have the same meaning as before. Our problem is to find the maximum $\mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$ of

$$\left| \sum_{r=1}^{m} z_r \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt \right|^2$$

if $\varphi(t)$ ranges over all $L^2(-\infty, \infty)$ functions of unit norm. We shall evaluate the quantity $\mu(\mathbf{z}, \mathbf{x}, \mathbf{k})$ in a way analogous to the solving of Problem 2a in Part I.

Using the notation

$$\varepsilon_k(t) = \begin{cases} 1 & \text{if } |t| \leqq k, \\ 0 & \text{if } t > k \end{cases}$$

we have

$$\left| \sum_r z_r \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt \right|^2 = \left| \sum_r z_r \int_{-\infty}^{\infty} \varepsilon_{k_r}(t) \varphi(t) e^{itx_r}\, dt \right|^2 =$$

$$= \left| \int_{-\infty}^{\infty} \varphi(t) \sum_r z_r \varepsilon_{k_r}(t) e^{itx_r}\, dt \right|^2 \leqq \int_{-\infty}^{\infty} |\varphi(t)|^2\, dt \int_{-\infty}^{\infty} \left|\sum_r z_r \varepsilon_{k_r}(t) e^{itx_r}\right|^2 dt =$$

$$= \int_{-\infty}^{\infty} \sum_{p,q=1}^{m} z_p \bar{z}_q \varepsilon_{k_p}(t) \varepsilon_{k_q}(t) e^{it(x_p - x_q)}\, dt =$$

$$= \sum_{p,q=1}^{m} z_p \bar{z}_q \frac{2 \sin \min(k_p, k_q)(x_p - x_q)}{x_p - x_q} = \mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$$

and the sign of equality stands if $\varphi(t) = \varphi(t, \mathbf{z}) = c \sum z_r \varepsilon_{k_r}(t) e^{itx_r}$ where c is the norming constant

$$\left\{ \int_{-\infty}^{\infty} \left|\sum z_r \varepsilon_{k_r}(t) e^{itx_r}\right|^2 dt \right\}^{-1/2} = \{\mu(\mathbf{z}, \mathbf{x}, \mathbf{k})\}^{-1/2}.$$

Summing up we have

$$(3.1)\qquad \left|\sum_r z_r \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt\right|^2 \leqq \left|\sum_r z_r \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}) e^{itx_r}\, dt\right|^2 =$$

$$= \mu(\mathbf{z}, \mathbf{x}, \mathbf{k}) \leqq M(\mathbf{x}, \mathbf{k}) \sum |z_r|^2, \quad \text{if} \quad \int_{-\infty}^{\infty} |\varphi(t)|^2\, dt = 1$$

where $M(\mathbf{x}, \mathbf{k})$ is the greatest eigenvalue of the matrix of the positive semidefinite form $\mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$. We now state

LEMMA 4.

$$\max_{\|\varphi(t)\|=1} \sum_r \left| \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt \right|^2 = M(\mathbf{x}, \mathbf{k}).$$

Indeed substituting

$$z_r = \overline{\int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt}$$

into (3. 1) we get by division with $\sum |z_r|^2$

$$(3.2)\qquad \sum \left| \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt \right|^2 \leqq M(\mathbf{x}, \mathbf{k}) \quad \text{if} \quad \|\varphi(t)\| = 1.$$

To show that the sign of equality is valid here for some φ let $z_1^*, \ldots, z_m^*$ be the components of a vector $\mathbf{z}^*$ for which

$$(3.3)\qquad \mu(\mathbf{z}^*; \mathbf{x}, \mathbf{k}) = \left| \sum z_r^* \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{ik_r t}\, dt \right|^2 = M(\mathbf{x}, \mathbf{k}) \sum |z_r^*|^2.$$

Such a vector exists and is an eigenvector of the matrix of the Hermitian form $\mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$ belonging to its greatest eigenvalue $M(\mathbf{x}, \mathbf{k})$. We state that

$$(3.4)\qquad z_r^* = \gamma \overline{\int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{itx_r}\, dt}$$

where γ is independent of r. Indeed if it were not so we should have by Cauchy's inequality and by (3. 3)

$$\sum_r \left| \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{itx_r}\, dt \right|^2 > \left| \sum_r z_r^* \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{itx_r}\, dt \right|^2 \Big/ \sum_r |z_r^*|^2 = M(\mathbf{x}, \mathbf{k})$$

in contradiction to (3. 2). Substituting now (3. 4) into (3. 3) and dividing by $\sum |z_r^*|^2$

we get

$$\sum_r \left| \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{itx_r} \, dt \right|^2 = M(\mathbf{x}, \mathbf{k})$$

thus completing the proof of Lemma 4.

4. We now introduce the m-vectors ξ and $\varkappa$ with components $\xi_r = \sigma^{-1} x_r$ and $\varkappa_r = \sigma k_r$ and state

(4. 1) $$M(\xi, \varkappa) = \sigma M(\mathbf{x}, \mathbf{k}).$$

This follows from the fact that the matrix of the form $\mu(\mathbf{z}, \xi, \varkappa)$ is σ times the matrix of $\mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$.

The quantity

(4. 2) $$\alpha_n(\mathbf{x}) = \sup_{\substack{0 \leqq k_r \leqq m \\ r=1,2,\ldots,n}} \frac{1}{m} M(\mathbf{x}, \mathbf{k})$$

is analogous to the $A_n(\mathbf{x})$ defined by (2. 4). (Cf. Lemma 4.) Hence it follows from (4. 1)

LEMMA 5. $\alpha_{\sigma n}(\xi) = \sigma \alpha_n(\mathbf{x})$.

5. In this section let x_r be $\pi(r - [m/2])/m$, k_r any non-negative number and $[\mathbf{k}]$ the m-vector of components $[k_1], \ldots, [k_m]$, where $[\beta]$ means the greatest integer contained in β. By comparing $M(\mathbf{x}, \mathbf{k})$ and $\Lambda(\mathbf{x}, [\mathbf{k}])$ defined in Section 2 we shall prove

LEMMA 6. $|M(\mathbf{x}, \mathbf{k}) - 2\Lambda(\mathbf{x}, [\mathbf{k}])| < 2m$.

It follows from (2. 2) that if

$$d_{pp} = k_p + \frac{1}{2}, \quad d_{pq} = \frac{\sin\{\min([k_p], [k_q]) + \frac{1}{2}\}(x_p - x_q)}{2 \sin \frac{1}{2}(x_p - x_q)}$$

then

(5. 1) $$\Lambda(\mathbf{x}, [\mathbf{k}]) = \max_{\Sigma |z_r|^2 = 1} \sum_{p,q=1}^{m} d_{pq} z_p \bar{z}_q$$

in perfect analogy with

(5. 2) $$\frac{1}{2} M(\mathbf{x}, \mathbf{k}) = \max_{\Sigma |z_r|^2 = 1} \sum_{p,q=1}^{m} \delta_{pq} z_p \bar{z}_q$$

where $\delta_{pp} = k_p$ and if $p \neq q$, $\delta_{pq} = \{\sin \min(k_p, k_q)(x_p - x_q)\}/(x_p - x_q)$. Let now $\Delta_{pq} = d_{pq} - \delta_{pq}$. We state that $|\Delta_{pq}| < 1$. This is obvious for the diagonal elements Δ_{pp}. For the non-diagonal elements the inequalities

$$0 < |x_p - x_q| < \pi \quad \text{and} \quad \left| \min([k_p], [k_q]) + \frac{1}{2} - \min(k_p, k_q) \right| \leqq \frac{1}{2}$$

hold. Now under the assumptions $0<|x|<\pi$ and $|k-k_1|<1/2$ one has

$$\left|\frac{\sin k_1 x}{2\sin\frac{x}{2}}-\frac{\sin kx}{x}\right| \leqq \left|\frac{\sin k_1 x}{2\sin\frac{x}{2}}-\frac{\sin kx}{2\sin\frac{x}{2}}\right|+\left|\frac{\sin kx}{2\sin\frac{x}{2}}-\frac{\sin kx}{x}\right| \leqq$$

$$\leqq \left|\cos\frac{k_1+k}{2}x\right|\left|\frac{\sin\frac{k_1-k}{2}x}{\sin\frac{x}{2}}\right|+|\sin kx|\left|\frac{1}{2\sin\frac{x}{2}}-\frac{1}{x}\right| \leqq$$

$$\leqq \left|\frac{\sin\frac{x}{4}}{\sin\frac{x}{2}}\right|+\left|\frac{1}{2\sin\frac{x}{2}}-\frac{1}{x}\right|<1.^{*}$$

Hence $|\Delta_{pq}|<1$ and

$$\left|\sum_{p,q=1}^{m}\Delta_{pq}z_p\bar{z}_q\right| \leqq \sum_{p,q=1}^{m}|\Delta_{pq}|\frac{|z_p|^2+|z_q|^2}{2}<m\sum_{1}^{m}|z_p|^2. \tag{5. 3}$$

A well known reasoning yields now Lemma 6: by (5. 1), (5. 2) and (5. 3)

$$\Lambda(\mathbf{x},[\mathbf{k}]) = \max_{\Sigma|z_r|^2=1}\left(\sum\Delta_{pq}z_p\bar{z}_q+\sum\delta_{pq}z_p\bar{z}_q\right) \leqq$$

$$\leqq \max\sum\Delta_{pq}z_p\bar{z}_q+\max\sum\delta_{pq}z_p\bar{z}_q = m+\frac{1}{2}M(\mathbf{x},\mathbf{k})$$

and similarly $\frac{1}{2}M(\mathbf{x},\mathbf{k}) \leqq m+\Lambda(\mathbf{x},\mathbf{k})$.

By taking into account the definitions of $a_n^{(m)}$ and $\alpha_n(\mathbf{x})$ at the end of Sections 2 and 4, respectively, it follows

LEMMA 7. *If* $x_r=\pi(r-c)/m$ $(r=1,2,\ldots,m)$ *then*

$$|\alpha_n(\mathbf{x})-2\alpha_n^{(m)}|<2.$$

We now choose $m=n$, $c=[n/2]$ and $\sigma=\pi/n$ in Lemma 5. Combining Lemmas 5 and 7 we get

$$\left|\frac{n}{\pi}\alpha_\pi(\xi)-2\alpha_n^{(n)}\right|<2 \tag{5. 4}$$

* Indeed for $|x|<\pi$ one has $\{\sin(x/4)\}/\{\sin(x/2)\} = \{2\cos(x/4)\}^{-1} \leqq \{2\cos(\pi/4)\}^{-1}<0.71$ and if $t^2<6$, the Maclaurin series of $\sin t$ is of Leibniz' type, hence for $0\leqq t\leqq\pi/2$ $(<6^{1/2})$ one has $t>\sin t>t-t^3/6$, or if $0<x<2\pi$

$$\frac{x}{2}>\sin\frac{x}{2}>\frac{x}{2}\left(1-\frac{x^2}{24}\right)$$

from which

$$0<\frac{1}{2\sin(x/2)}-\frac{1}{x}<\frac{1}{x(1-x^2/24)}-\frac{1}{x}=\frac{x}{24-x^2}<\frac{\pi}{24-\pi^2}<\frac{1}{4}.$$

if $\xi_r = r-[n/2]$ $(r=1, 2, \ldots, n)$. Now from (4. 2)

$$(5.5)\qquad \frac{n}{\pi}\alpha_\pi(\xi) = \frac{1}{\pi}\sup_\varphi \sum_{r=1}^{n} \max_{0\leqq \varkappa_r\leqq \pi}\left|\int_{-\varkappa_r}^{\varkappa_r}\varphi(t)e^{i(r-[m/2])t}\,dt\right|^2 \Big/ \int_{-\infty}^{\infty}|\varphi(t)|^2\,dt =$$

$$= \frac{1}{\pi}\sup_\varphi \sum_{s=1-[n/2]}^{n-[n/2]} \max_{0\leqq \varkappa_s\leqq \pi}\left|\int_{-\varkappa_s}^{\varkappa_s}\varphi(t)e^{ist}\,dt\right|^2 \Big/ \int_{-\infty}^{\infty}|\varphi(t)|^2\,dt$$

and the last supremum evidently remains unaltered if we restrict ourselves to functions $\varphi(t)$ with $\varphi(t)\equiv 0$, if $|t|>\pi$.

6. Now we are ready to prove the statement at the end of Section 1. First we suppose that $A_n<A$ for each integer n.

Then using in turn (5. 4), Lemmas 3 and 2 we have

$$\frac{n}{\pi}\alpha_\pi(\xi) < 2a_n^{(n)}+2 < 4A_n^{(2n)}+2 < (2+4\pi)A_n+2 < (2+4\pi)A+2$$

for every n, so by (5. 5)

$$\sup_\varphi \sum_{s=-\infty}^{\infty} \max_{0\leqq \varkappa_s\leqq \pi}\left|\int_{-\varkappa_s}^{\varkappa_s}\varphi(t)e^{ist}\,dt\right|^2 \Big/ \int_{-\pi}^{\pi}|\varphi(t)|^2\,dt < 2\pi\{(1+2\pi)A+1\},$$

or by (1. 4) $E(\varphi)$ is bounded if $\|\varphi\|=1$.

If, however $n\pi^{-1}\alpha_\pi(\xi)<\alpha$ for each integer n, then by (5. 4), Lemma 3 and (2. 5)

$$\alpha \geqq E(\varphi) \geqq n\pi^{-1}\alpha_\pi(\xi) > 2a_n^{(n)}-2 > 2A_n^{(2n)}-2 > A_n-2,$$

hence A_n is bounded.

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