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ON A PAPER OF SZELE AND SZENDREI ON GROUPS WITH COMMUTATIVE ENDOMORPHISM RINGS

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In this journal, twenty years ago, the late T. SZELE and J. SZENDREI characterized the major classes of abelian groups whose endomorphism ring is commutative [4]. In this paper, I extend the characterization by giving necessary and sufficient conditions for a mixed group to have a commutative endomorphism ring, and point out some of the difficulties involved in extending it further.

1. Notation

By a *group*, I shall always mean an additively written abelian group. If G, H are groups, Hom (G, H) means the group of homomorphisms of G into H; End (G) means the group of endomorphisms of G; and $\mathscr{E}(G)$ means the ring of endomorphisms of G.

If p is a prime number, and G a group, p is called a *relevant prime for* G if G has non-zero p-primary component. The p-primary component of a group G is denoted by G_p. If H is a subgroup of G, and $p^n H = H \cap p^n G$ for all integers n, then H is called *p-pure in* G. If S is a set of primes, H is *S-pure in* G if H is p-pure in G for all $p \in S$. If $a \in G$, the *p-height of a in* G, denoted by $h(a)(p)$, is the largest nonnegative integer n for which the equation $p^n x = a$ has a solution in G, if such an n exists. Otherwise, a has *infinite p-height in* G, denoted by $h(a)(p) = \infty$.

A group G is called *S-divisible*, for some set S of primes, if $pG = G$ for all $p \in S$.

If $f, g \in \mathscr{E}(G)$, then $[f, g]$ denotes the endomorphism $fg - gf$.

The (weak) direct sum is denoted by $\oplus$, the complete direct sum (direct product) by $\oplus^*$.

I denote by Z the group of integers, by Q the group of rationals, by Z_p the group of p-adic integers, by $C(n)$ the cyclic group of order n for arbitrary positive integer n, and by $C(p^\infty)$ the p-quasicyclic group.

Any notation not specifically mentioned above comes from L. FUCHS' book [3].

2. Szele and Szendrei's characterization

In modern notation, the characterization of [4] may be paraphrased as follows:

THEOREM 1. *Let* T *be a torsion group.* $\mathscr{E}(T)$ *is commutative if and only if* $T = \bigoplus_{p \in S} T_p$, *where* S *is an arbitrary set of primes, and* $T_p \cong C(p^{k_p})$, $0 < k_p \leqq \infty$.

THEOREM 2. *Let G be a mixed group without non-zero elements of infinite p-height for all relevant primes p. $\mathcal{E}(G)$ is commutative if and only if $\bigoplus_{p\in S} G_p \subset G \subset \bigoplus^*_{p\in S} G_p$, where S is an arbitrary set of primes; $G_p \cong C(p^{k_p})$, $0<k_p<\infty$; and G is S-pure in $\bigoplus^*_{p\in S} G_p$.*

THEOREM 3. *Let G be a mixed group such that $G = T\oplus U$, where T is the torsion subgroup of G. $\mathcal{E}(G)$ is commutative if and only if $T = \bigoplus_{p\in S} G_p$, where S is an arbitrary set of primes and $G_p \cong C(p^{k_p})$, $0<k_p<\infty$; $pU=U$ for all $p\in S$; and $\mathcal{E}(U)$ is commutative.*

The authors give an example of a mixed group G with $\mathcal{E}(G)$ commutative, which does not satisfy the hypotheses of either Theorem 2 or Theorem 3. Finally, they give a necessary, but not sufficient, condition, Theorem 4, and a sufficient, but not necessary condition, Theorem 5, for the commutativity of $\mathcal{E}(G)$.

THEOREM 4. *If G is mixed, and $\mathcal{E}(G)$ commutative, then the torsion subgroup $T \cong \bigoplus_{p\in S} C(p^{k_p})$, $0<k_p<\infty$; $p(G/T)=G/T$ for all $p\in S$; if $A=\{a\in G: h(a)(p)=\infty$ for all $p\in S\}$ then A is a torsion-free subgroup of G such that $pA=A$ for all $p\in S$, and G/A is a group of the type characterized in Theorem 2.*

The example $G = Q \bigoplus^*_p C(p)$, where the summation is taken over all primes, shows that the conditions of Theorem 4 are not sufficient to ensure the commutativity of $\mathcal{E}(G)$.

THEOREM 5. *Let G be mixed and satisfy the conclusions of Theorem 4. Assume furthermore that there exists a prime q such that $q(G/A)=G/A$; that A contains no non-zero element of infinite q-height; and that $\mathcal{E}(A)$ is commutative. Then $\mathcal{E}(G)$ is commutative.*

The example $G = Q\oplus C(p)$ shows that the conditions of Theorem 5 are not necessary for commutativity of $\mathcal{E}(G)$.

3. Necessary and sufficient conditions for commutativity of $\mathcal{E}(G)$

Let us call a group which satisfies the conditions of Theorem 2 a *semi-torsion group*. Clearly, for each infinite set S of primes, and each fixed $T = \bigoplus_{p\in S} C(p^{k_p})$, $0<k_p<\infty$, there are $2^{\aleph_0}$ non-isomorphic semi-torsion groups G such that $T\subset G \subset \bigoplus^*_{p\in S} C(p^{k_p})$. Semi-torsion groups have many interesting properties; for example, if G is semi-torsion, then so is End (G), and $G\cong \mathrm{Hom}\,(\mathrm{End}\,(G), G)$; furthermore, $G\cong \mathrm{End}\,(G)$ if and only if G is a subring containing the identity of the ring $\bigoplus^*_{p\in S} C(p^{k_p})$.

The conclusion of Theorem 4 is equivalent to the statement that G is an extension of a torsion-free group A by a semi-torsion group G/A such that, if S is the set of relevant primes for G/A, then A is the set of elements of G of infinite p-height for all $p\in S$.

Now an analysis of Szele and Szendrei's proofs shows that the conclusions of Theorem 4 can be strengthened, and the hypothesis of Theorem 5 weakened to give the following:

THEOREM. *Let G be a mixed group. Then $\mathcal{E}(G)$ is commutative if and only if there exists a semi-torsion group U with relevant prime set S, and a torsion-free group A such that*

(1) *G is an extension of A by U, and the torsion subgroup T of G is the torsion subgroup of U.*

(2) $A=\{a\in G:h(a)(p)=\infty$ *for all* $p\in S\}$.

(3) *Each pair of endomorphisms of A that can be extended to G commute.*

(4) *If $f, g\in\mathcal{E}(G)$, then the unique homomorphism $F:U/T\to G$ induced by $[f,g]$ is the zero map.*

PROOF. Assume $\mathcal{E}(G)$ is commutative. The conditions (1) and (2) follow from Theorem 4, together with the observation that the natural epimorphism $G\to U\to C(p^{k_p})$ splits, so $G\cong C(p^{k_p})\oplus H_p$, where H_p contains no elements of p-power order. Conditions (3) and (4) are trivially satisfied, since $[f,g]=0$ for all $f,g\in\mathcal{E}(G)$.

Conversely, let G be an extension of A by U satisfying conditions (1) through (4). Then G is a group satisfying the conclusions of Theorem 4, and possessing a fully invariant subgroup $A\oplus T$. If $f,g\in\mathcal{E}(G)$, then T is contained in the kernel of $[f,g]$ by Theorem 1, and A is contained in the kernel of $[f,g]$ by (3). Hence $[f,g]$ induces a unique homomorphism $F:U/T\to G$, which by (4) is the zero map, so $[f,g]=0$, and $\mathcal{E}(G)$ is commutative.

It is not known whether condition (3) can be replaced by the statement: $\mathcal{E}(A)$ is commutative.

The relationship between condition (4) and the hypothesis of Theorem 5 is elucidated by the following

LEMMA. *Let G be an extension of A by U satisfying conditions* (1), (2) *and* (3) *of the Theorem. Each of the following conditions is implied by its predecessor.*

(1) *There exists a prime q such that $qU=U$, and A contains no non-zero element of infinite q-height.*

(2) Hom $(U,A)=0$.

(3) *If $f,g\in\mathcal{E}(G)$, then the homomorphism $F:U/T\to G$ induced by $[f,g]$ is the zero map.*

PROOF. Clearly it suffices to prove (2) implies (3). Assume $[f,g]$ has finite p-height in End (G) for some $p\in S$. Then a suitable multiple has p-height zero; since $G=G_p\oplus H_p$, say, where $H_p=\{x\in G:h(x)(p)=\infty\}$ there exists $a\in G_p$ such that $0\neq[f,g](a)\in G_p$. But this is a contradiction, since by Theorem 1, the fully invariant subgroup G_p is contained in the kernel of $[f,g]$. Hence $[f,g]$ has infinite p-height for all $p\in S$, so maps G into A. Thus $F:U/T\to G$ is induced by some unique $F'\in$ Hom $(U,A)=0$.

Note that the example following Theorem 5 shows that (1) is strictly stronger than (2), but it is not known whether there exist groups with commutative endomorphism ring satisfying (3) but not (2). However, (2) is the most useful criterion for recognizing such groups.

EXAMPLE. For a fixed semi-torsion group U, and a fixed torsion-free group A such that $\mathcal{E}(A)$ is commutative, $\mathrm{Hom}(U, A) = \mathrm{Hom}(A, U) = 0$, there may be many non-isomorphic extension of A by U with commutative endomorphism ring.

Let S be a set of infinitely many, but not all, primes; let k_p, $p \in S$ be arbitrary positive integers, let $T = \bigoplus_{p \in S} C(p^{k_p})$, and let U be such that $T \subset U \subset \bigoplus^*_{p \in S} C(p^{k_p})$, and $U/T \cong Q$. Let A be a rank 1 torsion-free group divisible by a prime p if and only if $p \in S$ so $\mathcal{E}(A)$ is commutative; let G be any extension of A by U. Since $\mathrm{Hom}(T, A) = 0$, we have $\mathrm{Hom}(U, A) \cong \mathrm{Hom}(U/T, A) = 0$, so by the Lemma, the conditions of the Theorem are satisfied and $\mathcal{E}(G)$ is commutative. Now let us count the possibilities for G.

Since every extension of A by T splits, $\mathrm{Ext}(U, A) \cong \mathrm{Ext}(U/T, A) \cong \mathrm{Ext}(Q, A)$. The short exact sequence $0 \to Z \to Q \to Q/Z \to 0$ yields $0 = \mathrm{Hom}(Q, A) \to \mathrm{Hom}(Z, A) \cong$ $\cong A \to \mathrm{Ext}(Q/Z, A) \to \mathrm{Ext}(Q, A) \to \mathrm{Ext}(Z, A) = 0$ so we have an exact sequence $0 \to A \to \bigoplus^*_p \mathrm{Ext}(C(p^\infty), A) \to \mathrm{Ext}(Q, A) \to 0$, the summation being taken over all primes p. Now $\mathrm{Ext}(C(p^\infty), A) \cong Z_p$ if $pA \neq A$, and zero otherwise, so we have $0 \to A \to$ $\to \bigoplus^*_{p \notin S} Z_p \to \mathrm{Ext}(Q, A) \to 0$, so $\mathrm{Ext}(Q, A)$, and hence $\mathrm{Ext}(U, A)$ is uncountable. (I am indebted to Professor P. Neumann for this example.)

4. Commutativity of $\mathcal{E}(A)$ for torsion-free A

Of course, the theorems of the previous sections shed no light on the problem of characterizing the torsion-free groups with commutative endomorphism ring. The main reason for the difficulty of this problem seems to be the following: whereas increasing the complexity of the torsion subgroup of a group increases the number of endomorphisms, and hence only very simple torsion subgroups can exist in a group with commutative endomorphism ring, increasing the complexity of a torsion-free group decreases the number of endomorphisms, and hence very large and complicated torsion-free groups can have uncomplicated endomorphism rings; isomorphic, for example, to the integers. Indeed, CORNER [1] has shown that there are torsion-free groups of arbitrary countable rank with commutative endomorphism rings, and claims [2] to have extended this result to torsion-free groups of arbitrary rank $< \aleph_\iota$, the first strongly inaccessible cardinal.

For certain restricted classes of torsion-free groups, however, it is easy to characterize those members with commutative endomorphism rings:

(1) If G is completely decomposable, or a complete direct sum of rank 1 groups, then obviously $\mathcal{E}(G)$ is commutative if and only if the rank 1 components of G have pairwise incomparable types.

(2) Let G be a reduced torsion-free group such that G/pG has rank 1 or zero for all primes p. Then if $\bar{G}$ denotes the cotorsion completion of G, it is well known that G is a pure dense subgroup of $\bar{G} \cong \bigoplus^*_{p \in S} Z_p$, where S is the set of primes for which G/pG has rank 1. Furthermore each endomorphism of G extends uniquely to an endomorphism of $\bar{G}$, which is a multiplication in the ring $\bigoplus^*_{p \in S} Z_p$. Hence

$\mathscr{E}(G)$ is a subring of the ring $\mathscr{E}(\bar{G})$, and hence is commutative. (This interesting example is due to Professor C. Murley.)

Conversely, given a set S of primes, there are $2^{\aleph_0}$ pure dense subgroups of $\bigoplus^*_{p \in S} Z_p$, each having a commutative endomorphism ring.

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