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Design sensitivity analysis of nonlinear dynamic response of structural and mechanical systems

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Abstract This paper describes a unified variational theory for design sensitivity analysis of nonlinear dynamic response of structural and mechanical systems for shape, nonshape, material and mechanical properties selection, as well as control problems. The concept of an adjoint system, the principle of virtual work and a Lagrangian-Eulerian formulation to describe the deformations and the design variations are used to develop a unified view point. A general formula for design sensitivity analysis is derived and interpreted for usual performance functionals. Analytical examples are utilized to demonstrate the use of the theory and give insights for application to more complex problems that must be treated numerically.

Nomenclature

The notation for analysis of nonlinear structural mechanics problems tends to be tedious and complex. To facilitate reading of the paper, the nomenclature used in the development of various concepts is summarized here. Bold faced letters represent tensors or matrices, and a "•" between two letters implies direct tensor product.

a right superscript that identifies a quantity for the adjoint structure

$\mathbf{b}$ design variable vector

$\mathbf{B}$ a strain operator

dV differential volume in the undeformed configuration

$d\overline{V}$ differential volume in the fixed reference domain

$\mathbf{f}$ body force per unit undeformed volume

g integrand of the displacement specified boundary integral in the response functional

G integrand of the volume integral in the response functional

h integrand of the traction specified boundary integral in the response functional

J Jacobian of the transformation from the undeformed configuration to the reference volume

$\overline{J}$ area metric for transformation from the undeformed configuration to the reference volume

$\mathbf{n}$ unit normal vector to the surface $\overline{\Gamma}$

r left superscript or subscript for quantities in the reference domain

$\mathbf{R}$ surface traction

$\mathbf{R}^0$ prescribed surface traction

R subscript referring to the traction specified surface

$\mathbf{S}$ second Piola-Kirchhoff stress tensor at time t

$\mathbf{S}^a$ stress tensor for the adjoint structure

t time

T total time

T left superscript representing the quantity at the final time T

T right superscript indicating transpose of a vector or matrix

$\overline{T}$ Jacobian of the transformation for the time to the reference time-domain

u subscript referring to the displacement specified surface

$\mathbf{u}$ displacement field at time t

$\mathbf{u}^0$ prescribed displacement field

$\mathbf{u}^{a0}$ prescribed adjoint displacement field

$\mathbf{u}^a$ displacement field for the adjoint structure

${}^0x_i$ coordinates in the undeformed configuration

${}^rx_i$ coordinates of the particle of the body in the reference domain

V, $\overline{V}$ volume in the undeformed configuration and the reference domain

W^a virtual work expression in which arbitrary variations are replaced by the corresponding adjoint fields

$\mathbf{X}$ Jacobian matrix for the transformation from undeformed configuration to the reference volume

$\overline{X}$ inverse of the Jacobian matrix $\mathbf{X}$

$\mathbf{z}$ composite state vector consisting of displacement, velocity and acceleration fields

$\mathcal{A}$ augmented 'action' functional defined in (9)

α adjoint strain operator

∇ gradient operator

δ variational operator

$\tilde{\delta}\mathbf{e}^a$ an operator for the adjoint structure defined in (26)

$\hat{\delta}$ Dirac delta function

ε Green-Lagrange strain tensor at time t

ε^a strain tensor for the adjoint structure

γ Lagrangian multiplier for the terminal conditions

$I\!\!L$ augmented Lagrangian functional defined in (9)

ϕ function that specifies the terminal conditions

Γ surface in the undeformed configuration

$\overline{\Gamma}$ surface in the reference volume

Γ_R, Γ_u traction and displacement specified surfaces in the undeformed configuration

Φ functional for the constitutive law

ρ mass density at time t

Ψ performance functional in the space-time domain

ψ integrand of the performance functional Ψ

τ time in the reference time-domain

ζ terminal time in the reference time-domain

Derivatives

The 'comma' notation for partial derivatives is used, i.e. $G,_{\mathbf{u}} = \partial G/\partial \mathbf{u}$. An 'upper dot' represents material time derivative, i.e. $\ddot{\mathbf{u}} = \partial^2 \mathbf{u}/\partial t^2$. A 'prime' implies derivative with respect to the time measured in the reference time-domain, i.e. $\mathbf{u}' = \mathrm{d}\mathbf{u}/\mathrm{d}\tau$.

Design variations

The following varational notation is used for design variation of various quantities:

$\bar{\delta}(\,)$ total design variation of ( ); i.e. $\bar{\delta}(\,) = \frac{\mathrm{d}(\,)}{\mathrm{d}\mathbf{b}}\delta\mathbf{b}$

$\bar{\bar{\delta}}(\,)$ explicit design variation (partial derivative) of ( ); i.e. $\bar{\bar{\delta}}(\,) = \frac{\partial(\,)}{\partial \mathbf{b}}\delta\mathbf{b}$ for which state fields are frozen

$\tilde{\delta}(\,)$ design variation of the fields that implicitly depend on the design variables, such as displacements, strains, velocities, accelerations, etc.; also design variation of functionals with respect to the implicit state fields; for this variation, the explicit dependence on the design variables is frozen

1 Introduction

The subject of *design sensitivity analysis* (DSA) is concerned with the development of procedures for the calculation of performance functional gradients with respect to design variables. DSA represents an important tool for design improvement and is a necessary stage within the optimization process. Considerable work has been done recently in developing DSA methods for various classes of problems (Haug and Arora 1979; Adelman and Haftka 1986; Haug *et al.* 1986; Haftka and Adelman 1989; Cardoso and Arora 1988; Tsay and Arora 1990). Basically, the methods can be classified as either direct differentiation or adjoint methods (Arora and Haug 1979). Adjoint methods of sensitivity analysis may be viewed as the general Lagrangian multiplier method (Belegundu 1985). Shape and nonshape problems have been addressed independently in the literature. Material derivative idea and domain parameterization (or reference volume) concepts have been used for shape design sensitivity analysis (Dems and Mróz 1984; Haber 1986). The dynamic response problem has been addressed by Haug and Arora (1979), Hsieh and Arora (1984), Meric (1988), Haug and Mani (1984), Tortorelli and Lu (1990) and Choi and Wang (1990). Great interest has recently been shown in optimization and control of structures and so-called flexible systems. Structural and mechanical systems have been traditionally treated with separate formulations and flexible systems have been mostly included in the second group. Shape and control design problems have also been approached separately and independently; however the subject of integrated optimal control and design has been addressed recently (Belegundu 1987; Khot 1988). For a more detailed review of the subject, Adelman and Haftka (1986) and Haftka and Adelman (1989) should be consulted.

Using the concept of a fixed reference volume (Eulerian coordinates) for design variations, a Lagrangian formulation to describe the deformation of the continuum, the principle of virtual work and an adjoint structure, a unified variational theory of design sensitivity analysis has been developed for nonlinear static structures, including large strains and material nonlinearities (Cardoso and Arora 1988; Tsay and Arora 1990). Within the framework of this theory, there is no distinction between shape and nonshape design if volume integrals are used throughout the formulation. The theory has been discretized with isoparametric finite elements and applied to DSA and optimization of nonlinear structural systems (Arora and Cardoso 1989).

This paper extends the foregoing theory to nonlinear dynamics of structural and mechanical systems. In order to do that, virtual work is formulated as balance of virtual mechanical energy referred to the underformed configuration of the system. The virtual fields are replaced by the state fields of an adjoint structure. The idea of the fixed reference volume is extended to the time domain. An action integral of the performance functional augmented with the virtual work equation is transformed to the fixed space-time configuration and the Lagrangian approach of sensitivity analysis is applied to that integral. Simple analytical examples are used to show use of the theory and gain insights for its application to more complex systems.

2 Definition of the problem

Using the total Lagrangian formulation to describe the motion of the continuum, the equation of motion for the body at the time t is

$$\int(\rho\ddot{\mathbf{u}} \bullet \delta\mathbf{u} + \mathbf{S} \bullet \delta\varepsilon - \mathbf{f} \bullet \delta\mathbf{u})\,\mathrm{d}V - \int \mathbf{R} \bullet \delta\mathbf{u}\,\mathrm{d}\Gamma = 0\,, \qquad (1)$$

where all the quantities are referred to the initial or underformed configuration, δ represents variation of the state fields, '$\bullet$' refers to the standard tensor product, the upper dot '$\cdot$' refers to the material time derivative, ρ is the mass density at time $t = 0$, $\mathbf{u}$ is the displacement field, $\mathbf{S}$ is the second Piola-Kirchhoff stress measure, ε is the Green-Lagrange strain tensor, $\mathbf{f}$ is the body force per unit volume, $\mathbf{R}$ is the surface traction, V is the underformed volume of the body, and $\Gamma = \Gamma_R \cup \Gamma_u$ is the surface of the body; Γ_u and Γ_R being the parts of the surface where the displacements $\mathbf{u} = \mathbf{u}^0$ and the loads $\mathbf{R} = \mathbf{R}^0$ are prescribed, respectively; ${}^0\mathbf{u}$ and ${}^0\dot{\mathbf{u}}$ are the initial displacement and velocity, respectively. In the formulation, the left superscript will represent the time at which the quantity is measured, unless specified otherwise; no left superscript implies time t. A left subscript will represent the configuration of reference, no left subscript means configuration at $t = 0$.

The Green-Lagrange strain tensor is given as

$$\varepsilon = \frac{1}{2}[(\nabla \mathbf{u}^T) + (\nabla \mathbf{u}^T)^T + (\nabla \mathbf{u}^T)(\nabla \mathbf{u}^T)^T]\,. \tag{2}$$

The nonlinear stress-strain law, in general, may be written as

$$\mathbf{S} = \Phi(\varepsilon, \mathbf{b})\,, \tag{3}$$

where $\mathbf{b}$ is the design variable vector. It is important to note that, for many applications, the functional form for Φ is not known. In numerical implementations, the explicit form is not needed. Only an incremental stress-strain relation is required. For hereditary materials, Φ takes an integral form.

Note that an inertial reference frame is used in deriving the system equations (1)-(3). The procedure accounts for finite deformations and strains. In addition, the strain and stress measures are invariant under superposed rigid body motion. Therefore, the governing equations represent structural as well as mechanical systems.

Consider the general performance functional defined in the space-time domain as

$$\begin{aligned}\Psi &= \int \psi \, \mathrm{d}t\,,\\ &= \int \Big[\int G(\mathbf{Z}, \mathbf{b}, t) \, \mathrm{d}V(\mathbf{b}) + \int g(\mathbf{R}, \mathbf{b}, t) \, \mathrm{d}\Gamma_u(\mathbf{b}) + \\ &+ \int h(\mathbf{z}, \mathbf{b}, t) \, \mathrm{d}\Gamma_R(\mathbf{b}) \Big] \, \mathrm{d}t\,,\end{aligned} \tag{4}$$

where the vectors of the state fields at time t are given as

$$\mathbf{Z} = (\mathbf{S}, \varepsilon, \mathbf{z})\,, \quad \mathbf{z} = (\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}})\,. \tag{5}$$

Consider also a terminal condition

$${}^T\phi({}^T\mathbf{z}, {}^T\mathbf{b}, T) = 0\,, \tag{6}$$

where $[0, T]$ is the time interval of interest.

The DSA problem to be solved is to derive the total design variation of the functional (4) for a system represented by the equation of motion (1) and subject to the terminal condition (6).

3 Lagrangian form of design sensitivity analysis

Considering the total design variation of the functional Ψ in (4) with respect to the design $\mathbf{b}$, we obtain

$$\bar{\delta}\Psi = \bar{\bar{\delta}}\Psi + \tilde{\delta}\Psi\,, \tag{7}$$

where $\bar{\delta}$ represents total variation with respect to the design variables, and $\bar{\bar{\delta}}$ and $\tilde{\delta}$ represent explicit and implicit variations, respectively.

The basic idea of introducing an adjoint structure is to replace the implicit design variations of the state fields in (7) by explicit design variations and certain adjoint state fields. After replacing the arbitrary state fields by the adjoint fields, the virtual work principle of (1) may be written symbolically as

$$W^a = 0\,. \tag{8}$$

Using now the same Lagrangian methodology (Cardoso and Arora 1988b; Arora and Cardoso 1991), we form an extended 'action' functional

$$\mathbb{A} = \int \mathbb{L} \, \mathrm{d}t + \gamma^T \phi\,, \quad \mathbb{L} = \psi - W^a\,, \tag{9}$$

where γ is a multiplier. Since $\bar{\delta} W^a = \bar{\delta}\,({}^T\phi) = 0$, the total design variation of (9) gives

$$\bar{\delta}\mathbb{A} = \bar{\delta}\Psi\,. \tag{10}$$

Now if we require the implicit design variation of $\mathbb{A}$ to vanish, i.e.

$$\tilde{\delta}\mathbb{A} = 0\,, \tag{11}$$

then, the total design variation of the functional Ψ is given as

$$\bar{\delta}\Psi = \bar{\bar{\delta}}\mathbb{A}\,. \tag{12}$$

Equation (11) leads to the definition of the adjoint problem as explained in the sequel. Equation (12) shows that the total design variation of the performance functional Ψ is given by the explicit design variation of the action functional in (9).

4 Design sensitivity analysis

Before performing the design variations as stated in Section 3, we proceed with the transformation of the various quantities to a fixed reference domain (Haber 1986; Cardoso and Arora 1988). In the reference domain, the volume V is mapped onto a fixed volume $\overline{V}$ with the boundary as $\overline{\Gamma}$. This transformation of the independent variables needs to be introduced into all the governing equations and state fields. The Jacobian of the space transformation is given as

$$J = |\mathbf{X}|\,, \quad \mathbf{X} = \frac{\partial({}^0x_1, {}^0x_2, {}^0x_3)}{\partial({}^rx_1, {}^rx_2, {}^rx_3)}\,, \quad \overline{\mathbf{X}} = \mathbf{X}^{-1}\,. \tag{13}$$

The area metric $\overline{J}$ is defined as

$$\overline{J} = J \|\overline{\mathbf{X}}^T \mathbf{n}\|\,. \tag{14}$$

The time domain $t \in [0, T]$ is transformed to $\tau \in [0, \zeta]$ as

$$T = \overline{T}\zeta\,. \tag{15}$$

In the foregoing equations, superscript or subscript r refers to the reference coordinates and $\mathbf{n}$ is the unit normal to the surface $\overline{\Gamma}$. For oriented bodies such as bars or beams, J and $|\mathbf{X}|$ may be different from each other if we use volume integrals throughout the sensitivity analysis. The time Jacobian $\overline{T}$ may be the total time T if the time interval is transformed to the unit interval in reference configuration, it may be the inverse of the frequency in problems where the reference configuration is in the phase domain, or still it may be equal to t/ω if the reference configuration is in the frequency domain.

We replace the virtual state fields for the primary structure in (1) by certain adjoint fields identified with the superscript 'a'. Therefore, we have the following transformed equations.

Virtual work equation (8) at time t

$$W^a \equiv \int (\rho\ddot{\mathbf{u}} \bullet \mathbf{u}^a + \mathbf{S} \bullet \varepsilon^a - \mathbf{f} \bullet \mathbf{u}^a) J \,\mathrm{d}\overline{V} -$$

$$- \int \mathbf{R} \bullet \mathbf{u}^a \overline{J} \,\mathrm{d}\overline{\Gamma} = 0 \,, \tag{16}$$

where ε^a replaces $\delta\varepsilon$ after substitution of $\delta\mathbf{u}$ by $\mathbf{u}^a$. Equations that determine these fields will be derived later.

Green-Lagrange strain tensor

$$\varepsilon = \mathbf{B}(\mathbf{u}^T) \,,$$

$$\mathbf{B}() = \frac{1}{2}\{\overline{\mathbf{X}}^T[{}_r\nabla()] + [{}_r\nabla()]^T\overline{\mathbf{X}} + \overline{\mathbf{X}}^T[{}_r\nabla()][{}_r\nabla()\overline{\mathbf{X}}]\} \,. \tag{17}$$

Adjoint strain tensor

$$\varepsilon^a = \alpha(\mathbf{u}^a) \,, \quad \alpha() = \frac{1}{2}\{\overline{\mathbf{X}}^T[{}_r\nabla()] + [{}_r\nabla()]^T\overline{\mathbf{X}} +$$

$$+ \overline{\mathbf{X}}^T[{}_r\nabla()][{}_r\nabla\mathbf{u}^T]^T\overline{\mathbf{X}} + \overline{\mathbf{X}}^T[{}_r\nabla\mathbf{u}^T][{}_r\nabla()]^T\overline{\mathbf{X}}\} \,. \tag{18}$$

Velocity and acceleration fields

$$\dot{\mathbf{u}} = \mathrm{d}\mathbf{u}/\mathrm{d}t = (\mathrm{d}\mathbf{u}/\mathrm{d}\tau)(\mathrm{d}\tau/\mathrm{d}t) = \mathbf{u}'\overline{T}^{-1}, \quad \ddot{\mathbf{u}} = \mathbf{u}''\overline{T}^{-2}. \tag{19}$$

Functional for sensitivity analysis

$$\Psi = \int \Big[\int G(\mathbf{Z}, \mathbf{b}, \tau) J(\mathbf{b}) \,\mathrm{d}\overline{V} + \int g(\mathbf{R}, \mathbf{b}, \tau) \overline{J}(\mathbf{b}) \,\mathrm{d}\overline{\Gamma}_u +$$

$$+ \int h(\mathbf{z}, \mathbf{b}, \tau) \overline{J}(\mathbf{b}) \,\mathrm{d}\overline{\Gamma}_R \Big] \overline{T} \,\mathrm{d}\tau \,. \tag{20}$$

Note that the integrands in (20) are now given in terms of the reference coordinates.

Carrying out the design variations as indicated in (12), combining terms, and replacing certain terms by the adjoint stress $\mathbf{S}^a$ (to be defined later), using (9) and considering time $\overline{T}$ as an independent field, the total variation of the performance functional is

$$\bar{\bar{\delta}}\Psi = \int \bar{\bar{\delta}} \mathbb{L} \,\mathrm{d}t + \gamma \bar{\bar{\delta}}{}^T\phi =$$

$$= \int (\bar{\bar{\delta}}\psi - \bar{\bar{\delta}}W^a) \,\mathrm{d}t + \gamma \bar{\bar{\delta}}{}^T\phi =$$

$$= \int \Big[\int \{[G_{,b} \bullet \bar{\bar{\delta}}\mathbf{b} + (G_{,\overline{T}} + G\overline{T}^{-1}) \bar{\bar{\delta}}\overline{T} + \mathbf{u}^a \bullet (\bar{\bar{\delta}}\mathbf{f} - \ddot{\mathbf{u}}\bar{\bar{\delta}}\rho) -$$

$$-(\varepsilon^a - G_{,S}) \bullet \bar{\bar{\delta}}\Phi - \mathbf{S}^a \bullet \bar{\bar{\delta}}\varepsilon - \mathbf{S} \bullet \bar{\bar{\delta}}\varepsilon^a] J +$$

$$+ [G - \rho\ddot{\mathbf{u}} \bullet \mathbf{u}^a - \mathbf{S} \bullet \varepsilon^a + \mathbf{f} \bullet \mathbf{u}^a] \bar{\bar{\delta}} J \} \,\mathrm{d}\overline{V} +$$

$$+ \int \{[h_{,b} \bullet \bar{\bar{\delta}}\mathbf{b} + (h_{,\overline{T}} + h\overline{T}^{-1}) \bar{\bar{\delta}}\overline{T} + \mathbf{u}^a \bullet \bar{\bar{\delta}}\mathbf{R}^0]\overline{J} +$$

$$+ [h + \mathbf{R}^0 \bullet \mathbf{u}^a] \bar{\bar{\delta}}\overline{J}\} \,\mathrm{d}\overline{\Gamma}_R + \int \Big\{ [g_{,b} \bullet \bar{\bar{\delta}}\mathbf{b} + (g_{,\overline{T}} + g\overline{T}^{-1}) \bar{\bar{\delta}}\overline{T}]\overline{J} +$$

$$+ [g + \mathbf{u}^{a0} \bullet \mathbf{R}] \bar{\bar{\delta}}\overline{J} \Big\} \,\mathrm{d}\overline{\Gamma}_u \Big] \overline{T} \,\mathrm{d}\tau + \gamma ({}^T\phi_{,b} \bullet \bar{\bar{\delta}}\mathbf{b} + {}^T\phi_{,\overline{T}} \bar{\bar{\delta}}\overline{T}), \tag{21}$$

where $\bar{\bar{\delta}}\Phi$ stands for design variations of the stress-strain law (3) with respect to the material parameters, and explicit design variations of the primary and adjoint strains in (17) and (18), due to dependence on $\overline{\mathbf{X}}$, are given as

$$\bar{\bar{\delta}}\varepsilon = \bar{\bar{\delta}}\mathbf{B}(\mathbf{u}^T) \,, \quad \bar{\bar{\delta}}\varepsilon^a = \bar{\bar{\delta}}\alpha(\mathbf{u}^T) \,. \tag{22}$$

In derivations to follow, we will enforce

$$\mathbf{u}^{a0} = -g_{,\mathbf{R}} \text{ on } \Gamma_u \,, \tag{23}$$

as the prescribed displacement for the adjoint structure, and

$$\mathbf{S}^a = (\varepsilon^a - G_{,S}) \bullet \Phi_{,\varepsilon} - G_{,\varepsilon} \,, \tag{24}$$

is the constitutive relation for the adjoint structure.

Implicit design variation of the extended functional is

$$\tilde{\delta}\mathbb{A} = \int (\tilde{\delta}\mathbb{L}\overline{T} + \mathbb{L}\tilde{\delta}\overline{T}) \,\mathrm{d}\tau + \gamma\tilde{\delta}({}^T\phi) =$$

$$= \int \{(\tilde{\delta}\psi - \tilde{\delta}W^a)\overline{T} + \psi\tilde{\delta}\overline{T}\} \,\mathrm{d}\tau + \gamma\tilde{\delta}({}^T\phi) =$$

$$= \int \Big(\int \{-\rho\mathbf{u}^a \bullet \tilde{\delta}\ddot{\mathbf{u}} - \mathbf{S}^a \bullet \tilde{\delta}\varepsilon - \mathbf{S} \bullet \tilde{\delta}\eta^a + (G_{,z} + \mathbf{f}_{,z} \bullet \mathbf{u}^a) \bullet \tilde{\delta}\mathbf{z} +$$

$$+ (G\overline{T}^{-1} + G_{,\overline{T}} + \mathbf{f}_{,\overline{T}} \bullet \mathbf{u}^a)\tilde{\delta}\overline{T}\} J \,\mathrm{d}\overline{V} +$$

$$+ \int (g\overline{T}^{-1} + g_{,\overline{T}})\tilde{\delta}\overline{T}\,\overline{J} \,\mathrm{d}\overline{\Gamma}_u + \int \Big\{ (h_{,z} + \mathbf{R}^0{}_{,z} \bullet \mathbf{u}^a) \bullet \tilde{\delta}\mathbf{z} +$$

$$+ (h\overline{T}^{-1} + h_{,\overline{T}}\delta\overline{T} + \mathbf{R}^0{}_{,\overline{T}} \bullet \mathbf{u}^a)\delta\overline{T} \Big\} \overline{J} \,\mathrm{d}\overline{\Gamma}_R \Big) \overline{T} \,\mathrm{d}\tau +$$

$$+ \gamma({}^T\phi_{,z}\,\tilde{\delta}\mathbf{z} + {}^T\dot{\phi}\zeta\tilde{\delta}\overline{T}) \,, \tag{25}$$

where $\overline{T}$ is taken as implicitly dependent on the state fields, as for the minimum time control problems (an example illustrates this point later),

$$\tilde{\delta}\varepsilon^a = \tilde{\delta}\mathbf{e}^a + \tilde{\delta}\eta^a \,, \quad \tilde{\delta}\mathbf{e}^a = \alpha(\tilde{\delta}\mathbf{u}^a) \,,$$

$$\tilde{\delta}\eta^a = \frac{1}{2}\{\overline{\mathbf{X}}^T[{}_r\nabla\mathbf{u}^{a^T}][{}_r\nabla\delta\mathbf{u}^T]^T\overline{\mathbf{X}} +$$

$$+ \overline{\mathbf{X}}^T[{}_r\nabla\delta\mathbf{u}^T][{}_r\nabla\mathbf{u}^{a^T}]^T\overline{\mathbf{X}}\} \,, \tag{26}$$

and the following expression has been used since it constitutes the equation of motion of the primary structure:

$$\int(-\rho\ddot{\mathbf{u}}\bullet\tilde{\delta}\mathbf{u}^a-\mathbf{S}\bullet\tilde{\delta}\mathbf{e}^a+\mathbf{f}\bullet\tilde{\delta}\mathbf{u}^a)J\,\mathrm{d}\overline{V}+\int\mathbf{R}^0\bullet\tilde{\delta}\mathbf{u}^a\overline{J}\,\mathrm{d}\overline{\Gamma}_R=$$
$$=0\,.$$

Substituting now for $\mathbf{z}$ from (5), using the implicit design variations of velocity and acceleration in (19) as

$$\bar{\delta}\dot{\mathbf{u}}=\overline{T}^{-1}\bar{\delta}\mathbf{u}'-\overline{T}^{-2}\mathbf{u}'\tilde{\delta}\overline{T}=\overline{T}^{-1}(\bar{\delta}\mathbf{u}'-\dot{\mathbf{u}}\tilde{\delta}\overline{T})\,,$$
$$\bar{\delta}\ddot{\mathbf{u}}=\overline{T}^{-2}\bar{\delta}\mathbf{u}''-2T^{-3}\mathbf{u}''\tilde{\delta}\overline{T}=\overline{T}^{-2}\bar{\delta}\mathbf{u}''-2\overline{T}^{-1}\ddot{\mathbf{u}}\tilde{\delta}\overline{T}\,, \qquad (27)$$

and integrating by parts, (25) becomes

$$\begin{aligned}
\tilde{\delta}\mathcal{A}=&\int\Bigg[\int\{-\rho\ddot{\mathbf{u}}^a\bullet\bar{\delta}\mathbf{u}-\mathbf{S}^a\bullet\tilde{\delta}\varepsilon-\mathbf{S}\bullet\tilde{\delta}\eta^a+[G,_{u}-\dot{G},_{\dot{u}}+\\
&+\ddot{G},_{\ddot{u}}+(\mathbf{f},_{u}-\dot{\mathbf{f}},_{\dot{u}}+\ddot{\mathbf{f}},_{\ddot{u}})\bullet\mathbf{u}^a-(\mathbf{f},_{\dot{u}}-2\dot{\mathbf{f}},_{\ddot{u}})\bullet\dot{\mathbf{u}}^a+\\
&+\mathbf{f},_{\ddot{u}}\bullet\ddot{\mathbf{u}}^a]\bullet\bar{\delta}\mathbf{u}+[G+G,_{\overline{T}}\overline{T}-G,_{\dot{u}}\bullet\dot{\mathbf{u}}-2G,_{\ddot{u}}\bullet\ddot{\mathbf{u}}+\\
&+(\mathbf{f},_{\overline{T}}\overline{T}-\mathbf{f},_{\dot{u}}\bullet\dot{\mathbf{u}}-2\mathbf{f},_{\ddot{u}}\bullet\ddot{\mathbf{u}}+2\rho\ddot{\mathbf{u}})\bullet\mathbf{u}^a]\overline{T}^{-1}\tilde{\delta}\overline{T}\}\overline{J}\,\mathrm{d}\overline{V}+\\
&+\int\{g+g,_{\overline{T}}\overline{T}\}\overline{T}^{-1}\tilde{\delta}\overline{T}\,\overline{J}\,\mathrm{d}\overline{\Gamma}_u+\int\{h,_{u}-\dot{h},_{\dot{u}}+\ddot{h},_{\ddot{u}}+\\
&+(\mathbf{R}^0,_{u}-\dot{\mathbf{R}}^0,_{\dot{u}}+\ddot{\mathbf{R}}^0,_{\ddot{u}})\bullet\mathbf{u}^a-(\mathbf{R}^0,_{\dot{u}}-2\dot{\mathbf{R}}^0,_{\ddot{u}})\bullet\dot{\mathbf{u}}^a+\\
&+\mathbf{R}^0,_{\ddot{u}}\bullet\ddot{\mathbf{u}}^a]\bullet\bar{\delta}\mathbf{u}+[h+h,_{\overline{T}}\overline{T}-h,_{\dot{u}}\bullet\dot{\mathbf{u}}-2h,_{\ddot{u}}\bullet\ddot{\mathbf{u}}+\\
&+(\mathbf{R}^0,_{\overline{T}}\overline{T}-\mathbf{R}^0,_{\dot{u}}\bullet\dot{\mathbf{u}}-2\mathbf{R}^0,_{\ddot{u}}\bullet\ddot{\mathbf{u}})\bullet\mathbf{u}^a]\overline{T}^{-1}\tilde{\delta}\overline{T}\}\overline{J}\,\mathrm{d}\overline{\Gamma}_R\Bigg]\overline{T}\,\mathrm{d}\tau+\\
&+\int\Big\{[\rho(^T\dot{\mathbf{u}}^a)+{}^TG,_{\dot{u}}-{}^T\dot{G},_{\ddot{u}}+(^T\mathbf{f},_{\dot{u}}-{}^T\dot{\mathbf{f}},_{\ddot{u}})\bullet(^T\mathbf{u}^a)-\\
&-{}^T\mathbf{f},_{\ddot{u}}\bullet(^T\dot{\mathbf{u}}^a)]\bullet\tilde{\delta}(^T\mathbf{u})+\\
&+[-\rho(^T\mathbf{u}^a)+{}^TG,_{\ddot{u}}+{}^T\mathbf{f},_{\ddot{u}}\bullet(^T\mathbf{u}^a)]\bullet\tilde{\delta}(^T\dot{\mathbf{u}})\Big\}J\,\mathrm{d}\overline{V}+\\
&+\int\Big\{[^Th,_{\dot{u}}-{}^T\dot{h},_{\ddot{u}}+(^T\mathbf{R}^0,_{\dot{u}}-{}^T\dot{\mathbf{R}}^0,_{\ddot{u}})\bullet(^T\mathbf{u}^a)-\\
&-{}^T\mathbf{R}^0,_{\ddot{u}}(^T\dot{\mathbf{u}}^a)]\bullet\tilde{\delta}(^T\mathbf{u})+\\
&+[^Th,_{\ddot{u}}+{}^T\mathbf{R}^0,_{\ddot{u}}(^T\mathbf{u}^a)]\bullet\tilde{\delta}(^T\dot{\mathbf{u}})\Big\}\overline{J}\,\mathrm{d}\overline{\Gamma}_R+\\
&+\gamma[^T\phi,_{u}\bullet\tilde{\delta}(^T\mathbf{u})+{}^T\phi,_{\dot{u}}\bullet\tilde{\delta}(^T\dot{\mathbf{u}})+\\
&+{}^T\phi,_{\ddot{u}}\bullet\tilde{\delta}(^T\ddot{\mathbf{u}})+{}^T\dot{\phi}\zeta\tilde{\delta}\overline{T}]\,,
\end{aligned} \qquad (28)$$

where initial conditions have been assumed satisfied which eliminates certain terms at $t=0$. Using the condition of (11) in (28), the adjoint problem is defined as

$$\int\{\rho\ddot{\mathbf{u}}^a\bullet\bar{\delta}\mathbf{u}+\mathbf{S}^a\bullet\tilde{\delta}\varepsilon+\mathbf{S}\bullet\tilde{\delta}\eta^a\}J\,\mathrm{d}\overline{V}=$$
$$\begin{aligned}
&=\int[G,_{u}-\dot{G},_{\dot{u}}+\ddot{G},_{\ddot{u}}+(\mathbf{f},_{u}-\dot{\mathbf{f}},_{\dot{u}}+\ddot{\mathbf{f}},_{\ddot{u}})\bullet\mathbf{u}^a-\\
&-(\mathbf{f},_{\dot{u}}-2\dot{\mathbf{f}},_{\ddot{u}})\bullet\dot{\mathbf{u}}^a+\mathbf{f},_{\ddot{u}}\bullet\ddot{\mathbf{u}}^a]\bullet\bar{\delta}\mathbf{u}J\,\mathrm{d}\overline{V}+\\
&+\int\{h,_{u}-\dot{h},_{\dot{u}}+\ddot{h},_{\ddot{u}}+(\mathbf{R}^0,_{u}-\dot{\mathbf{R}}^0,_{\dot{u}}+\ddot{\mathbf{R}}^0,_{\ddot{u}})\bullet\mathbf{u}^a-\\
&-(\mathbf{R}^0,_{\dot{u}}-2\dot{\mathbf{R}}^0,_{\ddot{u}})\bullet\dot{\mathbf{u}}^a+\mathbf{R}^0,_{\ddot{u}}\bullet\ddot{\mathbf{u}}^a\}\bullet\bar{\delta}\mathbf{u}\overline{J}\,\mathrm{d}\overline{\Gamma}_R\,,
\end{aligned} \qquad (29)$$

$$\begin{aligned}
&\int\rho(^T\dot{\mathbf{u}}^a)\bullet\tilde{\delta}(^T\mathbf{u})J\,\mathrm{d}\overline{V}=\int\{-{}^TG,_{\dot{u}}+{}^T\dot{G},_{\ddot{u}}-\\
&-(^T\mathbf{f},_{\dot{u}}-{}^T\dot{\mathbf{f}},_{\ddot{u}})\bullet(^T\mathbf{u}^a)+{}^T\mathbf{f},_{\ddot{u}}\bullet(^T\dot{\mathbf{u}}^a)\}\bullet\tilde{\delta}(^T\mathbf{u})\overline{J}\,\mathrm{d}\overline{V}+\\
&+\int\{-{}^Th,_{\dot{u}}+{}^T\dot{h},_{\ddot{u}}-(^T\mathbf{R}^0,_{\dot{u}}-{}^T\dot{\mathbf{R}}^0,_{\ddot{u}})\bullet(^T\mathbf{u}^a)+\\
&+{}^T\mathbf{R}^0,_{\ddot{u}}\bullet(^T\dot{\mathbf{u}}^a)\}\bullet\tilde{\delta}(^T\mathbf{u})\overline{J}\,\mathrm{d}\overline{\Gamma}_R-\gamma(^T\phi,_{u})\bullet(\tilde{\delta}^T\mathbf{u})\,,
\end{aligned} \qquad (30)$$

$$\begin{aligned}
&\int\rho(^T\mathbf{u}^a)\bullet\tilde{\delta}(^T\dot{\mathbf{u}})J\,\mathrm{d}\overline{V}=\\
&=\int\{^TG,_{\ddot{u}}+{}^T\mathbf{f},_{\ddot{u}}\bullet(^T\mathbf{u}^a)\}\bullet\tilde{\delta}(^T\dot{\mathbf{u}})J\,\mathrm{d}\overline{V}+\\
&+\int\{^Th,_{\ddot{u}}+{}^T\mathbf{R}^0,_{\ddot{u}}\bullet(^T\mathbf{u}^a)\}\bullet\tilde{\delta}(^T\dot{\mathbf{u}})\overline{J}\,\mathrm{d}\overline{\Gamma}_R+\\
&+\gamma(^T\phi,_{\dot{u}})\bullet\tilde{\delta}(^T\dot{\mathbf{u}})\,,
\end{aligned} \qquad (31)$$

$$\gamma(^T\phi,_{\ddot{u}})\bullet\tilde{\delta}(^T\ddot{\mathbf{u}})=0\,, \qquad (32)$$

$$\begin{aligned}
&\int\Bigg[\int\{G+G,_{\overline{T}}\overline{T}-G,_{\dot{u}}\bullet\dot{\mathbf{u}}-2G,_{\ddot{u}}\bullet\ddot{\mathbf{u}}+\\
&+(\mathbf{f},_{\overline{T}}\overline{T}-\mathbf{f},_{\dot{u}}\bullet\dot{\mathbf{u}}-2\mathbf{f},_{\ddot{u}}\bullet\ddot{\mathbf{u}}+2\rho\ddot{\mathbf{u}})\bullet\mathbf{u}^a\}J\,\mathrm{d}\overline{V}+\\
&+\int\{h+h,_{\overline{T}}\overline{T}-h,_{\dot{u}}\bullet\dot{\mathbf{u}}-2h,_{\ddot{u}}\bullet\ddot{\mathbf{u}}+\\
&+(\mathbf{R}^0,_{\overline{T}}\overline{T}-\mathbf{R}^0,_{\dot{u}}\bullet\dot{\mathbf{u}}-2\mathbf{R}^0,_{\ddot{u}}\bullet\ddot{\mathbf{u}})\bullet\mathbf{u}^a\}\overline{J}\,\mathrm{d}\overline{\Gamma}_R+\\
&+\int\{g+g,_{\overline{T}}\overline{T}\}\overline{J}\,\mathrm{d}\overline{\Gamma}_u\Bigg]\tilde{\delta}\overline{T}\,\mathrm{d}\tau+\gamma(^T\dot{\phi})\zeta\tilde{\delta}\overline{T}=0\,.
\end{aligned} \qquad (33)$$

The adjoint equations (29)-(33) solve for the adjoint state fields required for design variation calculations in (21). Equations (30)-(32) are the terminal conditions for (29), and (33) solves for the Lagrangian multiplier γ. Damping forces are included as part of the applied body forces $\mathbf{f}$ and surface forces $\mathbf{R}^0$. Comparing (29) with (1), we see that the adjoint structural masses and primary structural masses are identical. Also comparing (29) with the incremental form of (1) (Cardoso and Arora 1988), we observe that the adjoint stiffness is the tangential stiffness of the primary structure. We may note, however, that the adjoint system is *dependent* on the trajectory of the primary system. This means that the history of the primary state fields must be memorized in order to have it available for the *backward integration* of the adjoint problem. Also note that the unknown fields appear on the right-hand side of (29). This implies that iterations will be necessary to solve (29). The other option would be to keep all the unknown terms on the left-hand side of (29) and deal with asymmetric operators which can also be computationally expensive. If the performance functional is linearly dependent on the state fields, as is the case for many structural response functionals, and the primary system is linear, then the adjoint system is not dependent on the state fields. This leads to considerable simplifications in numerical computations.

In (21), no distinction has been made between design and control variables. Usually, control is included in forces $\mathbf{f}$ and $\mathbf{R}^0$. Thus, design variations $\bar{\bar{\delta}}\mathbf{f}$ and $\bar{\bar{\delta}}\mathbf{R}^0$ may be viewed to include control variations, if we keep control variations inside the time integral. This means that the continuum formulation in time does not provide sensitivities with respect to control in the sense of total derivatives $\bar{\delta}\Psi/\bar{\delta}\mathbf{b}$, but only control variations $\bar{\delta}\psi$. If $\mathbf{b}$ is not dependent on time (design variable), then $\bar{\delta}\Psi/\bar{\delta}\mathbf{b} = \int(\bar{\delta}\psi/\bar{\delta}\mathbf{b})\,\mathrm{d}t$. For a time-discretized system there is no such distinction.

5 Examples

5.1 Nonlinear freely vibrating structure

Consider the two-bar structure shown in Fig. 1, with an attached concentrated mass M at the centre node. The mass of the bars is neglected in comparison with the mass M. Consider the free vibration of the structure induced by an initial displacement ${}^0w$ of the central node. Transformation to the reference domain is shown in the figure. The design variables for the problem are $\mathbf{b} \equiv (M, E, A, L)$. The Green-Lagrange strain measure and the stress in the members are given, respectively, as

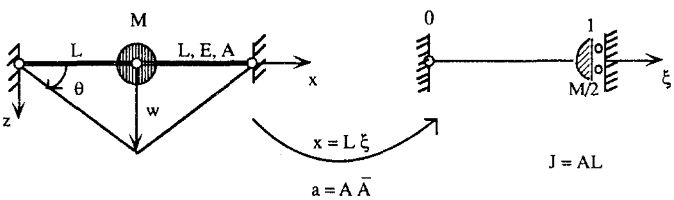

Fig. 1. Two-bar vibrating structure and mapping of half the structure to a control volume

$$\varepsilon = \frac{1}{2}[({}^tL^2 - L^2)/L^2] = \frac{1}{2}w^2L^{-2}\,, \quad S = E\varepsilon\,. \tag{34}$$

This is a geometrically nonlinear problem where the frequency of free vibration depends on the initial conditions. The velocity at the centre is calculated as (Panovko 1971)

$$\dot{w}^2 = \frac{1}{2}EAL^{-3}M^{-1}({}^0w^4 - w^4)\,. \tag{35}$$

It is required to perform design sensitivity analysis for the velocity of the central node at the time $t = p$, when the corresponding displacement vanishes for the first time (one fourth of the period of vibration). The functional for sensitivity analysis is given as

$$\Psi = \int\int_0^1\int_{\bar{A}}(AL)^{-1}\dot{w}\hat{\delta}(\xi-1)\hat{\delta}(t-p)AL\,\mathrm{d}\bar{A}\,\mathrm{d}\xi\,\mathrm{d}t =$$

$$= \int\int_0^1\int_{\bar{A}} -(AL)^{-1}w\hat{\delta}(\xi-1)\dot{\hat{\delta}}(t-p)AL\,\mathrm{d}\bar{A}\,\mathrm{d}\xi\,\mathrm{d}t =$$

$$= \int\int GJ\,AL\mathrm{d}\bar{V}\,\mathrm{d}t\,,$$

$$G = -(AL)^{-1}w\hat{\delta}(\xi-1)\dot{\hat{\delta}}(t-p)\,, \qquad J = AL\,,$$

$$G_{,w} = -(AL)^{-1}\hat{\delta}(\xi-1)\dot{\hat{\delta}}(t-p)\,, \tag{36}$$

where the symbol $\hat{\delta}$ represents a Dirac delta function. The adjoint problem is given from (29) as

$$\frac{1}{2}M\ddot{w}^a \bullet \delta w + \int(S^a\delta\varepsilon + S\delta\eta^a)J\,\mathrm{d}\bar{V} = \int G_{,w}\,\delta wJ\,\mathrm{d}\bar{V}\,, \tag{37}$$

with the time boundary conditions

$$^p\dot{w}^a = {}^pw^a = 0\,. \tag{38}$$

Substituting into (37) the quantities

$$\delta\varepsilon = \delta\Big(\frac{1}{2}w^2L^{-2}\Big) = wL^{-2}\delta w\,,$$

$$\varepsilon^a = ww^aL^{-2}\,, \quad S^a = E\varepsilon^a\,,$$

$$\delta\varepsilon^a = wL^{-2}\delta w^a + w^aL^{-2}\delta w = \delta e^a + \delta\eta^a\,, \tag{39}$$

we have the adjoint problem as

$$\frac{1}{2}M\ddot{w}^a + \frac{3}{2}EAL^{-3}w^2w^a = -\dot{\hat{\delta}}(t-p)\,. \tag{40}$$

The solution for the adjoint problem is

$$w^a = 2(M\Omega^a)^{-1}\int_0^t -\dot{\hat{\delta}}(\sigma-p)\sin\Omega^a(t-\sigma)\,\mathrm{d}\sigma =$$

$$= 2M^{-1}\cos\Omega^a(t-p),$$

$$\text{where } (\Omega^a)^2 = 3M^{-1}EAL^{-3}w^2. \tag{41}$$

The adjoint total axial displacement and the displacement at any point are given as

$$u^a(L) = w^a\sin\theta = w^awL^{-1}\,, \quad u^a(\xi) = w^awL^{-1}\xi\,. \tag{42}$$

The general design sensitivity equation (21) reduces to

$$\bar{\delta}\Psi = \frac{1}{2}\bar{\delta}M \int_0^p -\ddot{w}w^a\,\mathrm{d}t + \int_0^p \Big[\int_0^1 \int_{\bar{A}} \{[-\varepsilon^a \bar{\bar{\delta}}\Phi - S^a \bar{\bar{\delta}}\varepsilon - S\bar{\bar{\delta}}\varepsilon^a +$$

$$+\bar{\bar{\delta}}G]J + [G - S\varepsilon^a]\bar{\delta}J\}\,\mathrm{d}A\,\mathrm{d}\xi\Big]\,\mathrm{d}t\,. \tag{43}$$

Various quantities for (43) are given in (39) and (41), and

$$\bar{\delta}J = L\bar{\delta}A + A\bar{\delta}L\,, \quad \bar{\bar{\delta}}GJ + G\bar{\delta}J = 0\,,$$

$$\bar{\bar{\delta}}\Phi = \varepsilon\bar{\delta}E\,, \quad \bar{\bar{\delta}}\varepsilon = \frac{1}{2}w^2\bar{\delta}L^{-2} = -w^2L^{-3}\bar{\delta}L\,,$$

$$\bar{\bar{\delta}}\varepsilon^a = ww^a\bar{\delta}L^{-2} = -ww^aL^{-3}\bar{\delta}L\,,$$

$$\int_0^p -w^a\ddot{w}\,\mathrm{d}t = -M^{-1\,p}\dot{w}\,. \tag{44}$$

Substituting these quantities into (43), we obtain

$$\bar{\delta}\Psi = ({}^p\dot{w})\Big[-\frac{1}{2}M^{-1}\bar{\delta}M + \frac{1}{2}E^{-1}\bar{\delta}E + \frac{1}{2}A^{-1}\bar{\delta}A -$$

$$-\frac{3}{2}L^{-1}\bar{\delta}L\Big]\,, \tag{45}$$

which can also be obtained directly from (35).

5.2 Harmonic oscillator

Consider the simple harmonic oscillator shown in Fig. 2, where the time T required for the mass M to move its initial pre-compressed resting position ${}^0u = 0$ to its final position, ${}^Tu$ is the quantity for design sensitivity analysis with respect to the vector $\mathbf{b} \equiv (M, K, a)$, where a is the pre-loaded distance. The motion and response of the system are

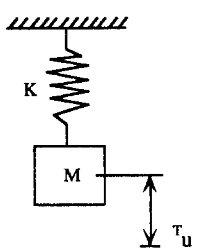

Fig. 2. Harmonic oscillator

$$M\ddot{u} + Ku = Ka\,, \quad {}^0u = {}^0\dot{u} = 0\,, \quad u = a(1-\cos\omega t)\,,$$

$$\omega^2 = K/M\,. \tag{46}$$

The functional for sensitivity analysis is given as

$$\Psi \equiv T = \int_0^T \mathrm{d}t = \int_0^1 GT\,\mathrm{d}\tau\,, \quad G = 1\,. \tag{47}$$

The adjoint problem of (29)-(33) is given as

$$M\ddot{u}^a + Ku^a = 0\,, \quad M({}^T\dot{u}^a) = -\gamma\,({}^T\phi_{,u}) \equiv -\gamma\,,$$

$${}^Tu^a = 0\,, \quad \int_0^1 G\,\mathrm{d}\tau + \gamma({}^T\dot{\phi})\zeta \equiv 1 + \gamma({}^T\dot{u}) = 0\,, \tag{48}$$

and its solution gives

$$\gamma = -({}^T\dot{u})^{-1}\,, \quad u^a = [M\omega\,({}^T\dot{u})]^{-1}\sin\omega(t-T)\,. \tag{49}$$

Now, the total design variation of the total time functional of (47) is performed using (21), which for this case is simplified as

$$\bar{\delta}\Psi = \bar{\bar{\delta}}M\int_0^T -u^a\ddot{u}\,\mathrm{d}t + \bar{\bar{\delta}}K\int_0^T(-u^au + au^a)\,\mathrm{d}t +$$

$$+\bar{\bar{\delta}}aK\int_0^T u^a\,\mathrm{d}t\,. \tag{50}$$

Substituting (46) and (49) into (50), we have

$$\int_0^T u^a\,\mathrm{d}t = -[M\omega\,({}^T\dot{u})]^{-1}\int_0^T \sin\omega(t-T)\,\mathrm{d}t =$$

$$= [K\,({}^T\dot{u})]^{-1}(1-\cos\omega T) = (Ka\,{}^T\dot{u})^{-1}u\,,$$

$$\int_0^T -u^au\,\mathrm{d}t =$$

$$= -[M\omega\,({}^T\dot{u})]^{-1}a\int_0^T[(1-\cos\omega T)\,\sin\omega(t-T)]\,\mathrm{d}t =$$

$$= [K\,({}^T\dot{u})]^{-1}\Big[a(1-\cos\omega T) - \frac{1}{2}T\,({}^T\dot{u})\Big] =$$

$$= -\frac{1}{2}TK^{-1} + [K\,({}^T\dot{u})]^{-1}a(1-\cos\omega T)\,,$$

$$\int_0^T -u^a\ddot{u}\,\mathrm{d}t = -[M\,({}^T\dot{u})]^{-1}\omega a\int_0^T \cos\omega t\,\sin\omega(t-T)\,\mathrm{d}t =$$

$$= \frac{1}{2}[M\,({}^T\dot{u})]^{-1}a\omega T\,\sin\omega T = \frac{1}{2}TM^{-1}\,.$$

Therefore, the total design variation of Ψ from (50) is

$$\bar{\delta}\Psi = \frac{1}{2}T(M^{-1}\bar{\bar{\delta}}M - K^{-1}\bar{\bar{\delta}}K) + u({}^T\dot{u}a)^{-1}\bar{\bar{\delta}}a\,. \tag{51}$$

This result can be verified by first calculating the total time from (46) as

$$T = \omega^{-1}\cos^{-1}[1 - ({}^T u/a)]\,, \quad \sin\omega T = (a\omega)^{-1}\,({}^T\dot{u})\,,$$

and then taking its direct design variation as

$$\bar{\delta}\Psi \equiv \bar{\delta}T = -T\omega^{-1}\bar{\bar{\delta}}\omega + u({}^T\dot{u}a)^{-1}\bar{\bar{\delta}}a =$$

$$= \frac{1}{2}T(M^{-1}\bar{\bar{\delta}}M - K^{-1}\bar{\bar{\delta}}K) + u({}^T\dot{u}a)^{-1}\bar{\bar{\delta}}a\,,$$

which matches with (51).

5.3 Nonlinear system with harmonic loading

The equation of motion of a nonlinear system harmonically excited is given as

$$M\ddot{u} + k_0 u + ku^3 = P\sin\omega t\,, \tag{52}$$

where the driving frequency is taken as $\omega = \sqrt{k_0/M}$ to simplify future calculations. This equation may represent a nonlinear pendulum where the function sin u is approximated as $[u-(1/6)u^3]$. The method of direct linearization gives an approximate equivalent governing equation and its steady-state solution as

$$M\ddot{u} + Ku = P\,\sin\omega t\,, \quad u = U\,\sin\omega t\,, \tag{53}$$

where

$$K = k_0 + (5/7)kU^2\,, \quad U = (K - M\omega^2)^{-1}P\,. \tag{54}$$

Manipulating (54), we have for the maximum amplitude of vibrations at the time $t = q = \pi/(2\omega)$

$$U = \{(7/5)P/k\}^{1/3}\,. \tag{55}$$

Consider the displacement at the time q (maximum displacement) as the functional requiring sensitivity analysis, and the design vector as $\mathbf{b} \equiv (M, k_0, k, P)$. The direct variation of the functional in (55) gives

$$\bar{\delta}({}^q u) \equiv \bar{\delta}U = \frac{1}{3}U(P^{-1}\bar{\delta}P - k^{-1}\bar{\delta}k)\,. \tag{56}$$

For the adjoint method of sensitivity analysis, we may write the functional as

$$\Psi \equiv \int_0^T u\hat{\delta}(t-q)\,\mathrm{d}t \equiv \int_0^\zeta u\omega\hat{\delta}\Big(\tau - \frac{\pi}{2}\Big)\omega^{-1}\,\mathrm{d}\tau = \int G\overline{T}\,\mathrm{d}\tau, \tag{57}$$

$$G = u\omega\hat{\delta}\Big(\tau - \frac{\pi}{2}\Big)\,, \quad T = \overline{T}\zeta\,, \quad \overline{T} = \omega^{-1}\,,$$

where we have used the time transformation $\tau = \omega t$ or $t = \omega^{-1}\tau$.

The 'action' extended functional is given, for this case, as

$$\mathcal{A} = \int_0^{\pi/2} \Big\{u\omega\hat{\delta}\Big(\tau - \frac{\pi}{2}\Big) - u^a\Big[M\omega^2 u'' + k_0 u + ku^3 - P\,\sin\tau\Big]\Big\}\omega^{-1}\,\mathrm{d}\tau\,, \tag{58}$$

where $\ddot{u} = \omega^2 u''$, $u'' = \mathrm{d}^2u/\mathrm{d}\tau^2$. For total design variation of the functional Ψ, (21) is reduced as

$$\bar{\bar{\delta}}\Psi \equiv \bar{\bar{\delta}}\mathcal{A} = \int_0^{\pi/2} [(-u^a u^3)\bar{\bar{\delta}}k + (u^a\,\sin\tau)\bar{\bar{\delta}}P]\omega^{-1}\,\mathrm{d}\tau\,, \tag{59}$$

where

$$G\bar{\bar{\delta}}\overline{T} + \bar{\bar{\delta}}G\overline{T} \equiv u\omega\hat{\delta}\Big(\tau - \frac{\pi}{2}\Big)\bar{\bar{\delta}}\omega^{-1} + u\hat{\delta}\Big(\tau - \frac{\pi}{2}\Big)\bar{\bar{\delta}}\omega = 0\,,$$

$$u = U\,\sin\tau\,, \quad u'' = -U\,\sin\tau = -u\,, \tag{60}$$

have been used. The adjoint system is governed by (29)–(31) as

$$(u^a)'' + (\Omega^a)^2 u^a = G_{,u} \equiv \omega\hat{\delta}\Big(\tau - \frac{\pi}{2}\Big)\,,$$

$${}^q u^a = {}^q\,(u^a)' = 0\,, \quad (\Omega^a)^2 = 1 + 3kM^{-1}\omega^{-2}u^2\,, \tag{61}$$

which has a solution as

$$u^a = -(M\Omega^a)^{-1}\omega\sin\Omega^a\Big(\tau - \frac{\pi}{2}\Big)\,. \tag{62}$$

Substituting the adjoint field of (62) into (59), we have

$$\int_0^{\pi/2} u^a\,\sin\tau\bar{\delta}P\omega^{-1}\,\mathrm{d}\tau = (1/3)UP^{-1}\bar{\delta}P\,,$$

$$\int_0^{\pi/2} -u^a u^3\bar{\delta}k\omega^{-1}\,\mathrm{d}\tau = (1/3)Uk^{-1}\bar{\delta}k\,,$$

$$\bar{\delta}\Psi = (1/3)U(P^{-1}\bar{\delta}P - k^{-1}\bar{\delta}k)\,, \tag{63}$$

which verifies (56). The first of equations (63) has been checked by numerical integration for different values of the system parameters. The second equation may be checked by using the result of the first, the equation of motion, and $\ddot{u} = -\omega^2 u$.

If P is a control variable, (59) and (61) are still valid, now with different values for u and u^a. In this case we could obtain the control variation of Ψ as

$$\bar{\delta}\Psi = \int_0^{\pi/2} (u^a\,\sin\tau)\bar{\delta}P\omega^{-1}\,\mathrm{d}\tau\,. \tag{64}$$

The sensitivity of Ψ with respect to control P would be taken with respect to the point control $P(\tau_i)$, $\tau_i \in [0, \pi/2]$, as

$$\bar{\delta}\Psi/\bar{\delta}P(\tau_i) = \omega^{-1}u^a\,\sin\tau\,, \quad \text{for } \tau \geq \tau_i\,,$$

$$\bar{\delta}\Psi/\bar{\delta}P(\tau_i) = 0\,, \quad \text{for } \tau < 0\,,$$

where $\bar{\delta}P(\tau)/\bar{\delta}P(\tau_i) = \hat{\delta}(\tau - \tau_i)$ has been used (Arora 1989).

6 Conclusions

A unified formulation for design sensitivity analysis of nonlinear dynamic structures has been derived that accounts for shape and nonshape design variables, selection of material parameters and controls. Structural and mechanical systems have been addressed with the same formulation that is explained as follows.

1. The Lagrangian description of the motion referred to an inertial reference frame that accounts for finite deformations and strains is used. Appropriate strain and stress measures that are invariant under any rigid body motion are used. This allows one to treat mechanical systems within the nonlinear structural dynamics formulation. Since a time-like parameter is already used in nonlinear static analysis, all one needs to do to unify structures and mechanical systems is to add the inertial term to the principle of virtual work. This allows the use of the same nonlinear analysis code for mechanical and structural systems.

2. Using Eulerian coordinates (fixed reference coordinates) for design sensitivity analysis, volume integrals are transformed to the reference domain and free domain problems are transformed into the fixed domain problems. Using the volume integrals throughout the derivations, shape and nonshape problems are unified in the same formulation.

3. Extending the dimension of the problem domain to include time, the design space becomes included in the control space. This way the design variables are simply the control variables that are not dependent on time. Using time integrals throughout the derivations, control and design problems are unified in the same formulation.

4. Extending the dimension of the fixed reference domain to include a parametric time variable, integrals in time domain are also transformed to the fixed reference domain. This way the free optimal time control problems are transformed into fixed time optimal control problems.

5. The adjoint structure concept is a major unifying factor for design sensitivity analysis, since it does not depend on the type of design variables but just on the type of the performance functional. However, the adjoint problem is a terminal value problem that needs to be integrated backwards in time. This implies that the adjoint system cannot be defined until the analysis problem has been completely solved. Most of the information generated during the analysis phase is also needed for solution of the adjoint system. This may make numerical implementation of the method somewhat tedious compared to the direct differentiation method.

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