Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 20 (1—2), (1969), pp. 25—37.

SOME REMARKS ON A PROPERTY OF TOPOLOGICAL CARDINAL FUNCTIONS

By
A. HAJNAL and I. JUHÁSZ (Budapest)

Introduction

In the paper [1] one of the authors has introduced the concept of the Darboux property of topological cardinal functions. In [1] several results and problems were stated. The main aim of this paper is to give some further results and simpler proofs for the results of [1].

In § 2 Theorems 1 and 2 give some information on the Darboux property of the weight function on the classes of T_1- and T_5-spaces respectively. However the results are still incomplete.

The rest of the theorems in this § deal with the density function and give an almost complete discussion of its behaviour on the different classes of spaces. We point out Problem 2 which remains unsolved.

In § 3 we prove Theorem 5 concerning linearly ordered spaces which settles the Darboux property of the weight (and density) function on these spaces. Without giving exact references we mention that at least in special cases the result must be contained in some theorems of W. Sierpiński and D. Kurepa concerning the Suslin problem.

In § 4 we introduce a new class of spaces lying between T_2- and T_3-spaces, called strongly Hausdorff spaces, and we prove a special result relevant to a problem stated by J. DE GROOT [2].

§ 1. Notations. Definitions

$|H|$ denotes the cardinality of the set H. We assume that each ordinal is the set of all smaller ordinals.

$\xi, \eta, \zeta, \ldots$ denote ordinals;

$\alpha, \beta, \varphi, \psi, \ldots$ denote cardinals (i.e. initial ordinals);

λ will always denote a limit cardinal.

α^+ denotes the immediate successor of the cardinal α.

If η is a limit ordinal, $\mathrm{cf}(\eta)$ is the least cardinal, which is cofinal with η.

The cardinal α is said to be *regular* if $\mathrm{cf}(\alpha)=\alpha$ and *singular* otherwise.

A regular limit cardinal is said to be *inaccessible.*

A limit cardinal λ is said to be a *strong limit cardinal* if $\alpha<\lambda$ implies $2^\alpha<\lambda$. (We sometimes write $\exp \alpha$ for 2^α.)

A strong limit inaccessible cardinal is called *strongly inaccessible.*

We will often make use of the generalized continuum hypothesis which will be briefly referred to as G.C.H. ω_ξ denotes the increasing sequence of infinite cardinals, $\omega_0=\omega$.

Capital letters $K, H, \dots, X, Y, \dots$ denote sets, $R, S, D, \dots$ denote topological spaces.

The class of all topological spaces will be denoted by $\mathcal{T}$, while the class of T_i-spaces will be denoted by $\mathcal{T}_i$, $i=0, \dots, 5$, respectively. $\mathcal{L}$ denotes the class of linearly ordered spaces provided with the usual interval topology.

A *topological cardinal function* is a function defined on a certain class of topological spaces with cardinal values.

In this paper we will consider the following cardinal functions.

The *weight function* w, defined as usual by

$$w(R) = \max \{\omega, \min \{|\mathfrak{B}|: \text{ for the open bases } \mathfrak{B} \text{ of } R\}\}.$$

The *density function* d:

$$d(R) = \max \{\omega, \min \{|S|: \text{ for } S \subset R, \bar{S} = R\}\}.$$

The *spread function* s, where

$$s(R) = \max \{\omega, \sup \{|D|: D \subset R \text{ where } D \text{ is a discrete subspace of } R\}\}.$$

The space R will be said *left separated (right separated)* if there exist a well-ordering $\{x_\xi\}_{\xi<\varphi} = R$ of the points of R and a sequence $\{U_\xi\}_{\xi<\varphi}$ of type φ of open subsets of R such that $x_\xi \in U_\xi$ and $x_\eta \notin U_\xi$ for $\eta < \xi$ [$x_\eta \notin U_\xi$ for $\eta > \xi$] for every $\xi < \varphi$ respectively.

To have a brief notation we introduce the following symbols.

Let Φ be a cardinal function defined on the class $\mathcal{C}$ of topological spaces;

$$(*) \qquad (\Phi, \mathcal{C}) \to \alpha$$

denotes that the following statement is true.

For each $R \in \mathcal{C}$, $\Phi(R) > \alpha$ implies that there exists a subspace $S \subset R$ such that $\Phi(S) = \alpha$.

$(\Phi, \mathcal{C}) \nrightarrow \alpha$ denotes the negation of the above statement.

If $(\Phi, \mathcal{C}) \to \alpha$ holds for every [regular] α then Φ is said to have the [regular] Darboux property on $\mathcal{C}$.

$$(**) \qquad [\Phi, \mathcal{C}] \to \lambda$$

denotes that the following statement is true:

If for each $\alpha < \lambda$ there exists a subspace $S \subset R$ with $\alpha \leq \Phi(S) < \lambda$ then there exists a subspace $S_0 \subset R$ with $\Phi(S_0) = \lambda$.

$[\Phi, \mathcal{C}] \nrightarrow \lambda$ denotes the negation of this statement.

If Φ has the Darboux property on $\mathcal{C}$ and $[\Phi, \mathcal{C}] \to \lambda$ holds for every λ then Φ is said to possess the *closed Darboux property* on $\mathcal{C}$.

The concepts of (regular, closed) Darboux properties were formulated in [1]. The introduction of the symbols $(*)$ and $(**)$ depending on the parameters α, λ enables us to give a more detailed analysis of these properties.

§ 2. The Darboux properties of the weight and density functions

First we are going to deal with the weight function w. In this case we know negative results only, except some trivial positive facts.

THEOREM 1. *If λ is a singular cardinal, then*

$$(w, \mathcal{T}_1) \nrightarrow \lambda.$$

PROOF. Let H be a set of potency λ, provided with the topology whose non-trivial (i.e. different from H) closed sets are exactly those of cardinality not greater than $\mathrm{cf}(\lambda)$. As every one-point set is closed in H, it is a T_1-space, indeed.

For each $K \subset H$, let $w^*(K)$ be the smallest cardinal β such that there exists a system $\mathfrak{L}$, $|\mathfrak{L}| = \beta$ of non-trivial closed subsets of K, with the property that every non-trivial closed subset of K is contained in one of the elements of $\mathfrak{L}$. Since each base for the closed sets in K has this property we get immediately

$$w^*(K) \leqq w(K).$$

On the other hand, if $\mathfrak{L}$ is the above mentioned system of power $w^*(K)$, let $\mathfrak{B}$ be the system of all sets of the form $Z \setminus \{x\}$, where $Z \in \mathfrak{L}$ and $x \in K$. Then $|\mathfrak{B}| \leqq |\mathfrak{L}| \cdot \mathrm{cf}(\lambda) = w^*(K)\,\mathrm{cf}(\lambda)$, because $Z \in \mathfrak{L}$ implies $|Z| \leqq \mathrm{cf}(\lambda)$. At the same time $\mathfrak{B}$ is a base for the closed sets in K for if S is an arbitrary non-trivial closed set in K, then there is a set $Z \in \mathfrak{L}$ with $S \subset Z$, and so we get

$$S = \bigcap_{x \in Z \setminus S} (Z \setminus \{x\}).$$

These considerations show immediately

$$w(K) \leqq \mathrm{cf}(\lambda) \cdot w^*(K)$$

and so $w^*(K) \geqq \mathrm{cf}(\lambda)$ implies

$$w(K) = w^*(K)$$

and $w^*(K) < \mathrm{cf}(\lambda)$ implies $w(K) \leqq \mathrm{cf}(\lambda) < \lambda$. Now assume $w^*(K) \geqq \mathrm{cf}(\lambda)$. We will prove that $w^*(K) = w(K) \neq \lambda$. Assume on the contrary, that $w^*(K) = \lambda$ and let $\mathfrak{L}$ be the required set-system of power λ. Let the cardinals α_ξ be chosen for each $\xi < \mathrm{cf}(\lambda)$ such that $\alpha_\xi < \alpha_\eta$ if $\xi < \eta$ and

$$\sum_{\xi < cf(\lambda)} \alpha_\xi = \lambda.$$

The system $\mathfrak{L}$ can be represented in the form

$$\mathfrak{L} = \bigcup_{\xi < cf(\lambda)} \mathfrak{L}_\xi,$$

where $\xi < \eta$ implies $\mathfrak{L}_\xi \subset \mathfrak{L}_\eta$ and $|\mathfrak{L}_\xi| = \alpha_\xi$. Then $|\mathfrak{L}_\xi| < w^*(K)$ for each $\xi < \mathrm{cf}(\lambda)$, and so one can find a non-trivial closed subset of K, say S_ξ, with $S_\xi \not\subset Z$ for each $Z \in \mathfrak{L}_\xi$. Let

$$S = \bigcup_{\xi < cf(\lambda)} S_\xi.$$

Then, by definition, $|S_\xi| \leqq \mathrm{cf}(\lambda)$ and so $|\bigcup_{\xi < \mathrm{cf}(\lambda)} S_\xi| = |S| \leqq \mathrm{cf}(\lambda)$, i.e. S is a closed subset of K. Now if Z is an arbitrary element of $\mathfrak{L}$, then $Z \in \mathfrak{L}_\xi$ for some $\xi < \mathrm{cf}(\lambda)$, and so

$$S_\xi \subset S \not\subset Z,$$

which contradicts the definition of $\mathfrak{L}$, consequently

$$w^*(K) = w(K) \neq \lambda.$$

Finally, we have to prove $w(H) > \lambda$. Let, indeed, $\mathfrak{B}$ be an arbitrary family of non-trivial subsets of H with $|\mathfrak{B}| < \lambda$. Then $|\bigcup \mathfrak{B}| \leqq |\mathfrak{B}| \,\mathrm{cf}(\lambda) < \lambda$, hence $w(H) \geqq \lambda$. This obviously implies $w(H) > \lambda$.

THEOREM 2. $2^\alpha > \alpha^+$ *implies*

$$(w, \mathcal{T}_5) \not\rightarrow \alpha^+.$$

PROOF. Let D_α be the discrete topological space of power α and βD_α its Stone—Čech compactification. It has been proved by B. POSPÍŠIL (see [3]) that there exists a point $p \in \beta D_\alpha \setminus D_\alpha$ whose every base of neighbourhoods has the cardinality 2^α, in other words, the character $\chi(p, \beta D_\alpha)$ of p in βD_α equals to 2^α. It follows from this that the character $\chi(p, R)$ of p in $D_\alpha \cup \{p\} = R$ is also 2^α because βD_α is regular and R is dense in it. Trivially R belongs to $\mathcal{T}_5$. Now let $A \subset D_\alpha$, $\bar{A}$ its closure in R and $\bar{A}^\beta$ its closure in βD_α. It is well-known that $\bar{A}^\beta$ is open-and-closed in βD_α and so $\bar{A} = \bar{A}^\beta \cap R$ is also open in R. But then $p \in \bar{A}$ implies

$$\chi(p, \bar{A}) = \chi(p, R) = 2^\alpha.$$

Let $S \subset R$ be an arbitrary subspace of R. Then there are three possibilities: (i), (ii) and (iii).

(i) $p \notin S$, then S is obviously discrete, and so $w(S) = |S| \leqq \alpha$.

(ii) $p \in S$ but $p \notin \overline{S \setminus \{p\}}$; then S is discrete, too, thus $w(S) = |S| \leqq \alpha$.

(iii) $p \in S$ and $p \in \overline{S \setminus \{p\}}$; it means that $S \setminus \{p\}$ is dense in S, so (as we have seen above)

$$\chi(p, S) = \chi(p, \overline{S \setminus \{p\}}) = 2^\alpha$$

which immediately gives us $w(S) = 2^\alpha$. Hence every subspace of R has a weight either at most α or 2^α. This proves Theorem 2.

COROLLARY. *If G. C. H. fails then w does not possess the regular Darboux property on $\mathcal{T}_5$.*

After this manuscript had been completed we obtained a result saying $(w, \mathcal{T}_2) \not\rightarrow \alpha^+$, if $\alpha^+ = 2^\alpha$. This result is going to be published in our joint paper "On hereditarily α-separable and α-Lindelöf spaces" in the *Annales Univ. Sci. Budapest*, **11** (1968).

From this result, together with the above Theorem 2 we can get easily that for any α, which is not strong limit or inaccessible, $(w, \mathcal{T}_2) \not\rightarrow \alpha$ holds. However, we still do not know the answer to the following problem.

PROBLEM 1. Is $(w, \mathcal{T}_i) \rightarrow 2^\alpha$ true for $i \geqq 3$, $\alpha \geqq \omega$?

The following cardinal function we shall consider is the density. In this case at least assuming G.C.H. we can give a rather complete discussion of the symbols $(*)$ and $(**)$. The only problem left open is the one stated on p. 34.

The following Lemma 1 was first published in [6] (Theorem II). We give here a new proof of it, which does not make use of transfinite induction. The same idea will be used in the proof of Theorem 5.

LEMMA 1. *Each $R\in\mathcal{T}$ contains a left separated subspace $S\subset R$ with $|S|\geqq d(R)$.*

PROOF. Let

$$R=\{q_\xi:\ \xi<\mu\} \tag{1}$$

be an arbitrary well-ordering of R. A point $q\in R$ will be called minimal if it has a neighbourhood U_q, whose minimal element in the above well-ordering is q. Let

$$S=\{p_\xi:\ \xi<\varrho\} \tag{2}$$

be the well-ordering of the set of all minimal points of R induced by the well-ordering (1). Then S is dense in R and so $d(R)\leqq|S|$. Indeed, if G is an arbitrary non-void open set in R then there exists a point $q\in G$ with a minimal suffix in the well-ordering (1). Hence, by definition, $q\in S$ and so $G\cap S\neq\emptyset$.

On the other hand it is trivial, that if $p_\xi\in S$ and U_ξ is the neighbourhood of p_ξ whose first element is p_ξ then U_ξ does not contain any predecessors of p_ξ.

COROLLARY. *If α is regular then*

$$(d,\mathcal{T})\to\alpha.$$

PROOF. Let, indeed, $R\in\mathcal{T}$ and $d(R)>\alpha$. According to Lemma 1 there is a sequence $S=\{p_\xi:\ \xi<\varrho\}$ of points of R such that $\varrho\geqq\alpha$ and every p_ξ has a neighbourhood U_ξ not containing any points p_η, $\eta<\xi$. Let

$$T=\{p_\xi\in S:\ \xi<\alpha\}.$$

We state that $d(T)=\alpha$. $d(T)\leqq\alpha$ is trivial since $|T|=\alpha$. On the other hand, if $K\subset T$ and $|K|<\alpha$, then there exists an ordinal $\eta<\alpha$ such that $\xi<\eta$ for each $p_\xi\in K$, because of the regularity of α. But then $p_\xi\notin U_\eta$ for each $p_\xi\in K$, which shows that K is not dense in T and so $d(T)\geqq\alpha$ i.e. $d(T)=\alpha$.

LEMMA 2. *If λ is a strong limit cardinal, $|R|=\lambda$ and $R\in\mathcal{T}_2$ then $d(R)=\lambda$.*

PROOF. It is well-known (see e.g. [4]) that $R\in\mathcal{T}_2$ implies $|R|\leqq\exp\exp d(R)$. Since λ is a strong limit cardinal $d(R)<\lambda$ would imply $|R|\leqq\exp\exp d(R)<\lambda$, which is impossible. So $d(R)=\lambda$.

COROLLARY 1. *For each strong limit cardinal λ*

$$(d,\mathcal{T}_2)\to\lambda$$

holds.

COROLLARY 2. *If λ is a strong limit cardinal then*

$$[d,\mathcal{T}_2]\to\lambda.$$

PROOF. If for every $\alpha<\lambda$ there exists a subspace S of the space R for which $d(R)\geqq\alpha$, then obviously $|R|\geqq\lambda$.

THEOREM 3. *If* $\operatorname{cf}(\lambda)=\omega$ *then*

$$(d, \mathcal{T}_2)\rightarrow\lambda.$$

PROOF. Let, indeed, $R\in\mathcal{T}_2$, $d(R)>\lambda$, then according to Lemma 1 there exists a left separated subset $R'\subset R$ of the power λ^+. In what follows we are going to consider only this subspace R'. Let $\mathfrak{G}$ be the system of all sets $G\subset R'$ being open in R' and having a cardinality not greater than λ. We will distinguish two cases (i) and (ii):

(i) $|\bigcup\mathfrak{G}|=\lambda^+$. Then we define a sequence $\{q_\xi\colon \xi<\lambda^+\}$ of points of R' by transfinite induction on ξ as follows. Let $R'=\{p_\nu\colon \nu<\lambda^+\}$ be a well-ordering of R' and let U_ν be a neighbourhood of p_ν not containing any predecessors of p_ν.

Now let $p_{\nu_0}=q_0$ be the first element of $\bigcup\mathfrak{G}$ and let G_0 be an arbitrary element of $\mathfrak{G}$ with $q_0\in G_0$. Assume that the points q_η and their neighbourhoods G_η are defined already for all η less than some $\xi<\lambda^+$. Then

$$\Big|\bigcup_{\eta<\xi} G_\eta\Big| \leqq |\xi|\cdot\lambda = \lambda$$

and so $\bigcup\mathfrak{G}\setminus\bigcup\{G_\eta\colon \eta<\xi\}\neq\emptyset$; we choose the first element p_{ν_ξ} of the above non-void set as q_ξ. G_ξ will be an arbitrary element of $\mathfrak{G}$ containing q_ξ. Put $D=\{q_\xi\colon \xi<\lambda^+\}$. We prove that D is discrete. Let us consider the neighbourhood $V_\xi=U_{\nu_\xi}\cap G_\xi$ of $q_\xi(=p_{\nu_\xi})$ for $\xi<\lambda^+$. Since by definition $q_\eta\notin U_{\nu_\xi}$ if $\eta<\xi$ and $q_\eta\notin G_\xi$ if $\eta>\xi$, $V_\xi\cap D=\{q_\xi\}$ for $\xi<\lambda^+$. Hence D is discrete. Thus R' and so R also contain a discrete subspace of potency λ, which is of density λ, too.

(ii) $|\bigcup\mathfrak{G}|<\lambda^+$. Then let $R''=R'\setminus\bigcup\mathfrak{G}$. Obviously each non-void open subset of R'' has the cardinality λ^+.

Now because of $\operatorname{cf}(\lambda)=\omega$ there are regular cardinals α_k $(k<\omega)$ such that

$$\lambda=\sum_{k<\omega}\alpha_k.$$

Since every infinite T_2-space contains infinitely many pairwise disjoint, non-void, open sets we can choose non-void subsets G_k $(k<\omega)$ open in R'' such that $G_k\cap G_l=\emptyset$ if $k\neq l$. As we have seen above

$$|G_k|=\lambda^+$$

for each $k<\omega$. By Lemma 1 and the proof of its Corollary for every $k<\omega$ there exists an $S_k\subset G_k$ with $d(S_k)=|S_k|=\alpha_k$ (because α_k is regular).

Now let $S=\bigcup_{k<\omega} S_k$. Since $|S|=\lambda$ it is sufficient to prove $d(S)\geqq\lambda$. Let $M\subset S$ be an arbitrary dense subset of S. Then $M\cap S_k$ is dense in S_k, too, because $M\setminus(M\cap S_k)\subset\bigcup_{l\neq k} G_l\subset R''\setminus G_k$, and so none of the points of $S_k\subset G_k$ is a cluster point of $M\setminus(M\cap S_k)$. But then $d(S_k)=\alpha_k$ implies $|M\cap S_k|=\alpha_k$ and so

$$|M|=\sum_{k<\omega}|M\cap S_k|=\sum_{k<\omega}\alpha_k=\lambda$$

which proves our statement.

Theorem 3 is one of the new results of this paper. The problem stated originally in [1] still remains open for singular cardinals λ with $\mathrm{cf}(\lambda) > \omega$. The simplest unsolved problem is

PROBLEM 2. Is $(d, \mathcal{T}_2) \to \omega_{\omega_1}$ true?

(Note that assuming G.C.H. the answer is yes by Corollary 1 of Lemma 2.) Corollary 2 of Lemma 2 implies assuming G.C.H. that d has the closed Darboux property on $\mathcal{T}_2$. We will point out that without assuming G.C.H. we cannot solve the following

PROBLEM 3. Is $[d, \mathcal{T}_2] \to \omega_\omega$ true?

This should be compared with the remark made after the proof of Theorem 7.

The following theorem shows that for T_1-spaces the above result does not remain true.

THEOREM 4. *For every singular* λ

$$(d, \mathcal{T}_1) \nrightarrow \lambda \quad \textit{and} \quad [d, \mathcal{T}_1] \nrightarrow \lambda.$$

PROOF. Let us consider the topology on the set λ^+ whose non-void open sets are exactly those of the form $[\varrho, \lambda^+) \setminus \{\varrho_1, ..., \varrho_k\}$ where

$$[\varrho, \lambda^+) = \{\sigma : \varrho \leqq \sigma < \lambda^+\} \quad \text{and} \quad \varrho, \varrho_1, ..., \varrho_k < \lambda^+, \ k < \omega.$$

Let R denote this space which is obviously a T_1-space.

Let now S be an arbitrary infinite subspace of R and let $\tau(S)$ be its order-type as a subset of λ^+. It is well-known that $\tau(S)$ has a unique decomposition

$$\tau(S) = \zeta(S) + k(S),$$

where $\zeta(S)$ is a limit ordinal and $k(S) < \omega$.

Now let H be the set of the last $k(S)$ elements of S and let C be an arbitrary cofinal subset of $S \setminus H$. Then it is obvious that for every $\xi \in S \setminus H$ and $\xi_1, ..., \xi_l < \lambda^+$

$$S \cap ([\xi, \lambda^+) \setminus \{\xi_1, ..., \xi_l\}) \cap (C \cup H) \neq \emptyset,$$

i.e. $C \cup H$ is dense in S.

On the other hand, if T is dense in S, then $T \setminus H$ must be cofinal with $S \setminus H$, because $\varrho \in S \setminus H$ and $\varrho > \sigma$ for every $\sigma \in T \setminus H$ would imply

$$T \cap ([\varrho, \lambda^+) \setminus H) = \emptyset.$$

Hence we have got the result $d(S) = \mathrm{cf}(\zeta(S))$. So e.g. $d(R) = \mathrm{cf}(\zeta(R)) = \mathrm{cf}(\lambda^+) = \lambda^+$, and since $\mathrm{cf}(\zeta)$ is always a regular cardinal, none of the subspaces of R have the density λ.

On the other hand, since for every regular cardinal $\alpha < \lambda$ there is a subspace of the density α, the same example shows that the second statement of our Theorem holds, as well.

§ 3. A theorem on ordered spaces

The main aim of this section is to prove Theorem 5.

THEOREM 5. *If $R \in \mathscr{L}$, then for each $\alpha < d(R)$ there exists a discrete subspace of R, which is of power α. Hence $d(R) \leqq s(R)^+$.*

PROOF. Let $\alpha < s(R)$ be arbitrary, and assume that R does not contain a discrete subspace of power α. The original order relation of R will be denoted by $<$ while $\prec$ is chosen to denote an arbitrary well-ordering of R.

As usual a set

$$(x, y) = \{z \in R : x < z < y\}$$

is called an open interval of R.

An element $p \in R$ is called normal if there exists an open interval (x, y) containing p, such that

$$p \prec z \quad \text{for every} \quad z \in (x, y) \setminus \{p\}.$$

It is easy to see that the set N of all normal elements is dense in R. Let, indeed, (x, y) be an arbitrary non-void interval of R, and p be the first element of (x, y) with respect to the well-ordering $\prec$. Then p is a normal element by definition. Thus $|N| \geqq d(R) > \alpha$.

For every $p \in N$ let I_p denote the maximal convex set containing p as the first element with respect to $\prec$. Of course, I_p contains p in its interior.

Now let N^* be the collection of all such sets I_p for $p \in N$. It is trivial that $p \neq q$, $p, q \in N$ implies $I_p \neq I_q$ since the least elements of I_p resp. I_q are different. We define a partial ordering $<^*$ of N^* as follows,

$$I_q <^* I_p \quad \text{iff} \quad p \in I_q.$$

As $p \in I_q$ obviously implies $q \prec p$ and so $q \prec z$ for every $z \in I_p$, $I_q <^* I_p$ implies $I_p \subset I_q$, since $I_p \cap I_q \neq 0$ and so $I_p \cup I_q$ is a convex set containing q as its first element, and therefore $I_p \cup I_q \subset I_q$. From this remark we get immediately that the relation $<^*$ is transitive.

Let now $q_1, q_2, p \in N$, $q_1 \prec q_2$ and $I_{q_1} <^* I_p$ and $I_{q_2} <^* I_p$. Then $p \in I_{q_1} \cap I_{q_2}$ and so $I_{q_1} \cup I_{q_2}$ is a convex set containing both q_1 and q_2. But then clearly $I_{q_1} \cup I_{q_2} \subset I_{q_1}$ which implies

$$I_{q_1} <^* I_{q_2}.$$

From these considerations it follows immediately that for every $I_p \in N^*$ the segment

$$S_p = \{I_q \in N^* : I_q <^* I_p\}$$

is well-ordered by the relation $<^*$, because for $I_{q_1}, I_{q_2} \in S_p$

$$q_1 \prec q_2 \Leftrightarrow I_{q_1} <^* I_{q_2}.$$

So the partially ordered set $(N^*, <^*)$ is a ramification system (or tree) in the sense of [7]. Let now $A \subset N$ be a set for which any two elements of the set-system

$$A^* = \{I_p : p \in A\}$$

are not comparable with respect to $<^*$. Then A is a discrete subspace of R since

I_p is a neighbourhood of p which — by definition — does not contain any other points of A. Consequently we obtain $|A|=|A^*|<\alpha$.

Now it is very easy to see that every ramification system of power greater than or equal to α^+, and not containing α pairwise incomparable elements contains a chain of length α, i.e. a set of power α, every two elements of which are comparable (see e.g. [7]). Let $C^*\subset N^*$ be a chain of length α and

$$C=\{p\in N\colon I_p\in C^*\}.$$

We can assume that the order-type of C (by $\prec$) is α. For every $p\in C$ let p^+ be the successor of p in C with respect to $\prec$. For every $p\in C$ let us choose an element

$$x_p\in I_p\setminus I_{p^+}\neq\emptyset.$$

Since I_{p^+} is convex, either $x_p>z$ or $z>x_p$ for each $z\in I_{p^+}$; in the first case we call x_p a right point and in the second case a left one. The set of all right points is denoted by H^r and that of the left points by H^l. Of course $|H^l\cup H^r|=|C|=\alpha$ and thus either $|H^r|=\alpha$ or $|H^l|=\alpha$.

Assume, for instance, $|H^l|=\alpha$. Then for $x_{p_1}, x_{p_2}\in H^l$ we get

$$x_{p_1}<x_{p_2}\Leftrightarrow p_1\prec p_2.$$

Let indeed $p_1\prec p_2$, then $p_1^+\preceq p_2$ so $I_{p_2}\subset I_{p_1^+}$ i.e. $x_{p_2}\in I_{p_1^+}$ which implies

$$x_{p_1}<x_{p_2}$$

by the definition of H^l.

Thus we have got a subset of R of potency α, whose original ordering $<$ coincides with the well-ordering $\prec$. According to our assumption R can not contain α isolated points and so we have a subset $H\subset H^l$ of power α not containing any isolated points and whose *induced* ordering $<$ is a well-ordering. We shall denote by x^+ the successor of $x\in H$ in H with respect to $<$.

Since H does not contain any isolated points, one of the intervals (x, x^+) and (x^+, x^{++}) is not void. Consequently there are α distinct non-void open intervals of the form (x, x^+), $x\in H$ which is in contradiction to our assumption, since these intervals are pairwise disjoint as well. Hence we have proved the existence of a discrete subspace of power α in R.

An analogous consideration leads to the same result, if $|H^r|=\alpha$, however then $<$ coincides with the converse of $\prec$.

As an immediate consequence of Theorem 5 we get

COROLLARY 1. *The cardinal function d has the Darboux property on $\mathscr{L}$.*

We also prove

COROLLARY 2. *For every singular cardinal λ*

$$[d, \mathscr{L}]\to\lambda.$$

PROOF. If for cofinally many $\alpha<\lambda$ there exist subspaces of $R\in\mathscr{L}$ of the corresponding density, then by Theorem 5 one can find discrete subspaces, whose cardinalities are cofinal with λ, too. But it is easy to see that if a linearly ordered space contains an infinite discrete subspace, then it contains as many pairwise disjoint open intervals as the cardinality of this discrete subspace.

On the other hand, ERDŐS and TARSKI [5] proved that *in every topological space* the least cardinal for which the space does not contain as many pairwise disjoint open subsets, is always regular. So if R contains α disjoint open intervals for each $\alpha<\lambda$, then it contains λ disjoint open intervals, too. Hence R contains a discrete subspace of cardinality (or density) λ, too.

We do not know whether d has the closed Darboux property on $\mathscr{L}$ since we cannot solve the following.

PROBLEM 4. Does Corollary 2 of Theorem 5 hold for inaccessible λ's as well? Note that in [5] an example is given showing that the theorem we used for the proof of Corollary does not remain true for inaccessible cardinals greater than ω.

In order to get similar results about the Darboux property of the weight function on $\mathscr{L}$, we have to make some preliminary remarks about the relation between the weight and density of the ordered spaces.

Let $R\in\mathscr{L}$. A pair $\langle x, y\rangle$ of two distinct points $x, y\in R$ is called a gap in R, if the open interval (x, y) is empty (i.e. y is the successor of x), but neither x nor y is isolated. Let $U(R)$ be the set of all gaps in R, and $g(R)=|U(R)|$.

LEMMA 3. *If* $R\in\mathscr{L}$, *then* $w(R)=d(R)+g(R)$.

PROOF. We can assume $|R|\geqq\omega$. Now let $S\subset R$ be a dense subset of cardinality $d(R)$ and H be the set of the endpoints of all the gaps in R. First we will show that the open intervals (a, b), where a, b belong to $S\cup H$, plus the isolated points, which of course all belong to S, constitute a base for R, which evidently implies

$$w(R)\leqq|S\cup H|\leqq d(R)+g(R).$$

Let, indeed, $x\in R$ be not isolated, and (p, q) be any open interval containing x. Now, if x does not have either a predecessor or a successor, then we can find $a\in S$ with $a\in(p, x)$, and $b\in S$ with $b\in(x, q)$, hence $x\in(a, b)\subset(p, q)$. If x has a predecessor, say a, then a certainly belongs to $S\cup H$, because either it is isolated, or it constitutes a gap with x, since the latter is not isolated. Since in this case x has no successor, we can find $b\in S$ with $b\in(x, q)$, and then $x\in(a, b)\subset(p, q)$ holds again. The case, when x has a successor can be settled quite analogously.

In order to get the converse inequality

$$w(R)\geqq d(R)+g(R),$$

it is obviously enough to prove $w(R)\geqq g(R)$. This follows, however, immediately from the observation that, if $\langle x, y\rangle\in U(R)$ is an arbitrary gap, then any base of R has to contain a set with x as last element and a set with y as first element. Thus our Lemma is proved.

COROLLARY 3. *w has the Darboux property on $\mathscr{L}$.*

PROOF. Let $R\in\mathscr{L}$ and $w(R)>\alpha$. Then we have the following two possibilities a) and b), respectively.

a) $w(R)=d(R)$. In this case it follows immediately from Theorem 5 that R contains a discrete subspace of cardinality — hence of weight — α.

b) $w(R)=g(R)$. In this case let $U_1\subset U(R)$ be a set of gaps with $|U_1|=\alpha$, and H_1 be the set of all endpoints of the gaps belonging to U_1. Obviously, $|H_1|=\alpha$

as well. One can see easily that the subspace $H_1 \subset R$ is of the weight α, and this completes the proof of the corollary.

Since the weight function is monotone, and for monotone cardinal functions the Darboux property and the closed Darboux property are equivalent (see e.g. [1]), we also have that w has the closed Darboux property on $\mathscr{L}$.

§ 4. Strongly Hausdorff spaces

We will say that a Hausdorff space R is strongly Hausdorff if for each infinite subset $S \subset R$ one can select a sequence $\{x_i\}_{i<\omega}$ of points of S and a sequence $\{U_i\}_{i<\omega}$, $x_i \in U_i$ of neighbourhoods such that $i \neq j < \omega$ implies $U_i \cap U_j = \emptyset$. The following theorem shows that this class of spaces is wide enough.

THEOREM 6. *Every Uryson space, hence every regular Hausdorff space, is strongly Hausdorff.*

PROOF. Let R be an Uryson space and let $S \subset R$ be an arbitrary infinite subspace of it. Let x_0 and y_0 be two arbitrary points of S and U_0 and V_0 a closed neighbourhood of x_0 and y_0 respectively that are disjoint. We can assume $S \setminus U_0$ is infinite. Assume that the points $x_i \in S$ and their neighbourhoods U_i have been already defined for each $i < k$ $(k > 0)$ in such a way that $S \setminus \left(\bigcup_{i<k} \overline{U}\right)$ is infinite. Then we can choose two points $x_k, y_k \in S \setminus \left(\bigcup_{i<k} \overline{U}_i\right)$, and two disjoint neighbourhoods U_k and V_k of x_k and y_k, respectively which are contained in the open set $R \setminus \bigcup_{i<k} \overline{U}_i$, and which have disjoint closures in R. We can also assume that

$$\left(S \setminus \bigcup_{i<k} \overline{U}_i\right) \setminus \overline{U}_k = S \setminus \bigcup_{i<k+1} \overline{U}_i$$

is infinite since

$$S \setminus \bigcup_{i<k} \overline{U}_i = \left[\left(S \setminus \bigcup_{i<k} \overline{U}_i\right) \setminus \overline{U}_k\right] \cup \left[\left(S \setminus \bigcup_{i<k} \overline{U}_i\right) \setminus \overline{V}_k\right],$$

and the roles of x_k and y_k are perfectly symmetric. The sequence $\{x_i\}_{i<\omega}$ defined by induction on k obviously satisfies the requirements having the pairwise disjoint neighbourhoods U_i.

On the other hand, the following example shows that there are Hausdorff spaces which are not strongly Hausdorff.

EXAMPLE. Let the set R consists of two kinds of elem nts: $R = P \cup H$, where $P \cap H = \emptyset$. Both P and H are countable, the elements of P are denoted by $x_0, \ldots, x_k, \ldots$ $(k < \omega)$, while H is regarded as the set of all quadruples (j, l, m, n) where $j, l, m, n < \omega$. For the topology in R, the points of H are assumed to be isolated and a neighbourhood base $\mathfrak{B}_k = \{V_{r,s}^{(k)} : r, s < \omega\}$ for x_k is defined as follows:

$$V_{r,s}^{(k)} = \{x_k\} \cup \{(k, l, m, n) : l > r\} \cup \{(j, l, m, k) : j < k, \quad l \leq k, \ m > s\}.$$

It is easy to see that

$$V_{r_1,s_1}^{(k)} \cap V_{r_2,s_2}^{(k)} = V_{r,s}^{(k)},$$

where $r = \max\{r_1, r_2\}$ and $s = \max\{s_1, s_2\}$.

Furthermore

$$\bigcap_{r,s<\omega} V_{r,s}^{(k)} = \{x_k\},$$

since for every $(j, l, m, n) \in H$ either $j=k$ and then $(j, l, m, n)=(k, l, m, n) \notin V_l^{(k)}$ or $k \neq j$ and then $(j, l, m, n) \notin V_{r,m}^{(k)}$.

Finally if $k_1 < k_2$ then x_{k_1} and x_{k_2} have disjoint neighbourhoods since for example

$$V_{k_2,s}^{(k_1)} \cap V_{r,s}^{(k_2)} = \emptyset$$

for every $r, s < \omega$.

This altogether shows that R is a Hausdorff space. But R is not strongly Hausdorff, indeed, since if $\{x_{k_t}: t<\omega\}$ is an arbitrary sequence of points from P, $k_{t_1} < k_{t_2}$ if $t_1 < t_2$, and $V_{r,s}^{(k_0)}$ is an arbitrary neighbourhood of x_{k_0}, then

$$V_{r,s}^{(k_0)} \cap V_{p,q}^{(k_t)} \neq \emptyset$$

for each $p, q < \omega$ whenever $t > 0$ and $k_t > r$, because then for example

$$(k_0, k_t, q+1, k_t) \in V_{r,s}^{(k_0)} \cap V_{p,q}^{(k_t)}.$$

Finally we are going to show an application of the notion introduced above. First we need a lemma, which is, however, interesting in itself, too.

LEMMA 4. *Let R be an arbitrary topological space with $|R|=\alpha>\omega$, and $\beta<\alpha$. Then either R contains a discrete subspace of power α, or the set S_β of all points $x \in R$ having a neighbourhood U_x with $|U_x|<\beta$ is of cardinality less than α.*

PROOF. Assume $|S_\beta|=\alpha$. Then we can define a set mapping F on S_β as follows:

$$F(x) = U_x \setminus \{x\}.$$

Thus $|F(x)|<\beta<\alpha$ holds for all $x \in S_\beta$, hence a theorem proved by A. Hajnal (which is also known as Ruziewicz' conjecture, see e.g. [8]) can be applied, and we can get a free subset $S \subset S_\beta$ with $|S|=\alpha$. This means, however, that $x \notin U_y$ holds for each pair of distinct points $x, y \in S$, i.e. S is a discrete subspace of power α.

THEOREM 7. *Let $\operatorname{cf}(\lambda)=\omega$, $\lambda>\omega$, and assume that for each $\alpha<\lambda$ the strongly Hausdorff space R contains a discrete (or right separated, or left separated, respectively) subspace, of cardinality α. Then there exists a discrete (or right separated; or left separated, resp.) subspace of power λ in R as well.*

PROOF. Let $\{\alpha_k: k<\omega\}$ be such a strictly increasing sequence of regular cardinal numbers, for which

$$\sum_{k<\omega} \alpha_k = \lambda, \quad \text{and} \quad \alpha_0 > \omega.$$

Let R_k be a discrete (or right separated, or left separated) subspace of R with $|R_k|=\alpha_k$ $(k<\omega)$, and let

$$R' = \bigcup_{k<\omega} R_k.$$

Let us apply now Lemma 4 to R' with $\beta=\alpha_k$. We get then that we can assume, for each $k<\omega$, less than λ points of R' have neighbourhoods of cardinality $<\alpha_k$. Indeed, otherwise we should know the existence of a discrete subspace of power λ, and our theorem would be proved.

We shall define a sequence of pairwise distinct elements of R' by induction as follows: Let x_0 be any point in R', every neighbourhood of which is of power $\geqq \alpha_0$. (The existence of such a point is assured by the foregoing remark.) Assume, x_l has already been defined for each $l<k<\omega$. Then we can choose such a point $x_k \in R' \setminus \{x_0, \dots, x_{k-1}\}$, every neighbourhood of which has a cardinality $\geqq \alpha_k$, analogously as x_0 was chosen.

Since R is strongly Hausdorff, we can select such an infinite subsequence $\{x_{k_l}\}_{l<\omega} \subset \{x_k\}_{k<\omega}$, whose elements have pairvise disjoint open neighbourhoods (in R, hence in R' as well).

Let U_l be the neighbourhood of x_{k_l} in R', mentioned above. Hence $|U_l| \geqq \alpha_{k_l}$, according to the construction of the x_k's. Now

$$U_l = U_l \cap R' = U_l \cap \Big(\bigcup_{k<\omega} R_k\Big) = \bigcup_{k<\omega} (U_l \cap R_k),$$

hence there exists a $k_0 < \omega$ with

$$|U_l \cap R_{k_0}| \geqq \alpha_{k_l}.$$

In other words: U_l contains a discrete (or right separated, or left separated, resp.) subspace S_l of cardinality $\geqq \alpha_{k_l}$. But then $S = \bigcup_{l<\omega} S_l$ is a discrete (or right separated, or left separated, resp.) subspace of cardinality λ, which completes the proof.

Let us denote the class of strongly Hausdorff spaces by $\mathscr{T}_2^*$. Then a similar reasoning as in the proofs of the above theorem and lemma would yield us the following relation:

$$[d, \mathscr{T}_2^*] \to \lambda \ (\mathrm{cf}(\lambda) = \omega).$$

Note that J. DE GROOT [2] stated the problem whether each T_2-space R contains a right separated or discrete subspace of maximal cardinality. Thus Theorem 7 is a partial answer to his question.

(Received 28 July 1967)

ANALIZIS I. TANSZÉK,
EÖTVÖS LORÁND TUDOMÁNYEGYETEM,
BUDAPEST, VIII., MÚZEUM KRT. 6–8

References

[1] I. JUHÁSZ, Über ein Mächtigkeitsproblem für topologische Räume, *Annales Univ. Sci. Budapest, Sectio Math.*, **8** (1965), pp. 75—82.
[2] J. DE GROOT, Discrete subspaces of Hausdorff spaces, *Bull. Acad. Sci. Pol.*, **13** (1966), pp. 537—544.
[3] B. POSPÍŠIL, On bicompact spaces, *Publ. Fac. Sci. Univ. Masaryk*, **270** (1939).
[4] B. POSPÍŠIL, *Časopis pro Pestov. Mat. a Fys.*, **37** (1937—38), pp. 89—96.
[5] P. ERDŐS and A. TARSKI, On families of mutually exclusive sets, *Annals of Math.*, **44** (1943), pp. 315—329.
[6] I. JUHÁSZ, On extremal values of mappings. I, *Annales Univ. Sci. Budapest, Sectio Math.*, **6** (1963), pp. 39—42.
[7] G. KUREPA, On the cardinal number of ordered sets and of symmetrical structures in dependence of the cardinal numbers of its chains and antichains, *Glaznik Mat. Fiz. i Astr.*, **14** (1952), pp. 183—203.
[8] A. HAJNAL, Proof of a conjecture of S. Ruziewicz, *Fund. Math.*, **50** (1961), pp. 123—128.