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STRUCTURES OF CONTINUOUS FUNCTIONS. I

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§ 1. Introduction

There are numerous theorems in the literature concerning representation of certain maps (functionals) defined on sets of continuous functions. As an example we shall quote the following two.

KAKUTANI—RIESZ THEOREM. *If X is a Hausdorff compact space and $C(X)$ is the set of all real-valued continuous functions defined on X, then every linear positive functional φ on $C(X)$ (i.e., a real-valued map φ on $C(X)$ satisfying: $\varphi(f+g) = \varphi(f)+\varphi(g)$ and $\varphi(f) \geqq 0$ for $f \geqq 0$) admits the integral representation*

$$\varphi(f) = \int f\, d\mu$$

where μ is a Baire measure in X.

MAZUR THEOREM. *If X is a separable metric space and F is a subring of $C(X)$ such that F contains all constant functions on X, F is closed under inversion (i.e., if $f \in F$ and $f(p) \neq 0$ for every $p \in X$, then $1/f \in F$), and F satisfies the following condition*
(1) *if $f_1, f_2, \dots$ are members of F such that $0 \leqq f_n(p) \leqq 1$ for every $p \in X$ and every n, then there exists a sequence of positive numers α_n such that $\sum_n \alpha_n < +\infty$ and $\sum \alpha_n \cdot f_n \in F$,*

then every linear multiplicative functional φ on F is either identically equal to 0 or admits the following trivial representation

$$\varphi(f) = f(p_0)$$

where p_0 is a fixed point of X.

The purpose of this paper is to provide a general framework for the discussion of such representation theorems. These theorems exhibit the following pattern (precise definitions of the terms involved will be given in the next section): we consider a topological space E on which certain algebraic operations and/or relations are defined — we will refer to such an E as a topological algebraic structure. Given an arbitrary space X we denote by $C(X, E)$ the set of all continuous functions f with $f\colon X \to E$. Every operation (relation) in E gives rise to a "pointwisely defined" operation (relation, respectively in $C(X, E)$); $C(X, E)$ becomes therefore an algebraic structure. Let F be a substructure of $C(X, E)$; we are concerned with representation of homomorphisms (functionals) of F; i.e., maps φ of F that preserve the given operations or relations. Note that in such a general setting, although we deal with continuous functions only, we do not exclude the case of a discrete X and in this case F is simply a substructure of the direct product of copies of E. In fact, the thechnique of structures of continuous functions is applicable to problems which — in their original formulation — involve no topology (see Sections 6 and 7 of the present paper). Finally, note that there are two essentially different types of representation: in Mazur's theorem the representation formula involves only one point of the space; in the Kakutani—Riesz theorem the value of $\varphi(f)$ depends upon values of f on a whole set of points.

The present paper is the first in the series "Structures of continuous functions". Three papers of this series, III, IV, V, ([11], [17], [18]) have been already published. (In III, E is the set of integers considered as a ring and as a lattice; in IV, E is the lattice of the real numbers and V contains one result concerning the case of arbitrary E.) The second paper of this series was intended as a summary of results on E-compact spaces; however, in view of a rapid development in this area its publication was continuously delayed. A partial summary of related results will be published outside this series [13].

§ 2. Structures of continuous functions

The purpose of this section is to provide necessary definitions. By an *algebraic structure* we mean a triplet

$$\{E; \{o_0, \dots, o_\xi, \dots\}_{\xi<\alpha}; \quad \{\varrho_0, \dots, \varrho_\eta, \dots\}_{\eta<\beta}\}, \tag{1}$$

where E is a set, o_ξ are operations on E, and ϱ_η are relations on E. We do not assume that these operations and relations are finitary. Whenever no confusion seems possible the structure (1) will be denoted simply by E. The type of the structure (1) is the pair of transfinite sequences

$$\{\nu_0, \dots, \nu_\xi, \dots\}_{\xi<\alpha}; \quad \{\mu_0, \dots, \mu_\eta, \dots\}_{\eta<\beta} \tag{2}$$

such that o_ξ is a ν_ξ-ary operation and ϱ_η is an μ_η-ary relation. For structures of the same type it is possible to define the concept of a homomorphism and that of an isomorphism. Let E and E_1 be structures of the same type; let o_ξ and $\bar{o}_\xi$ (ϱ_η and $\bar{\varrho}_\eta$) be the corresponding operations (relations) in E and E_1, respectively. For simplicity of notation we will assume at this moment that these operations and relations are binary. A map $\varphi\colon E \to E_1$ is called a *homomorphism* provided that

$$\varphi(x_1 o_\xi x_2) = \varphi(x_1)\bar{o}_\xi\varphi(x_2) \quad \text{for every} \quad x_1, x_2, \in E \quad \text{and for every} \quad \xi<\alpha \tag{3}$$

and

(4) $x_1 \varrho_\eta x_2$ implies $\varphi(x_1)\bar{\varrho}_\eta\varphi(x_2)$ for every $x_1, x_2 \in E$ and for every $\eta<\beta$. φ is called an *isomorphism* provided that φ is one-to-one, φ satisfies (3), and φ satisfies (4) with "implies" replaced by "if and only if". Note that according to the above definitions a one-to-one homomorphism need not to be an isomorphism.

A *substructure* E_0 of (1) is a subset of E whose operations and relations are those of (1) restricted to E_0 and which is closed under all of the operations of (1).

A *topological algebraic structure* is a structure (1) in which E is a Hausdorff topological space and such that all the operations o_ξ are continuous (relative to the product topology in the corresponding power of E). In general, we will not make any topological assumptions on the relations of (1), however, it is sometimes useful to assume that they are closed (in the respective powers of E) or that they are E-compact.

If E is a topological algebraic structure and X is an arbitrary topological space, then by $C(X, E)$ we shall denote the algebraic structure consisting of all continuous functions $f\colon X \to E$; operations and relations in $C(X, E)$ are the pointwisely defined counterparts of the operations and relations in E. That is, if o_ξ is an operation and ϱ_η is a relation in E (assumed, for simplicity of notation, to be binary), then the pointwisely defined counterparts $o_\xi^{(X)}$ and $\varrho_\eta^{(X)}$ in $C(X, E)$ of o_ξ and ϱ_η, respectively, are defined as follows:

(4) $$h = f o_\xi^{(X)} g \quad \text{if, and only if,} \quad h(p) = f(p) o_\xi g(p) \quad \text{for every} \quad p \in X;$$

and

(5) $$f \varrho_\eta^{(X)} g \quad \text{if, and only if,} \quad f(p) \varrho_\eta g(p) \quad \text{for every} \quad p \in X.$$

The superscript X in $o_\xi^{(X)}$ and $\varrho_\eta^{(X)}$ will be omitted whenever possible. Note that the structure $C(X, E)$ is of the same type as E. Furthermore, the assumption that the operations o_ξ are continuous implies that $C(X, E)$ is closed with respect to the operations $o_\xi^{(X)}$.

Throughout the rest of the paper we shall use the following notations: E will be a topological algebraic structure, E_1 will be an algebraic structure of the same type as E; F will be a substructure of $C(X, E)$ and φ will be a homomorphism of F into E_1 (note that F and E_1 are of the same type).

To conclude this section observe that in the Kakutani—Riesz Theorem we have $E = E_1 =$ the ordered group of the reals (i.e., $E = E_1 = \{\mathcal{R}; +; \leqq\}$) and in the Mazur theorem, $E = E_1 =$ the ring of the reals (i.e., $E = E_1 = \{\mathcal{R}; +; \cdot\}$) (where $+, \cdot, \leqq$ denote, respectively, the addition, the multiplication, and the "less than or equal to" relation in the set $\mathcal{R}$ of the reals).

§ 3. Supports and weak supports

Our main tool in dealing with the representation problem will be the concept of a support and that of a weak support.[1] The algebraic structure will not enter into the considerations of this section (so one may consider E as a plain topological space and E_1 as a plain set; thus φ is an arbitrary map with $\varphi\colon F \to E_1$).

A closed set $A \subset X$ is called a *support* of φ provided that for every $f, g \in F$ the equality $f|A = g|A$ implies $\varphi(f) = \varphi(g)$. A is called a *weak support* of φ provided that for every open set $U \subset X$ with $A \subset U$ and for every $f, g \in F$, the equality $f|U = g|U$ implies $\varphi(f) = \varphi(g)$.

Obviously, a support of φ is a weak support of φ (the converse is not necessarily true, see Examples 4. 3 and 4. 4). The concept of a weak support admits a natural and useful generalization: if εX is an extension of X (i.e., εX is a Hausdorff superspace of X in which X is dense), then a closed subset A of εX is called a weak support prof φ ovided that for every open subset U of εX with $A \subset U$ the equality $f|U \cap X = g|U \cap X$ implies $\varphi(f) = \varphi(g)$ (for every $f, g \in F$). An analogous generalization of the concept of a support, is, of course, superfluous.

[1] These concepts were introduced in [18].

Note that the empty set is a support of φ iff φ is a constant map. Any superset of a support (a weak support) of φ is again a support (a weak support) of φ, in particular, the whole space X is always a support of φ. We shall therefore be interested in the existence of a smallest support[2] or a smallest weak support.[2]

Note that the existence of a one-point support completely solves the representation problem; indeed we have the following.

3. 1. *Suppose that F contains all constant functions from $C(X, E)$ and let p_0 be a point from X. $\{p_0\}$ is a support of φ if, and only if, φ can be represented in the form*

$$\varphi(f)=\alpha(f(p_0)) \quad \textit{for every} \quad f\in F,$$

where α is a fixed homomorphism of E into E_1.

§ 4. The compact case

In this section X will be assumed to be a Hausdorff space. We shall give a few sufficient conditions for the existence of smallest supports and weak supports in the case of a compact X as well as discuss a few counter-examples.

Let $\mathfrak{C}$ be a multiplicative[3] base for closed subsets of X.[4] We shall say that φ *has the property Π relative to* $\mathfrak{C}$ (in symbols: $\Pi(\varphi, \mathfrak{C})$ holds) provided that the intersection of two supports of φ from C is a support of φ.

4. 1. THEOREM. *Let X be compact. If φ has the property Π relative to $\mathfrak{C}$, then the intersection of all supports of φ from $\mathfrak{C}$ is the smallest weak support of φ.*

PROOF. Let

$$\mathfrak{Z}_\varphi=\{A\colon A\in\mathfrak{C} \quad \text{and } A \text{ is a support of } \varphi\},\ Z_\varphi=\bigcap\mathfrak{Z}_\varphi.$$

It is easy to see that the class $\mathfrak{Z}_\varphi$ is multiplicative. Let U be an open subset of X with $Z_\varphi\subset U$ and let $f|U=g|U$, $f, g\in F$. Since $\mathfrak{Z}_\varphi$ is multiplicative (and X is compact), there is an $A\in\mathfrak{Z}_\varphi$ with $A\subset U$. We have $f|A=g|A$, therefore $\varphi(f)=\varphi(g)$. Thus Z_φ is a weak support of φ.

Now, Z_φ is the smallest weak support of φ. Indeed, assume that Z is a weak support of φ and assume that $Z_\varphi\not\subset Z$. Let $p_0\in Z_\varphi\setminus Z$. There is an open set G such that $Z\subset G$ and $p_0\notin\bar{G}$. Since $\mathfrak{C}$ is a base for closed sets, there is an $A\in\mathfrak{C}$ such that $\bar{G}\subset A$ and $p_0\notin A$. Since $Z\subset \operatorname{Int} A$, A is a support of φ. Thus $A\in\mathfrak{Z}_\varphi$, hence $Z_\varphi\subset A$, contrary to the fact that $p_0\in Z_\varphi$ and $p_0\notin A$.

We shall now give sufficient conditions for $\Pi(\varphi, \mathfrak{C})$.

We say that F *has the property* (K) *relative to* $\mathfrak{C}$ (in symbols: $K(F, C)$ holds) provided that the following condition is satisfied

for every $A, B\in\mathfrak{C}$ and for every $f, g\in F$ with $f|A\cap B=g|A\cap B$ there exists an $h\in F$ such that $f|A=h|A$ and $g|B=h|B$.

[2] A smallest support (weak support) of φ is a support (weak support) which is contained in every support (weak support) of φ.

[3] A class $\mathfrak{C}$ of sets is said to be *multiplicative* provided that $A, B\in\mathfrak{C}$ implies $A\cap B\in\mathfrak{C}$.

[4] A base for closed sets is a class of closed sets such that every closed set is an intersection of some members of this class.

4. 2. THEOREM. *If F has the property (K) relative to $\mathfrak{C}$, then φ has the property Π relative to $\mathfrak{C}$.*

PROOF. Let $A, B \in \mathfrak{C}$ and let A and B be supports of φ. If $f|C = g|C$, where $f, g \in F$ and $C = A \cap B$, then there is an $h \in F$ with $f|A = h|A$ and $g|B = h|B$. Since A and B are supports of φ, we have $\varphi(f) = \varphi(h)$ and $\varphi(g) = \varphi(h)$, hence $\varphi(f) = \varphi(g)$. Thus $\mathfrak{C}$ is a support of φ.

Note that $F = C(X, E)$ has the property (K) relative to an additive[5] (and multiplicative) class $\mathfrak{C}$ of closed subsets of X whenever X has the following extension property: for every $A \in \mathfrak{C}$ every continuous function $f: A \to E$ admits a continuous extension $f: X \to E$ (e.g., $F = C(X, E)$ has the property (K) relative to the class of all closed subsets of X whenever X is normal and E is an absolute (metric retract). Indeed, in this case we define $h_0(p) = f(p)$ for $p \in A$ and $h_0(p) = g(p)$ for $p \in B$ (so that $h_0: A \cup B \to E$) and then take a continuous extension h of h_0 with $h: X \to E$. A particular case of the above is: if X is 0-dimensional, then $F = C(X, E)$ has always the property K relative to the class $\mathfrak{C}$ of all open-closed subsets of X. Consequently, *if X is a 0-dimensional compact space, then every map φ on $C(X, E)$ has a smallest weak support.*

We shall now consider a few examples.

4. 3. EXAMPLE. Let $E = E_1$ be the lattice of the real numbers (i.e., $E = E_1 = \langle \mathscr{R}; \vee, \wedge \rangle$, where $\vee$, and $\wedge$ stand for maximum and minimum, respectively) and let $F = C(X, E)$ (i.e., F is the lattice of real-valued continuous functions on X). If X is compact, then every (lattice-) homomorphism $\varphi: C(X, E) \to E$ has a one-point weak support (consequently, φ has a smallest weak support). If X is infinite (completely regular, but not necessarily compact), then there is a homomorphism φ without a one-point support (for these results see [17]). Such a φ has the property Π relative to the class of all closed subsets of X (obvious by 4. 2) and this implies that φ does not have a smallest support.

4. 4. EXAMPLE. Let X be the closed unit interval, let $E = \mathscr{R}$ and let F consist of all continuously differentiable functions on X. Let $\mathfrak{C}$ be the class of all finite unions of intervals of the form $[a, b]$ where $0 \leqq a < b \leqq 1$, where a is rational, b is irrational or $b = 1$. $\mathfrak{C}$ is an additive and multiplicative base for closed subsets of X. F has the property K relative to $\mathfrak{C}$ (on the other hand, F has the property K relative to neither the class of all closed subsets of X nor the class of all finite unions of arbitrary closed subintervals of X). Thus (by 4. 2 and 4. 1), every map $\varphi: F \to E_1$, where E_1 is an arbitrary set, has a smallest weak support. The map φ, defined by $\varphi(f) = f'(x_0)$, (x_0 — a fixed point of X) has $\{x_0\}$ as its smallest weak support; φ does not have a smallest support.

4. 5. EXAMPLE. Let $X = [0, 1]$, let $E = E_1$ be the ring of the reals $\mathscr{R}$, and let F consist of all polynomials in $C(X, E)$. Define $\varphi(f) = f(2)$ for every $f \in F$. F is a subring of $C(X, E)$ and F contains all constant functions in $C(X, E)$. φ is a (ring-) homomorphism of F. Every infinite closed subset of X is a support of φ whereas no finite subset of X is. If $\mathfrak{C}$ is an arbitrary multiplicative base for closed subsets of X, then φ does not have the property Π relative to $\mathfrak{C}$ ($\mathfrak{C}$ contains infinite disjoint

[5] A class $\mathfrak{C}$ of sets is said to be *additive* provided that $A, B \in \mathfrak{C}$ implies $A \cup B \in \mathfrak{C}$.

sets). Thus, F does not have the property K (relative to any such $\mathfrak{C}$). Every one-point set is a weak support of φ; consequently φ does not have a smallest weak support.

4. 6. EXAMPLE. Let $\mathscr{R}_\alpha$, where α is an ordinal, denote the ordered product $[0, 1)\times S(\alpha)$[6] (ordered according to second coordinates). Elements of $\mathscr{R}_\alpha$ of the form $(0, \xi)$ will be denoted by ξ. $\mathscr{R}_\alpha$ will be considered as a lattice. $\mathscr{R}_\alpha$ is connected relative to its order topology.

Let $X=[0, 1]$ and let α be a fixed ordinal with $\alpha>\Omega$. Let $F=C(X, \mathscr{R}_\alpha)$. From the connectedness of X it is easy to infer that

(i) if $f\in C(X, \mathscr{R}_\alpha)$ and $f(x_0)<\Omega$ for some $x_0\in X$, then $f(x)<\Omega$ for every $x\in X$.

Let $F_1=\{f\in F: f(x)<\Omega$ for every $x\in X\}$, $F_2=\{f\in F: f(x)\geqq\Omega$ for every $x\in X\}$. Clearly, $F_1\cap F_2=\emptyset$ and from (i) infer that $F=F_1\cup F_2$. Define $\varphi(f)=f(0)$ for $f\in F_1$ and $\varphi(f)=f(1)$ for $f\in F_2$. φ is a (lattice-) homomorphism of $C(X, \mathscr{R}_\alpha)$ (into the chain $\mathscr{R}_\alpha$). The set $A=\{0, 1\}$ is the smallest support of φ (as well as its smallest weak support). It follows that the intersection of any two supports of φ is again a support of φ. On the other hand, $F=C(X, \mathscr{R}_\alpha)$ does not have the property K relative to any multiplicative base $\mathfrak{C}$ for closed subsets of X.

We have seen that a smallest weak support need not exist. But if X is compact, then φ always has a minimal weak support (i.e., a weak support that does not contain properly another weak support). This follows immediately from the KURATOWSKI lemma ([5], statement (41), p. 88): repeating the proof of 4. 1 we can show that the intersection of a chain of weak supports is again a weak support.

We shall now turn to the existence of smallest supports. In some cases it is possible to prove that weak supports of φ are, in fact, its supports. This is, for instance, the case when φ is continuous (in a certain sense) and if functions from F that agree on a weak support A of φ can be approximated by functions that agree on neighborhoods of A. A formal statement to this effect can be formulated as follows.

Let A be a closed subset of X and let D be a directed set. Suppose that we can define a convergence $\underset{(1)}{\to}$ for D-nets (i.e., nets with D as the set of indices) of elements of F such that

(1) *for every $f, g\in F$ with $f|A=g|A$ there exist nets $\{f_n: n\in D\}$ and $\{g_n: n\in D\}$ of functions for F and a net $\{U_n: n\in D\}$ of open subsets of X such that $f_n\underset{(1)}{\to}f$, $g_n\underset{(1)}{\to}g$, $A\subset U_n$, and $f_n|U_n=g_n|U_n$.*

We have

4. 7. *Let A be a weak support of φ and suppose that convergences $\underset{(1)}{\to}$ and $\underset{(2)}{\to}$ of D-nets in F and E_1, respectively, are defined. If $\underset{(1)}{\to}$ satisfies condition* (1) *and φ is continuous relative to these convergences, then A is a support of φ.*

Let us mention some cases when a convergence satisfying (1) can be defined.

4. 8. *If E is a normed linear space, then the uniform convergence of sequences in $C(X, E)$ satisfies condition* (1) *(relative to any closed subset of X).*

[6] $S(\alpha)$ denotes the set of all ordinals $\xi<\alpha$.

PROOF. In a normed linear space closed spheres are retracts of the whole space; consequently, we can define a sequence of continuous functions $r_n\colon E\to E$, $n=1, 2, \ldots$, such that $\|r_n(e)\|\leqq\frac{1}{n}$ and $r_n(e)=e$ for $\|e\|\leqq\frac{1}{n}$. Let A be a closed subset of X and let $f, g\in C(X, E)$, $f|A=g|A$. Set $f_n(x)=f(x)+r_n(g(x)-f(x))$ and $g_n(x)=g(x)$ for $n=1, 2, \ldots$. Clearly, $f_n\to f$ and $g_n\to g$ uniformly on X; furthermore $f_n|U_n=g_n|U_n$, where $U_n=\left\{x\in X\colon\|f(x)-g(x)\|<\frac{1}{n}\right\}$; U_n is an open subset of X containing A.

Clearly, 4. 8 can be generalized to other types of linear topological spaces which have bases of (closed) neighborhoods that are retracts of the whole space. If such a space does not satisfy the first axiom of countability, then one has to consider convergence of uncountable nets.

4. 9. *Let E be a topological abelian group (written additively) having a base $\mathfrak{G}$ of neighborhoods of the zero-element* 0 *with* card $\mathfrak{G}\leqq m$. *Let X be* 0*-dimensional compact. The uniform convergence of nets of cardinality $\leqq m$ in $C(X, E)$ satisfies condition* (1) *(relative to any closed subset of X).*

PROOF. Consider $\mathfrak{G}$ as a directed set; G precedes G_1 iff $G\supset G_1$. Let $f, g\in C(X, E)$, $f|A=g|A$, where A is a closed subset of X. For every $G\in\mathfrak{G}$ there exists a closed-open subset U_G of X such that $A\subset U_G$ and $f(p)-g(p)\in G$ for $p\in U_G$. Define $f_G(p)=g(p)$ for $p\in U_G$, $f_G(p)=f(p)$ for $p\in X\setminus U_G$ and $g_G=g$.

A trivial case in which weak supports are is given by the following.

4. 10. *If either E or X is discrete, then every weak support of a $\varphi\colon F\to E_1$ is a support of φ.*

PROOF. If E is discrete, then the diagonal of $E\times E$ is open in $E\times E$; consequently, if two functions agree on a subset A of X, then they agree on an open superset of A. The case of a discrete X is obvious.

§ 5. Compact case: one-point weak supports

We shall now consider the following question: for what structures E and E_1 is it true that all homomorphisms $\varphi\colon C(X, E)\to E_1$, where X is an arbitrary Hausdorff compact space, have one-point weak supports? We conjecture (see [18]) that this question can be decided by examining finite spaces. A partial success, concerning only 0-dimensional compact spaces, has been obtained in [18]. We shall quote this result.[7]

We say that a topological algebraic structure E is an *s-algebra* provided that among the operations of E there is a binary operation o satisfying the following condition

(s) *for every compact subset C of E there exist elements 0_C and 1_C such that $0_C o e=0_C o e'$ for every $e, e'\in C$ and $1_C o e=e$ for every $e\in C$.*

[7] This result has been announced in [16].

Examples of s-algebras: every topological ring E with unit element is an s-algebra; one takes o to be the multiplication, 0_C and 1_C to be the zero element and the unit element of E, respectively. Every ordered set considered as a lattice with the order topology is an s-algebra; one takes, for instance, o to be the maximum ($\vee$) and $0_C = \sup C$, $1_C = \inf C$ for every compact subset C of E.

5. 1. THEOREM. *Let E be an s-algebra and let E_1 be an algebraic structure of the same type as E.*[8] *If every homomorphism $\varphi: C(\mathscr{D}_2, E) \to E_1$, where $\mathscr{D}_2$ is the two-point discrete space, has a one-point support, then every homomorphism $\varphi: C(X, E) \to E_1$, where X is an arbitrary Hausdorff 0-dimensional compact space, has a one-point weak support.*

As it was pointed out in [18] the above theorem fails if "weak support" is replaced by "support" in its conclusion. The theorem also fails if "0-dimensional" is removed from its assumption. Consider the chain $\mathscr{R}_\alpha$ ($\alpha > \Omega$) described in Example 4. 6 and let $E = E_1 = \mathscr{R}_\alpha$. It is easy to see that the assumptions of Theorem 5. 1 are satisfied, but its conclusion fails for X=the closed interval $[0, 1]$. But note also that $C(X, E)$, $X = [0, 1]$, does not have the property (K) (sec. 4); it appears that assumptions of this type would enable us to extend Theorem 5. 1 to arbitrary Hausdorff compact spaces.

§ 6. E-compact spaces

In the absence of compactness of X the study of supports become more difficult. In particular, it may happen that all functions in $C(X, E)$ can be continuously extended over some extension εX of X (in fact, $C(X, E)$ may turn out to be isomorphic to $C(\varepsilon X, E)$) and homomorphisms of $C(X, E)$ may have very simple supports in εX which, however, are not contained in X. To eliminate such difficulties one needs to assume that X coincides with some of its extensions; an exact formulation of this assumption is that X is E-compact (see statement 6. 3 below). An exposition of the various facts concerning E-compact spaces and the related concept of E-completely regular spaces can be found in [4], [2], [19], [12], [13]; the purpose of the present section is to state in a concise form some information that is relevant to our discussion. Only 6. 4 is proved since its proof cannot be found in the quoted literature.

A space X is said to be E-completely regular (E-compact) provided that, for some cardinal $\mathfrak{m}$, X is homeomorphic to a subspace (a closed subspace, respectively) of some topological power $E^{\mathfrak{m}}$ of E.

6. 1. *Every structure of continuous functions $C(X, E)$, where X is an arbitrary space is isomorphic to the structure $C(X', E)$, where X' is an E-completely regular space. In fact, there is a continuous map Φ of X onto X' such that the map $\overleftarrow{\Phi}$ defined by $\overleftarrow{\Phi}(g) = g\Phi$ for every $g \in C(X', E)$ is an isomorphism of $C(X', E)$ onto $C(X, E)$.*

From now on all spaces will be assumed to be Hausdorff. An *extension* of X is a pair $(X, \varepsilon X)$, where εX is a superspace of X in which X is dense. We will usually

[8] E_1 has therefore at least one binary operation, but we do not assume that E_1 is an s-algebra.

denote $(X, \varepsilon X)$ simple by εX. Two extensions εX and $\varepsilon_1 X$ are said to be equal in the sense of extensions (in symbols: $\varepsilon X \underset{\text{ext}}{=} \varepsilon_1 X$) provided that there exists a homeomorphism h of εX onto $\varepsilon_1 X$ such that $h(p)=p$ for every $p \in X$.

6. 2. *For every E-completely regular space X there exists an (unique up to* $\underset{\text{ext}}{=}$*) E-compact extension* $\beta_E X$ *of X such that every continuous function* $f \in C(X, Y)$ *where Y is an arbitrary E-compact space, admits a continuous extension* $f^* \in C(\beta_E X, Y)$.

6. 3. *Assume that X is E-completely regular. X is E-compact if, and only if,* $\beta_E X = X$.

According to 6. 2 we can define a map Ψ of $C(X, E)$ onto $C(\beta_E X, E)$ by setting $\Psi(f)=$ the continuous extension f^* of f with $f^* \in C(\beta_E X, E)$; X being dense in $\beta_E X$ implies the uniqueness of f^*. In most cases, Ψ turns out to be an isomorphism.

6. 4. *Let E be a topological algebraic structure such that all the relations of E are E-compact. Let X be E-completely regular. The map* Ψ *defined by*
$\Psi(f)=$ *the continuous extension* $f^* \in C(\beta_E X, E)$ *of* $f \in C(X, E)$*, is an isomorphism of* $C(X, E)$ *onto* $C(\beta_E X, E)$.

Proof. That Ψ preserves the operations follows easily from the continuity of the operations. Let ϱ be a relation in E; assume for simplicity of notations that ϱ is binary. Let $f, g \in C(X, E)$, let f^* and g^* be continuous extensions of f and g, respectively, with $f^*, g^* \in C(\beta_E X, E)$. We have to show that $f\varrho^{(X)}g$ iff $f^*\varrho^{(\beta_E X)}g^*$. The "if" part is obvious. Assume $f\varrho^{(X)}g$. Define a map h of X into $E \times E$ setting $h(p)=(f(p), g(p))$ for every $p \in X$. The assumption $f\varrho^{(X)}g$ implies that, in fact, $h \in C(X, \varrho)$. Consequently, h admits a continuous extension h^* with $h^* \in C(\beta_E X, \varrho)$. In other words, $h^*(p) \in \varrho$ for every $p \in \beta_E X$. But $h_1(p)=(f^*(p), g^*(p))$ is a continuous map of $\beta_E X$ into $E \times E$ which agrees with h^* on a dense subset of $\beta_E X$, hence $h_1(p)= =h^*(p)$ for every $p \in \beta_E X$. This implies that $h_1(p)=(f^*(p), g^*(p)) \in \varrho$ (i.e., $f^*(p)\varrho g^*(p)$) for every $p \in \beta_E X$; i.e., $f^*\varrho^{(\beta_E X)}g^*$.
Recall that every subspace of a finite power $\mathscr{R}^n$ of the reals $\mathscr{R}$ is $\mathscr{R}$-compact; in other words, every finitary relation in $\mathscr{R}$ is $\mathscr{R}$-compact. Consequently, as a particular case of 6. 4 we obtain:

For every completely regular space X the structures $C(X, \mathscr{R})$ *and* $C(\beta_{\mathscr{R}} X, \mathscr{R})$ *are isomorphic relative to all pointwisely defined operations and all finitary pointwisely defined relations.*

Note. If εX is an arbitrary extension of X, then Ψ is defined only on the substructure $F_{\varepsilon X}$ of $C(X, E)$ consisting of all those functions f in $C(X, E)$ that admit an extension belonging to $C(\varepsilon X, E)$. Again, the continuity of operations implies that Ψ preserves them; however, in this case Ψ need not preserve E-compact relations. For instance, let $E=\mathscr{R}$, $X=$ the open interval $(0, 1)$, $\varepsilon X=$ the closed interval $[0, 1]$. $F_{\varepsilon X}$ consists of all uniformly continuous functions on X. Ψ does not preserve the relation $<$; in fact, there are functions $f \in F_{\varepsilon X}$ such that $f(x)>0$ for every $x \in X$, but it is not true that $f^*(x)>0$ for every $x \in \varepsilon X$. On the other hand, an argument similar to that used in the proof of 6. 4 shows that in case of an arbitrary extension εX of X, Ψ preserves all relations that are closed in the respective powers of E.
The class of all E-completely regular (E-compact) spaces will be denoted by

$\mathfrak{C}(E)$ ($\mathfrak{K}(E)$, respectively). Note that $\mathfrak{K}(E)\subset\mathfrak{C}(E)$ and $\mathfrak{K}(E)=\mathfrak{K}(E_1)$ implies $\mathfrak{C}(E)=\mathfrak{C}(E_1)$.

A space E is called *admissible* if there is a compact space E^* with $\mathfrak{C}(E)=\mathfrak{C}(E^*)$. If E is admissible, then there exists a compact superspace E_1 of E with $\mathfrak{C}(E)=\mathfrak{C}(E_1)$ (for instance, $E_1=\beta_{E^*}E$).

6. 5. *Let E be an admissible space and let E_1 be a compact superspace of E with $\mathfrak{C}(E)=\mathfrak{C}(E_1)$. An E-completely regular space X is E-compact if, and only if, the following condition is satisfied*

for every $p_0\in\beta_{E_1}X\setminus X$ there is a continuous function $f\colon\beta_{E_1}X\to E_1$ such that $f[X]\subset E$ and $f(p_0)\notin E$.

Note that the extension $\beta_E X$ depends only upon the class of compactness of E; in other words,

6. 6. *If $\mathfrak{K}(E)=\mathfrak{K}(E_1)$, then for every E-completely regular X we have $\beta_E X \underset{\text{ext}}{=} \beta_{E_1}X$.*

Let us now discuss a few examples. If $E=\mathscr{I}$ (=the unit interval [0, 1]) or if E is the space of the reals $\mathscr{R}$, then $\mathfrak{C}(E)$ is the class of all (Hausdorff) completely regular spaces. $\mathfrak{C}(\mathscr{D})$, where $\mathscr{D}$ is a two-point discrete space, is the class of all (Hausdorff) 0-dimensional spaces; in fact, $\mathfrak{C}(E)=\mathfrak{C}(\mathscr{D})$ *iff E is a 0-dimensional space containing more than one point.* $\mathfrak{K}(\mathscr{I})$ is the class of all compact spaces. $\mathfrak{K}(\mathfrak{D})$ is the class of all 0-dimensional compact spaces; in fact $\mathfrak{K}(E)=\mathfrak{K}(\mathscr{D})$ *iff E is a 0-dimensional compact space containing more than one point.* In the next section we shall frequently refer to the class $\mathfrak{K}(\mathscr{N})$[9] where $\mathscr{N}$ is the space of non-negative integers (=the discrete space of cardinality $\aleph_0$). A discrete space is $\mathscr{N}$-compact iff its cardinality is non-measurable in the Ulam sense. We have $\mathfrak{K}(E)=\mathfrak{K}(\mathscr{N})$ *iff E is $\mathscr{N}$-compact and E contains a closed copy of $\mathscr{N}$.* Every $\mathscr{N}$-compact space is 0-dimensional; every Lindelöf 0-dimensional space is $\mathscr{N}$-compact. In particular, for every 0-dimensional non-compact subspace E of the reals $\mathscr{R}$ we have $\mathfrak{K}(E)=\mathfrak{K}(\mathscr{N})$.

§ 7. Non-compact case: $F=C(X, E)$

The purpose of the present and the next section is to show how the previously obtained results can be applied to the case of an arbitrary space X. No general theorems will be proved in these two sections; however, a general procedure will be described in rough terms and then illustrated by a few theorems concerning particular structures E and E_1. In this section we shall discuss the case when F is the whole structure $C(X, E)$; the case of substructures of $C(X, E)$ will be discussed in the next section.

We shall assume that E is admissible; let E_1 be a compact superspace of E. We denote by $C^*(X, E)$ the set of all functions f from $C(X, E)$ such that $f[X]$ is contained in a compact subset of E. (If $E=$ the space of integers, then $C^*(X, E)$ consists of all bounded functions in $C(X, E)$; however, if E = the space of rational numbers, then $C^*(X, E)$ does not contain all bounded functions.) By 6. 2 every $f\in C^*(X, E)$ admits an extension $f^*\in C(\beta_{E_1}X, E)$; in most cases $C^*(X, E)$ and $C(\beta_{E_1}X, E)$ are isomorphic.

[9] This case was first mentioned in [4].

A homomorphism $\varphi: C(X, E) \to E_1$ induces a homomorphism $\varphi^*: C(\beta_{E_1}X; E) \to \to E$; φ^* is defined by $\varphi^*(g) = \varphi(g|X)$ for every $g \in C(\beta_{E_1}X; E)$. Now, $\beta_{E_1}X$ is a compact space; suppose that we are able to prove that a set $A \subset \beta_{E_1}X$ is the smallest support or the smallest weak support of φ^*. Assuming that X is E-compact, we will try to prove that $A \subset X$; here we appeal to statement 6. 5. If $A \subset X$ is proved, then A is a support of φ (or weak support) restricted to $C^*(X, E)$; the last step is to show that A is a support of the whole φ. On the other hand, if X is not E-compact, then we will try to get a negative result: to show an existence of a φ which does not have such supports as those which exists in the case of a compact or E-compact X.

We shall now illustrate the above procedure.

To start with we shall reprove a theorem due essentially to BIAŁYNICKI—BIRULA and ŻELAZKO [1] (see also [7]).

7. 1a. THEOREM. *Let B an algebra over a field K, having the unit element e (both B and K are assumed to carry the discrete topology). If X is K-compact, then every homomorphism $\varphi: C(X, B) \to K$ has a one point support.*

PROOF. Assume first that X is a two-point space, $X = \{p_1, p_2\}$. If there is a homomorphism $\varphi: C(X, B) \to K$ such that none of the points p_i is a support of φ, then there are four functions f_i, g_i, $i = 1, 2$, such that $f_i(p_i) = g_i(p_i)$ and $\varphi(f_i) \neq \neq \varphi(g_i)$ for $i = 1, 2$. The function $f = (f_1 - g_1)(f_2 - g_2)$ is identically equal to 0, but $\varphi(f) = (\varphi(f_1) - \varphi(g_1))(\varphi(f_2) - \varphi(g_2)) \neq 0$ which is impossible. Thus, the conclusion of the theorem is satisfied for a two point space X; consequently, by Theorem 5. 1, if X is a 0-dimensional compact space, then every $\varphi: C(X, B) \to K$ has a one-point weak support. But B is discrete, hence by 4. 10, φ has a one-point support.

If K is finite, then the theorem is shown; in fact, in this case being K-compact is equivalênt to X being 0-dimensional Hausdorff compact. Assume therefore that K is infinite and let X be a K-compact space. We shall assume that K is contained in B. Let e be the unit element of B; e is also the unit element of K; let $C_0(X, K)$ denote the set of all constant functions $f: X \to K$. For every $k \in K$ we denote by $f^{(k)}$ the constant function on X whose value is k. We can assume that

$$(1) \qquad \varphi(f^{(k)}) = k \quad \text{for every} \quad k \in K;$$

indeed, φ restricted to $C_0(X, K)$ induces in a natural way an endomorphism of K, say α; this endomorphism does not vanish identically $(\varphi(f^{(e)}) \neq 0$; for otherwise $\varphi(f) = 0$ for every $f \in C(X, B))$, hence α is one-to-one; compose φ with α^{-1}. Clearly, if $\alpha^{-1} \circ \varphi$ has a one-point support then φ has also.

Let K_1 be the one-point compactification of K; K_1 is a compact superspace of K with $\mathfrak{C}(K_1) = \mathfrak{C}(K) = \mathfrak{C}(B)$. $C^*(X, B)$ consists of all functions in $C(X, B)$ having finitely many values. Each function $f \in C^*(X, B)$ admits a continuous extension $f^* \in C(\beta_{K_1}X, B)$. Let us set $\varphi^*(g) = \varphi(g|X)$ for every $g \in (\beta_{K_1}X, B)$ and, by the first part of the proof, φ^* has a one-point support $\{p_0\}$ in $\beta_{K_1}X$. We shall prove that $p_0 \in X$.

Assume that $p_0 \in \beta_{K_1}X \setminus X$. There is continuous function $g_0: \beta_{K_1}X \to K_1$ such that $g_0[X] \subset K$ and $g_0(p_0) = \infty$ (where ∞ is the ideal point of the one point compactification K_1 of K). Let $f_0 = g_0|X$; clearly $f_0 \in C(X, B)$. Let $k_0 = \varphi(f_0)$. There is a neighborhood U of p_0 such that $g_0(p)\neq k_0$ for every $p\in U$. Let $A=\{p\in\beta_{K_1}X: g_0(p)=k_0\}$; we have $A\cap U=\emptyset$. Take a $k_1\in K$ with $k_1\neq k_0$ and set $g_1(p)=k_1$ for $p\in A$ and $g_1(p)=k_0$ for $p\in\beta_{K_1}X\setminus A$. $g_1\in C(\beta_{K_1}X, B)$ and from (1) we infer that $\varphi^*(g_1)=k_0$. Setting $f_1=g_1|X$, we have $\varphi(f_1)=k_0$, consequently, $\varphi(f_0-f_1)=0$; but $(f_0-f_1)(p)\in K$ and $(f_0-f_1)(p)\neq 0$ for every $p\in X$; therefore f_0-f_1 has an inverse in $C(X, B)$. This contradicts the fact that $\varphi(f_0-f_1)=0$; hence $p_0\in X$.

It follows from the above that $\{p_0\}$ is a support of φ restricted to $C^*(X, B)$. Let f_1 and f_2 be two arbitrary functions in $C(X, B)$ with $f_1(p_0)=f_2(p_0)$. Let $A=\{p\in X: f_1(p)=f_2(p)\}$; A is a closed-open subset of X. Set $f_3(p)=e$ for $p\in A, f_3(p)=0$ for $p\in X\setminus A$. Then $f_3\in C^*(X, B)$, therefore $\varphi(f_3)=\varphi(f^{(e)})=e$. On the other hand, the function $(f_1-f_2)f_3$ is identically equal to 0, therefore $\varphi(f_1-f_2)\cdot\varphi(f_3) = \varphi((f_1-f_2)\cdot f_3) = 0$; therefore $\varphi(f_1-f_2) = 0$; thus $\varphi(f_1) = \varphi(f_2)$. Consequently, $\{p_0\}$ is a support of φ.

The following is the converse of 7. 1.a.

7. 1. b. THEOREM. *Let K be a field with the discrete topology. If X is not K-compact, then there exists a homomorphism φ: $C(X, K)\to K$ which does not have a one-point support in X.*

PROOF. Every function $f\in C(X, K)$ admits an extension $f^*\in C(\beta_K X, K)$; take a point $p_0\in\beta_K X\setminus X$ (note that $\beta_K X\neq X$) and let $\varphi(f)=f^*(p_0)$.

Theorem 7. 1.a and 7. 1.b such be compared with the results of [1] (or with a more general version of these results given in [7]). If K is finite, then (as it was already observed) X is K-compact iff X is compact; hence, in this case, a discrete X is K-compact iff X is finite. If $\aleph_0\leqq \text{card}\, K<\aleph_I$, where $\aleph_I$ is the first measurable cardinal (in the Ulam sense), then X is K-compact iff X is N-compact; hence, in this case, a discrete X is K-compact iff card $X<\aleph_I$. In general, setting $m=\text{card}\, K$, we have that a discrete X is K-compact iff card $X<\aleph(m)$. $\aleph(m)$ is used here in the sense of [7].

Theorem 7. 1.a is not the best one. The proof shows that this theorem remains valid if K is integral domain satisfying the condition

(2) *for every space X and every non-constant homomorphism φ: $C(X, K)\to K$, if $f\in C(X, K)$ and $f(p)\neq 0$ for every $p\in X$, then $\varphi(f)\neq 0$.*

It has been shown in [11] that the ring of integers satisfies (2) (see [11], § 5, (v)). Consequently, Theorem 7. 1.a is true if K is the ring of integers. (The last statement is more general than Theorem 2 in [11].)

REMARK 1. Condition (2) obviously implies the following one

(3) *for every non-constant endomorphism α of K we have $\alpha(k)\neq 0$ for every $k\in K, k\neq 0$.*

We do not know if (3) implies (2). It is easy to see that (3) is equivalent to

(3a) *every endomorphism α of K can be extended to an endomorphism $\tilde{\alpha}$ of $\tilde{K}$, where $\tilde{K}$ is the field of quotients of K.*

Similarly, (2) is equivalent to

(2a) *for every space X, every homomorphism $\varphi: C(X, K) \to K$ can be extended to a homomorphism $\tilde{\varphi}: C(X, \tilde{K}) \to \tilde{K}$, where $\tilde{K}$ is the field of quotients of K.*

We shall now discuss the case where $E = E_1$ is an ordered subgroup of the reals $\mathscr{R}$. In other words, we shall discuss maps φ of $C(X, E)$ into E. That preserve $+$ and $\leqq$. If $E = \mathscr{R}$, then such maps coincide with integrals (in the case of compact, or more generally, $\mathscr{R}$-compact, X); consequently, they need not to have finite supports. In contrast to this we shall show

7. 2. a. THEOREM. *If E is a proper ordered subgroup of the additive group of the reals $\mathscr{R}$ and X is an $\mathscr{N}$-compact space, then every homomorphism $\varphi: C(X, E) \to E$ has a finite support.*

One can assume without the loss of generality that E contains the number 1. This assumption will be kept throughout the following discussion.

The above theorem for the case of a $\mathscr{D}$-compact X has been announced in [10]. We shall start with the proof of this particular case. We need the following:

7. 3. LEMMA. *Let E be a subgroup of the additive group of the reals. Assume that there is a sequence $\alpha_1, \alpha_2, \dots$ of positive numbers such that*

(a) $\sum_n \alpha_n < +\infty$

and

(b) *for every sequence $x_n \in E$ with $x_n \to 0$ we have $\sum_n \alpha_n x_n \in E$.*

Then $E = \mathscr{R}$.

PROOF. Let $x \in E$, $x \neq 0$. Then $\alpha_n x \in E$ for $n = 1, 2, \dots$; hence E contains a sequence convergent to 0, therefore E is dense in $\mathscr{R}$.

Let c be an arbitrary real. By induction one can define a sequence $x_1, x_2, \dots$ of elements of E such that

(4) $$|\alpha_1 x_1 + \dots + \alpha_n x_n - c| < \min\left\{\frac{1}{2n}\alpha_n, \frac{1}{2(n+1)}\alpha_{n+1}\right\}.$$

Clearly, $\sum_n \alpha_n x_n = c$. It remains to show that $x_n \to 0$. We shall show that $|x_{n+1}| < \frac{1}{n+1}$ for $n = 1, 2, \dots$. Let $c_n = \alpha_1 x_1 + \dots + \alpha_n x_n - c$; we have

$$|c_n| < \min\left\{\frac{1}{2n}\alpha_n, \frac{1}{2(n+1)}\alpha_{n+1}\right\}.$$

Now

$$|x_{n+1}| = \frac{1}{\alpha_{n+1}}|\alpha_{n+1}x_{n+1}| = \frac{1}{\alpha_{n+1}}\left(|\alpha_{n+1}x_{n+1}| - |c_n| + |c_n|\right) \leqq$$

$$\leqq \frac{1}{\alpha_{n+1}}\left(|c_n + \alpha_{n+1}x_{n+1}| + |c_n|\right) = \frac{1}{\alpha_{n+1}}\left(|c_{n+1}| + |c_n|\right) \leqq$$

$$\leqq \frac{1}{\alpha_{n+1}}\left(\frac{1}{2(n+1)}\alpha_{n+1} + \frac{1}{2(n+1)}\alpha_{n+1}\right) = \frac{1}{\alpha_{n+1}} \cdot \frac{\alpha_{n+1}}{n+1} = \frac{1}{n+1}.$$

Proof of Theorem 7. 2.a *for a $\mathcal{D}$-compact X*. Let X be $\mathcal{D}$-compact (i.e., 0-dimensional and compact) and let $\varphi\colon C(X, E)\to E$ be a given homomorphism. By remarks after Theorem 4. 2, φ has a smallest weak support A. Let $\underset{(1)}{\to}$ be the uniform convergence of sequences in $C(X, E)$; by 4. 9, $\underset{(1)}{\to}$ satisfies condition (1) of §4. Clearly, φ is continuous relative to $\underset{(1)}{\to}$ and the usual convergence in E; consequently, A is the smallest support of φ. It remains to show that A is finite.

Assume A is infinite. There is a sequence $U_1, U_2, \ldots$ of mutually disjoint closed-open subsets of X with $U_n \cap A \neq \emptyset$ for $n=1, 2, \ldots$. Set $f_n(p)=1$ for $p\in U_n$ and $f_n(p)=0$ for $p\in X\setminus U_n$. Let $\alpha_n=\varphi(f_n)$. We have $\alpha_n\in E$ and $\alpha_n>0$ (if $\alpha_n=0$, then $X\setminus U_n$ would be a support of φ). On the other hand, $\varphi(f_1+\cdots+f_n)\leqq\varphi(g)$, where g is the function identically equal to 1; therefore the series $\sum_n \alpha_n$ is convergent. Let $x_1, x_2, \ldots$ be an arbitrary sequence of elements of E with $x_n\to 0$. The function f, defined by $f(p)=x_n$ for $p\in U_n$ and $f(p)=0$ for $p\in X\setminus\bigcup\{U_n\colon n=1, 2, \ldots\}$, belongs to $C(X, E)$; moreover, $f=\sum_n x_n\cdot f_n$, the convergence of the series being uniform. It follows that $\sum_n \alpha_n\cdot x_n=\sum_n x_n\cdot\varphi(f_n)=\sum_n \varphi(x_n\cdot f_n)=\varphi(f)\in E$; consequently, by Lemma 7. 3, $E=\mathcal{R}$, contrary to the assumption.

To complete the proof we need still two lemmas.

7. 4. Lemma. *For every $f\in C(X, E)$ there is a sequence $g_1, g_2, \ldots$ of functions from $C(X, E)$ such that each g_n has only finitely many values and the set of functions $nf-g_n$, $n=1, 2, \ldots$, is bounded in $C(X, E)$ (i.e., there is an $h\in C(X, E)$ such that $|nf-g_n|\leqq h$ for every n).*

Proof. Select a sequence of numbers $0<a_1<a_2<\ldots$ such that $a_n\notin E$ and $a_n\to\infty$. For every n select a $b_n\in E$ with $a_n^2<b_n$. Let $A_1=\{p\in X\colon |f(p)|<a_1\}$ and $A_n=\{p\in X\colon a_{n-1}<|f(p)|<a_n\}$ for $n=2, 3, \ldots$. The sets A_n are closed and open and $\bigcup_n A_n=X$. Define $h(p)=b_n+2$ for $p\in A_n$. Clearly, $h\in C(X, E)$ and

$$f^2(p)+2<h(p) \quad \text{for every} \quad p\in X.$$

Now, for a given n select $\alpha_0<\alpha_1<\ldots<\alpha_s$ so that

$$\alpha_0\leqq -n^2<n^2\leqq\alpha_s, \quad 1<\alpha_{i+1}-\alpha_i<2, \quad \alpha_i\setminus E.$$

Since $\alpha_{i+1}-\alpha_i>1$ (and $1\in E$), we can find $\beta_i\in E$ with $\alpha_i<\beta_i<\alpha_{i+1}$ for $i=0, 1, \ldots, s-1$. Set $B_i=\{p\in X\colon \alpha_i<nf(p)<\alpha_{i+1}\}$ for $i=0, 1, \ldots, s-1$. B_i are closed and open; the set $B=\bigcup\{B_i\colon i=0, \ldots, s-1\}$ is also closed and open. Set

$$g_n(p)=\beta_i \quad \text{for} \quad p\in B_i, \quad g_n(p)=0 \quad \text{for} \quad p\in X\setminus B.$$

We then have

$$|nf(p)-g_n(p)|\leqq h(p) \quad \text{for every} \quad p\in X.$$

Indeed, if $p\in B_i$ (for some i), then

$$|nf(p)-g_n(p)|\leqq\alpha_{i+1}-\alpha_i<2\leqq h(p);$$

on the other hand, if $p\in X\setminus B$, then $|nf(p)|\geqq n^2$, hence $|f(p)|\geqq n$, therefore $|nf(p)\leqq \leqq f^2(p)\leqq h(p)$.

We shall now consider additive maps of $C(X, E)$ into E that are bounded (i.e., they carry bounded sets of functions in $C(X, E)$ into bounded sets of numbers).

Every homomorphism of $C(X, E)$ into E is an additive bounded map; the difference of two additive bounded maps is again an additive bounded map.

7. 5. LEMMA. *Let $C^{**}(X, E)$ be the set of all functions in $C(X, E)$ that have only finitely many values. If two additive bounded maps of $C(X, E)$ into E agree on $C^{**}(X, E)$, then they agree everywhere on $C(X, E)$.*

PROOF. It suffices to show that if an additive bounded map ψ of $C(X, E)$ into E vanishes on $C^{**}(X, E)$, then ψ vanishes everywhere. Let f be an arbitrary function in $C(X, E)$. By Lemma 7. 4; there exists a sequence $g_1, g_2, \ldots$ of functions from $C^{**}(X, E)$ such that the set $nf - g_n$, $n=1, 2, \ldots,$ is bounded. Consequently, the set of numbers $\psi(nf-g_n)$, $n = 1, 2, \ldots,$ is bounded. But $\psi(g_n) = 0$, hence $\psi(nf-g_n) = n\psi(f)$; therefore $\psi(f)=0$.

Proof of Theorem 7. 2a *for the general case*. Recall the material of §6 and the remarks at the beginning of the present section. Let E_1 be a 0-dimensional compact superspace of E. We have $\beta_{E_1}X \underset{\text{ext}}{=} \beta_{\mathscr{D}}X$. Let φ be a homomorphism of $C(X, E)$ into E; we can assume that φ does not vanish identically. Since $C^*(X, E)$ is isomorphic to $C(\beta_{\mathscr{D}}X, E)$ (and the theorem is true in the compact case), we infer that φ restricted to $C^*(X, E)$ has a finite support A contained in $\beta_{\mathscr{D}}X$. Let $A = \{p_1, \ldots, p_k\}$. It is clear that we have

$$\varphi(f) = \alpha_1 f^*(p_1) + \cdots + \alpha_k f^*(p_k) \quad \text{for every} \quad f \in C^*(X, E) \tag{5}$$

where $\alpha_1, \ldots, \alpha_k$ are fixed numers and f^* denotes the continuous extension of f over $\beta_{\mathscr{D}}X$. We can assume that all α_i are positive.

We shall prove that $A \subset X$. Indeed, assume that $p_{i_0} \in \beta_{\mathscr{D}}X \setminus X$. Since X is $\mathscr{N}$-compact, there is a continuous function $f_0^*: \beta_{\mathscr{D}}X \to \mathscr{N}^*$ ($\mathscr{N}^* = \mathscr{N} \cup \{\infty\}$ is the one-point compactification of $\mathscr{N}$) such that $f_0^*(p_{i_0}) = \infty$ and $f_0^*(p) \in \mathscr{N}$ for every $p \in X$; see 6. 5. Clearly, it can be assumed that $f_0^*(p_i) = 0$ for $i \neq i_0$. Let $f_0 = f_0^*|X$; we have (in view of the assumption $1 \in E$) $f_0 \in C(X, E)$. Let $f_0^{(n)} = f_0 \wedge n$ for $n = 1, 2, \ldots$. Clearly, $f_0^{(n)} \in C^*(X, E)$, hence, from (5) we infer that $\varphi(f_0^{(n)}) = \alpha_{i_0} \cdot n$. But $0 \leqq f_0^{(n)} \leqq f_0$, therefore $0 \leqq \varphi(f_0^{(n)}) \leqq \varphi(f_0)$ for $n = 1, 2, \ldots$; and this implies that, contrary to the assumption, $\alpha_{i_0} = 0$. Thus $A \subset X$.

Knowing that $A \subset X$ we can rewrite (5) as follows

$$\varphi(f) = \alpha_1 f(p_1) + \cdots + \alpha_k(p_k) \quad \text{for every} \quad f \in C^*(X, E). \tag{6}$$

It suffices to show that (6) holds for every $f \in C^*(X, E)$. This, however, follows immediately from Lemma 7. 5. Indeed, the left-hand side of (6) defines a homomorphism of $C(X, E)$ which agrees with φ on $C^{**}(X, E)$ (in fact, on $C^*(X, E)$). Therefore the left-hand side of (6) agrees with φ everywhere on $C(X, E)$.

Theorem 7. 2a is shown.

The converse of Theorem 7. 2a is obvious.

7. 2b. THEOREM. *If X is not E-compact, then there exists a homomorphism $\varphi: C(X, E) \to E$ without a finite support.*

PROOF. It suffices to set

$$\varphi(f) = f^*(p_0) \quad \text{for every} \quad f \in C(X, E),$$

where p_0 is a fixed point of $\beta_E X \setminus X$ and f^* denotes the continuous extension of f with $f^*\colon \beta_E X \to E$. It is clear that no compact subset of X is a support of φ.

As a still another example of the above procedure one could mention a generalization of a result of Turowicz due to R. C. Moore. TUROWICZ [20] considers multiplicative functionals $\varphi\colon C(X, \mathscr{R}) \to \mathscr{R}$ that are continuous with respect to the uniform convergence and proves that if X is compact, then every such functional has a countable support — in fact, Turowicz obtains a representation formula for such functionals.[10] R. C. MOORE [6] proves that every such functional has a countable compact support in X (and hence is representable in Turowicz's form) iff X is $\mathscr{R}$-compact.[11]

In [2] BLEFKO proves a result[12] related to Theorems 7.1a and 7.1b and Theorem 2 in [11].

7. 6. THEOREM (BLEFKO). *Let $\mathscr{P}$ be the ring of rationals with the standard topology. Every homomorphism $\varphi\colon C(X, \mathscr{P}) \to \mathscr{P}$ has a one-point support in X if, and only if, X is $\mathscr{N}$-compact.*

The above seems to be the only result concerning a non-locally compact structure.

§ 8. Non-compact case: $F \subset C(X, E)$

When dealing with substructures F of $C(X, E)$ it can always be assumed that F separates points and closed sets of X. A formal statement to this effect is as follows. Let $f_1, \ldots, f_k$ be functions from X into E. We denote by $\langle f_1, \ldots, f_k \rangle$ the map of X into the product E^k whose value at a point $p \in X$, $\langle f_1, \ldots f_k \rangle(p)$, is the point $(f_1(p), \ldots, f_k(p))$ of E^k. A class F of continuous functions from X into E is called an *E-separating* class for X provided that for every closed set $A \subset X$ and every point $p \in X \setminus A$ there is a finite number of functions $f_1, \ldots, f_k$ from F such that $\langle f_1, \ldots, f_k \rangle(p) \notin \mathrm{cl}\, \langle f_1, \ldots, f_k \rangle [A]$, where cl denotes the closure in E^k. The following statement is a generalization of 6. 1.

8. 1. *Let $F \subset C(X, E)$. There exists an E-completely regular space X' and a continuous map Φ of X onto X' such that every $f \in F$ can be (uniquely) written in the form $f = g \circ \Phi$; furthermore, the class F' of all those $g \in C(X', E)$ for which $g \circ \Phi \in F$ is an E-separating class for X'.*

Thus, if we let (as in 6. 1) $\overleftarrow{\Phi}(g) = g \circ \Phi$ for every $g \in F'$, then $\overleftarrow{\Phi}$ is a one-to-one map of F' into F and obviously $\overleftarrow{\Phi}$ is an isomorphism relative to pointwisely defined operations and relations. In other words, F is isomorphic to an E-separating structure.

In the preceding section when studying the whole structure $C(X, E)$ we used certain relation between X and one of the maximal compactifications (statement 6. 5).

[10] Turowicz has formulated his result only for the case of a compact metric X. However, in [3], BOURGIN shows that the same procedure can be applied in case of arbitrary compact (Hausdorff) spaces.

[11] This result has been announced in [15].

[12] This result has been announced in [14].

Sometimes this procedure can be applied also to substructures of $C(X, E)$. For some substructures F of $C(X, E)$ it is possible to assign a compactification cX of X such that all homomorphisms of F have support of a certain type in X iff certain relation holds between X and cX. This procedure was applied in [8] to substructures of $C(X, \mathscr{R})$, where $\mathscr{R}$ is the ring of the reals; let us briefly recall the known facts.

If X is compact, then all homomorphisms of the ring $C(X, \mathscr{R})$ into $\mathscr{R}$ have one-point support. A subset P of a space X is said to be Q-closed in X provided that for every $p_0 \in X \setminus P$ there is a continuous function $f: X \to [0, 1]$ such that $f(p_0)=0$ and $f(p)>0$ for every $p \in P$. For an arbitrary (completely regular) space X all homomorphisms of the ring $C(X, \mathscr{R})$ into $\mathscr{R}$ have one-point supports in X iff X is Q-closed in βX. Consider now subrings F of $C(X, \mathscr{R})$ such that (a) F contains all constant functions on X, (b) F is inverse closed (i.e., if $f \in F$ and $f(p) \neq 0$ for every $p \in X$, then $1/f \in F$), and (c) F is closed with respect to uniform convergence. It was shown in [8] (Theorem 2) that to each subring F satisfying the above conditions it is possible to assign a compactification cX of X such that all homomorphisms of F have one-point supports in X iff X is Q-closed in cX. The compactification cX can be defined, for instance, as the smallest compactification such that all bounded functions in F can be continuously extended over cX.[13] It was shown in [9] that similar theorems hold true for some linear sublattices of $C(X, \mathscr{R})$.

In this section we shall give still another illustration of the above procedure. We shall obtain results paralleling those of [8] but concerning some subrings of $C(X, \mathscr{Z})$, where $\mathscr{Z}$ is the ring of integers (homomorphisms of the whole ring $C(X, \mathscr{Z})$ have been studied in [11]). These results, in turn, will be applied to obtain a characterization of the class of strongly non-measurable cardinals in the Ulam sense (see [12]).

8. 2a. THEOREM. *Let F be a subring of $C(X, \mathscr{Z})$ satisfying the following conditions:*

(a) *F contains all constant functions.*

(b) *$f \in F$ iff all truncations of f belong to F;*[14]

(c) *F is closed under composition with functions $\alpha: \mathscr{Z} \to \mathscr{Z}$ (i.e., for every $f \in F$ and for every $\alpha: \mathscr{Z} \to \mathscr{Z}$, the composition $\alpha \circ f$ belongs to F);*

(d) *F is $\mathscr{Z}$-separating.*

Let cX be the smallest compactification such that every function f in F admits a continuous extension $f^*: cX \to \mathscr{Z} \cup \{\pm\infty\}$. If X is Q-closed in cX, then every homomorphism $\varphi: F \to \mathscr{Z}$ has a one-point support in X.

The proof of this theorem is almost identical with that of Theorem 2 in [11]; let us only discuss the necessary changes. The compactification cX is 0-dimensional;[15]

[13] cX can also be defined as the smallest compactification such that every function f in F admits a continuous extension $f^*: cX \to \mathscr{R} \cup \{\pm\infty\}$, where $\mathscr{R} \cup \{\pm\infty\}$ is the (unique) two-point compactification of $\mathscr{R}$.

[14] The i-th truncation of f is defined by $f^{(i)} = -i \vee (f \wedge i)$.

[15] It is useful to formulate a general statement concerning such compactifications.

Let E be a compact space, let X be an E-completely regular space, and let F be an E-separating class for X.

(a) *There exists the smallest compactification cX of X having the property*

(i) *evevy $f \in F$ admits a continuous extension $f^*: cX \to E$.*

(b) *This compactification cX is E-completely regular.*

hence if $p_0 \in cX \setminus X$, then there is a continuous function $g: cX \to [0, 1]$ such that $g(p_0) = 0$ and $g(p) > 0$ for every $p \in X$. Using 0-dimensionality of cX we can modify g so that its values on X are of the form $1/n$. Taking the reciprocal of g we obtain a continuous function $f^*: cX \to \mathscr{X} \cup \{\pm\infty\}$ with $f^*(p_0) = \infty$ and $0 < f(p) < +\infty$ for every $p \in X$. It is now clear that the considerations of [11] can be applied if we shall show that F contains all functions f from $C(X, \mathscr{X})$ that admit continuous extensions $f^*: cX \to \mathscr{X} \cup \{\pm\infty\}$. This will be accomplished in the following two lemmas.

8. 3. LEMMA. *Let X be a compact space and let F be a subring of $C(X, \mathscr{X})$ that satisfies* (a) *and* (c) *of Theorem* 8. 2a. *If F distinguishes points of X (i.e., if for every $p, q \in X$ with $p \neq q$ there is an $f \in F$ with $f(p) \neq f(q)$), then $F = C(X, \mathscr{X})$.*

PROOF. A straightforward compactness argument shows that for each pair of disjoint closed subsets A and B of X there is an $f \in F$ with $f(p) = 0$ for $p \in A$ and $f(p) = 1$ for $p \in B$. Let g be an arbitrary function from $C(X, \mathscr{X})$; let $k_1, \dots, k_n$ be all the values of g. Let $A_i = g^{-1}[k_i]$. There are functions $f_1, \dots, f_n \in F$ such that $f_i(p) = 0$ for $p \in \bigcup \{A_j: j < i\}$ and $f_i(p) = 1$ for $p \in \bigcup \{A_j: j \geqq i\}$. Let $f = f_1 + \dots + f_n$. We have $f \in F$ and $f(p) = j$ for $p \in A_j$. It suffices to compose f with a function $\alpha: \mathscr{X} \to \mathscr{X}$ such that $\alpha(j) = k_j$ for $j = 1, 2, \dots, n$.

8. 4. LEMMA. *Under the notations and the assumptions of Theorem* 8. 2a, *F contains all functions f on X that admit continuous extensions $f^*: cX \to \mathscr{X} \cup \{\pm\infty\}$.*

PROOF. Let F^* be the set of all bounded functions in F. It follows directly from condition (d) that the class of all continuous extensions of members of F^* over cX distinguishes points of cX (use also footnote[15]). Consequently, by the preceding lemma, F^* contains all bounded function from $C(X, \mathscr{X})$ that admit continuous extensions over cX. The lemma now follows directly from condition (b).

Note that in the converse of Theorem 8. 2a we can relax the condition on F.

8. 2b. THEOREM. *Let F be an arbitrary subring of $C(X, \mathscr{X})$ that is $\mathscr{X}$-separating and let cX be defined as in* 8. 2a. *If X is not Q-closed in cX, then F admits a homomorphism $\varphi: F \to \mathscr{X}$ which does not have a one-point support in X.*

PROOF. There is a point $p_0 \in cX \setminus X$ such that for no continuous function $g: cX \to [0, 1]$ it is true that $g(p_0) = 0$ and $g(p) > 0$ for every $p \in X$. It is clear that for every continuous extension $f^*: cX \to \mathscr{X} \cup \{\pm\infty\}$ of an $f \in F$ we have $f^*(p_0) \in \mathscr{X}$. Consequently, the formula $\varphi(f) = f^*(p_0)$ for every $f \in F$ defines a homomorphism of F into $\mathscr{X}$. Clearly, φ does not have a compact support in X.

We are now ready to give the characterization of the class $\mathscr{X}$ of strongly non-measurable cardinals (see [12]).

(c) *This compactification cX can also be characterized as the compactification having property* (i) *and the following one*

(ii) *for every $p, q \in cX \setminus X$, if $p \neq q$, then there is an $f \in F$ such that $f^*(p) \neq f^*(q)$, where f^* is the continuous extension of f with $f^*: cX \to E$.*

(Note that the implication in (ii) holds for every $p, q \in cX$).

Verification of the above statement is routine.

In the proof of Theorem 8. 2a we apply this statement with $E = \mathscr{X} \cup \{\pm\infty\}$.

8. 5. THEOREM. *Let* $\mathfrak{m}$ *be a cardinal satisfying* $\mathfrak{m}^{\aleph_0}=\mathfrak{m}$ *and let* $X_{\mathfrak{m}}$ *be a discrete space of cardinality* $\mathfrak{m}$. *The following are equivalent*

(a) $\mathfrak{m}\in\mathcal{M}$;

(b) *there is a subring F of* $C(X_{\mathfrak{m}}, \mathcal{Z})$ *such that F is* $\mathcal{Z}$*-separating, every homomorphism* $\varphi\colon F\to\mathcal{Z}$ *has a one-point support in* $X_{\mathfrak{m}}$, *and card* $F=\mathfrak{m}$.

PROOF. Let $\mathfrak{m}\in\mathcal{M}$. By Theorems 4. 1 and 5. 1 in [12], there is a class H of continuous functions $h\colon\beta X_{\mathfrak{m}}\to[0,1]$ such that $h(p)>0$ for every $p\in X_{\mathfrak{m}}$ and every $h\in H$ and for every $p\in\beta X_{\mathfrak{m}}\setminus X_{\mathfrak{m}}$ there is an $h\in H$ with $h(p)=0$; furthermore, card $H=\mathfrak{m}$. Using 0-dimensionality of $\beta X_{\mathfrak{m}}$ we can assume that all the functions h in H have values of the form $1/n$ on $X_{\mathfrak{m}}$. Let F_0 be the class of the reciprocals of the restrictions of members of H to $X_{\mathfrak{m}}$; let F_1 be an arbitrary $\mathcal{Z}$-separating class for $X_{\mathfrak{m}}$ with card $F_1=\mathfrak{m}$. Let F be the smallest subring of $C(X_{\mathfrak{m}}, Z)$ containing $F_0\cup F_1$ and satisfying conditions (a), (b), and (c) of Theorem 8. 2a. From $\mathfrak{m}^{\aleph_0}=\mathfrak{m}$ we infer that card $F\leqq\mathfrak{m}$. It is easy to see that $X_{\mathfrak{m}}$ is Q-closed in the corresponding compactification $cX_{\mathfrak{m}}$ of $X_{\mathfrak{m}}$. Consequently, the conclusion follows directly from Theorem 8. 2a.

Conversely, assume that (b) is satisfied. Let $cX_{\mathfrak{m}}$ be the compactification corresponding to F. By Theorem 8. 2b, $X_{\mathfrak{m}}$ is Q-closed in $cX_{\mathfrak{m}}$. From card $F=\mathfrak{m}$ we infer that $cX_{\mathfrak{m}}$ has a base of cardinality $\mathfrak{m}$; in fact, the class of all continuous extensions $f^*\colon cX_{\mathfrak{m}}\to Z\cup\{\pm\infty\}$ of functions $f\in F$ is a $\mathcal{Z}\cup\{\pm\infty\}$-separating class for $cX_{\mathfrak{m}}$. Consequently, by Theorem 5. 1 in [12], $\mathfrak{m}\in\mathcal{M}$.

It is easy to see that if the cardinal $\mathfrak{m}$ in the above theorem is of the form $\mathfrak{m}=2^{\mathfrak{n}}$, then we can find a ring F satisfying (b) which is closed relative to any system of $\mathfrak{m}$ operations each having $\leqq\mathfrak{n}$ arguments.

Theorem 8. 5 was announcend in [12]. As it was pointed out in [12], a similar theorem can be proved for subrings of $C(X_{\mathfrak{m}}, \mathcal{R})$ (where $\mathcal{R}$ is the ring of the reals). In general, with the aid of the class **M** one can prove for various structures E the existence of substructures F of $C(X_{\mathfrak{m}}, E)$ (i.e., of direct products of copies of E) such that F has essentially the same homomorphisms into E as $C(X_{\mathfrak{m}}, E)$ but F is not isomorphic to any $C(X, E)$. Furthermore, for sufficiently large Ulam non-measurable cardinals, F can assumed to be closed relative to large systems of operations of huge numbers of arguments. This indicates the impossibility of axiomatic description of direct products of E by means of formulas (of possibily infinite length) involving only elements and homomorphisms of $C(X_{\mathfrak{m}}, E)$, provided that the number of these formulas and their length is Ulam non-measurable. More remarks on this subject will be published later.

§ 9. Concluding remarks

In Section 7 we used the substructure $C^*(X, E)$ to reduce the study of supports to the compact case. Sometimes a different procedure is possible. If E admits a compact superstructure E^*, then $C(X, E)$ is isomorphic to a substructure of $C(\beta_{E^*}X, E^*)$. The same is true for substructures of $C(X, E)$. We can therefore use $C(\beta_{E^*}X, E)$ to reduce the study of supports to the compact case. This procedure can be used, for instance, when $C(X, E)$ is considered as a lattice of continuous functions with values in a chain E; indeed every chain E can be extended to a compact chain. In fact, this procedure has been used implicitly by several authors in the study of homomorphisms of lattices of continuous functions. The author plans to publish a paper containing further applications; it will be shown that results similar to those discussed in the preceding section can also be obtained for some sublattices of $C(X, \mathscr{R})$.

Representation theorems for homomorphisms frequently lead to the so-called "homeomorphism theorems". The first such theorem is due to Banach: if X and Y are compact metric spaces and $C(X, \mathscr{R})$ and $C(Y, \mathscr{R})$ are isomorphic as Banach spaces, then X and Y are homeomorphic. We shall say that a structure E is (topologically) *determining*, provided that for every E-compact spaces X and Y the isomorphism of $C(X, E)$ and $C(Y, E)$ implies the homeomorphism of X and Y. It follows from 6. 4 that *if the relations of E are E-compact, then the class of all E-compact spaces is a maximal class of spaces in which the above implication may hold.* There is a group of theorems asserting that the various structures on the set $\mathscr{R}$ of the reals are determining. Perhaps the best known is the one in which $\mathscr{R}$ is considered as a ring; at the same time, this is the weakest theorem in this direction. In fact, if Φ is a ring-isomorphism between $C(X, \mathscr{R})$ and $C(Y, \mathscr{R})$, then Φ is an isomorphism relative to all pointwisely defined operations and relations. The strongest out of presently known theorems is the one where $\mathscr{R}$ is considered as a lattice. It would be interesting to see whether this is, in fact, the strongest possible theorem in this direction. The question can be formulated as follows. Suppose that $E=\{\mathscr{R}; \{0_0, \ldots, 0_\xi, \ldots\}_{\xi<\alpha}; \{\varrho_0, \ldots, \varrho_\eta\}_{\eta<\beta}\}$ is a determining structure on the reals $\mathscr{R}$. Is it true that every isomorphism between $C(X, E)$ and $C(Y, E)$ is, in fact, a lattice-isomorphism?

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