Acta Mathematica Academiae Scientiarum Hungaricae Tomus 17 (3—4), pp. 419—422 (1966)

A PROOF OF SAINT-VENANT'S THEOREM ON TORSIONAL RIGIDITY

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1. Let D be a simply connected plane domain, A its area, ϱ and σ the radius and the area, respectively, of its greatest inscribed circle, P the torsional rigidity of D defined by

$$P = \sup_f \; 4\Big[\iint_D f\,dx\,dy\Big]^2 \Big/ \iint_D (f_x^2+f_y^2)\,dx\,dy \tag{1}$$

($f=0$ on the boundary, f continuous, f_x and f_y piecewise continuous inside D).

More than a hundred years ago B. DE SAINT-VENANT [5] conjectured that of *all domains D of equal area A the circular one has the greatest torsional rigidity*, i.e.

$$P \leqq P_{\text{circle}} = \frac{A^2}{2\pi}. \tag{2}$$

DE SAINT-VENANT supported (2) by ample physical evidence, yet the first rigorous proof of this inequality was given only comparatively late, in 1948, by G. PÓLYA [3]. Another proof of (2) by H. DAVENPORT is incorporated in [4], p. 121—122.

We shall give here a proof of (1) in the case of simply connected domains, which partly runs parallel with FABER's and KRAHN's proof of RAYLEIGH's conjecture as expounded in a previous paper [2], partly uses an idea of PÓLYA and SZEGŐ to be found in [4], p. 100—102.

As a side result we get *an upper estimation of P in terms of A and ϱ, namely*

$$P < 4\varrho^2 A. \tag{3}$$

If $\pi\varrho^2 = \sigma < A/8$, this estimation is sharper than (2). It cannot be discussed here whether the constant 4 in (3) can be replaced by a less number or not. For convex domains, anyhow, one has the sharper estimation

$$P \leqq \frac{4}{3}\varrho^2 A,$$

where the constant 4/3 is the best possible one [1].

2. If the boundary of D is sufficiently smooth, e.g. D is a polygonal domain, then in the definition (1) of P the symbol sup can be exchanged into max, i.e. there exists a maximalizing function $v=v(x, y)$ vanishing on the boundary for which one has

$$P = 4\Big[\iint_D v\,dx\,dy\Big]^2\Big/\iint_D (v_x^2+v_y^2)\,dx\,dy. \tag{4}$$

In this section we shall assume that D is a domain of this kind. Then, without restricting the generality, one can assume that v satisfies the differential equation $v_{xx}+v_{yy}+2=0$ ([4], p. 88). Hence v cannot have local minima inside D and is always positive there.

The level lines of $v(x, y)$ will be labelled by a parameter τ just as in [2]. The meaning of the parameter τ is as follows. If the function $v(x, y)$ assumes on one of its level lines the value z and $D(\tau)$ is a domain consisting of those points of D where $v(x, y)>z$, then the area of $D(\tau)$ should be equal to τ. The boundary of $D(\tau)$ will be termed the level line C_τ.

We introduce now in D instead of the coordinates x and y the new coordinates τ and s where s is an arc length counted from appropriate points of the possibly disconnected level line C_τ and ranging from 0 to $L(\tau)$, the total length of C_τ. Further we introduce the notation

$$\Delta = \begin{vmatrix} \dfrac{\partial x}{\partial \tau} & \dfrac{\partial y}{\partial \tau} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{vmatrix} \tag{5}$$

and the function $\chi(\tau)$ defined by

$$\chi(\tau)=v(x, y) \quad \text{on} \quad C_\tau. \tag{6}$$

Obviously $\chi(\tau)$ decreases monotonically in $0\leqq\tau\leqq A$ and $\chi(A)=0$.

Then we have just in the same way as in [2]

$$\iint_D v\,dx\,dy = \int_0^A \chi(\tau)\int_0^{L(\tau)} |\Delta|\,ds\,d\tau \tag{7}$$

and

$$\iint_D (v_x^2+v_y^2)\,dx\,dy = \int_0^A \chi'^2(\tau)\int_0^{L(\tau)} \frac{ds}{|\Delta|}\,d\tau. \tag{8}$$

Since Δ may vanish on the boundary of D and likewise $1/\Delta$ may vanish in points where $\partial v/\partial x=\partial v/\partial y=0$ (these points are either of finite number or they are countable) the integrals on the right sides are possibly improper ones.

We recall further the following formulae of [2]:

$$\int_0^{L(\tau)} |\Delta|\,ds = 1, \quad \int_0^{L(\tau)} \frac{ds}{|\Delta|}\int_0^{L(\tau)} |\Delta|\,ds \geqq \Big(\int_0^{L(\tau)} ds\Big)^2 = \{L(\tau)\}^2$$

and

$$L(\tau) \geqq M(\tau) = \begin{cases} \sqrt{4\pi\tau} & \text{if} \quad 0\leqq\tau\leqq\sigma, \\ \dfrac{\tau}{\varrho}+\pi\varrho & \text{if} \quad \sigma\leqq\tau\leqq A. \end{cases} \tag{9}$$

The last inequalities are consequences of the classical isoperimetric inequality and its refined form due to BONNESEN, a short proof of which may be found in [6]. Using these inequalities in (7) and (8) we have

$$(7') \qquad \iint_D v\,dx\,dy = \int_0^A \chi(\tau)\,d\tau = -\int_0^A \tau\chi'(\tau)\,d\tau$$

since $\chi(A)=0$, and

$$\iint_D (v_x^2+v_y^2)\,dx\,dy = \int_0^A \chi'^2(\tau)\int_0^{L(\tau)} \frac{ds}{|\Delta|}\,d\tau =$$

$$(8') \qquad = \int_0^A \chi'^2(\tau)\int_0^{L(\tau)} \frac{ds}{|\Delta|}\int_0^{L(\tau)} |\Delta|\,ds\,d\tau \geqq$$

$$\geqq \int_0^A \chi'^2(\tau)L^2(\tau)\,d\tau \geqq \int_0^A \chi'^2(\tau)M^2(\tau)\,d\tau.$$

Now we have using (7′) and (8′)

$$(10) \qquad \frac{P}{4} = \frac{\left(\iint_D v\,dx\,dy\right)^2}{\iint_D (v_x^2+v_y^2)\,dx\,dy} \leqq \frac{\left(\int_0^A \chi'(\tau)\tau\,d\tau\right)^2}{\int_0^A \chi'^2(\tau)M^2(\tau)\,d\tau} \leqq \int_0^A \left(\frac{\tau}{M(\tau)}\right)^2 d\tau$$

by Schwarz's inequality. Using the explicit expression (9) of $M(\tau)$ one obtains easily

$$(11) \qquad \int_0^A \left(\frac{\tau}{M(\tau)}\right)^2 d\tau = \int_0^\sigma \frac{\tau^2}{4\pi\tau}\,d\tau + \int_0^A \left(\frac{\tau}{\frac{\tau}{\varrho}+\pi\varrho}\right)^2 d\tau = \frac{\sigma^2}{\pi}\, g\left(\frac{A}{\sigma}\right),$$

where

$$g(\xi) = \xi - \frac{3}{8} - \frac{1}{\xi+1} - 2\log\frac{\xi+1}{2}.$$

We remark here that if $h(\xi)=\xi^2-8g(\xi)$ then $h(1)=0$, $h'(\xi)=2\xi(1-\xi)^2\cdot$ $\cdot(1+\xi)^{-2}$, so that from (10) and (11) we have for any polygonal domain

$$(12) \qquad A^2-2\pi P \geqq A^2-8\sigma^2 g(A/\sigma) = \sigma^2 h(A/\sigma) =$$

$$= \sigma^2 \int_1^{A/\sigma} 2\xi\left(\frac{1-\xi}{1+\xi}\right)^2 d\xi > 0$$

since $A>\sigma$.

3. Let now D be any simply connected domain for which P is defined by (1). We want now to show that the inequality

$$A^2-2\pi P \geqq \sigma^2 h(A/\sigma) \tag{12'}$$

still holds. For sake of simplicity we consider only those domains D which can be approximated by a sequence of circumscribed polygonal domains D_n $(n=1, 2, \ldots)$ with areas A_n, areas of the greatest inscribed circles σ_n, and torsional rigidities P_n. We suppose (i) $D_n \supseteqq D$ from which one has $P_n \geqq P$ and (ii) $A_n \to A$ which implies $\sigma_n \to \sigma$.

Then, by (12)

$$A^2-2\pi P \geqq A^2-2\pi P_n = (A_n^2-2\pi P_n)-(A_n^2-A^2) \geqq \sigma_n^2 h(A_n/\sigma_n)-(A_n^2-A^2)$$

and since for $n\to\infty$ the right hand side tends to $\sigma^2 h(A/\sigma)$ we obtain (12'). The integral representation of $h(A/\sigma)$ in (12) shows that the right hand side of (12') is positive unless $A/\sigma=1$ i.e. unless D is a circular disk. On the other hand we get $A^2=2\pi P$ in this exceptional case (formula (2)), so SAINT-VENANT's theorem is proved.

4. Now we turn to the proof of inequality (3). In the case of polygonal domains this follows from observing that for $\xi \geqq 1$ one has $g(\xi)<\xi-3/8$, consequently the right side of (11) is certainly less than

$$\frac{\sigma^2}{\pi}\left(\frac{A}{\sigma}-\frac{3}{8}\right)=\frac{\sigma A}{\pi}-\frac{3}{8}\frac{\sigma^2}{\pi},$$

hence from (10)

$$\frac{P}{4}< A\varrho^2-\frac{3\sigma^2}{8\pi}.$$

Repeating the argument of Section 3 we have for any simply connected domain D

$$\frac{P}{4}\leqq A\varrho^2-\frac{3\sigma^2}{8\pi}< A\varrho^2.$$

(Received 28 September 1965)

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