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A q-Integral Representation of Rogers' q-Ultraspherical Polynomials and Some Applications

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Abstract. A q-integral representation of Rogers' q-ultraspherical polynomials $C_n(x;\beta|q)$ is obtained by using Sears' summation formula for balanced nonterminating ${}_3\phi_2$ series. It is then used to give a simple derivation of the Gasper–Rahman formula for the Poisson kernel of $C_n(x;\beta|q)$. As another application it is shown how this representation can be directly used to give an asymptotic expansion of the q-ultraspherical polynomials.

1. Introduction

The continuous q-ultraspherical polynomials $C_n(x;\beta|q)$, introduced by L. J. Rogers in 1895 [11] and recently studied by Askey and Ismail [2, 3], Bressoud [5], Gasper [6], Rahman [10], and Gasper and Rahman [7], can be defined by the generating function

$$(1.1)\qquad \frac{(\beta te^{i\theta}, te^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} C_n(x;\beta|q)t^n, \qquad |q|<1, \quad |t|<1,$$

where $x=\cos\theta$, $0\le\theta\le\pi$, and

$$(1.2)\qquad (a;q)_\infty = (1-a)(1-aq)(1-aq^2)\dots,$$

$$\frac{(a_1,a_2,\dots,a_n;q)_\infty}{(b_1,b_2,\dots,b_m;q)_\infty} = \frac{(a_1;q)_\infty(a_2;q)_\infty\dots(a_n;q)_\infty}{(b_1;q)_\infty(b_2;q)_\infty\dots(b_m;q)_\infty}.$$

Use of the q-binomial theorem

$$(1.3)\qquad \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \qquad |q|<1, \quad |x|<1,$$

where $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty = (1-a)(1-aq)\dots(1-aq^{n-1})$, immediately leads to the familiar basic hypergeometric series representation of

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$C_n(x,\beta|q)$ (see [3])

$$C_n(x;\beta|q)=\sum_{k=0}^{n}\frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}\cos(n-2k)\theta. \tag{1.4}$$

This is a very simple basic hypergeometric series representation of the q-ultraspherical polynomials that leads to many interesting results. For example, Askey and Ismail [3] used it to prove the orthogonality relation

$$\begin{aligned}&\int_{-1}^{1}C_m(x;\beta|q)C_n(x;\beta|q)W_\beta(x|q)(1-x^2)^{-1/2}\,dx\\&\quad=\frac{2\pi(1-\beta)}{1-\beta q^n}\frac{(\beta^2;q)_n}{(q;q)_n}\frac{(\beta,\beta q;q)_\infty}{(q,\beta^2;q)_\infty}\delta_{m,n},\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}W_\beta(x|q)&=\prod_{n=0}^{\infty}\frac{1-2(2x^2-1)q^n+q^{2n}}{1-2\beta(2x^2-1)q^n+\beta^2q^{2n}}\\&=\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\beta e^{2i\theta},\beta e^{-2i\theta};q)_\infty},\qquad x=\cos\theta,\end{aligned} \tag{1.6}$$

and Gasper [6] used it to give a very simple proof of Rogers' linearization formula [11] for the product of two q-ultraspherical polynomials. There are, of course, other representations of $C_n(x;\beta|q)$ that are also very useful. For example, Askey and Ismail [3] showed that

$$\begin{aligned}&C_n(x;\beta|q)\\&\quad=\frac{(\beta^2;q)_n}{(q;q)_n}\beta^{-n/2}{}_4\phi_3\left[\begin{matrix}q^{-n},\beta^2q^n,\beta^{1/2}e^{i\theta},\beta^{1/2}e^{-i\theta}\\ \beta q^{1/2},-\beta q^{1/2},-\beta\end{matrix};q,q\right],\end{aligned} \tag{1.7}$$

$0\le\theta\le\pi$, where

$$\begin{aligned}&{}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,\dots,b_r\end{matrix};q,z\right]\\&\quad=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_n}{(q,b_1,b_2,\dots,b_r;q)_n}z^n\end{aligned} \tag{1.8}$$

whenever the series converges. There is also an infinite series representation, which is a q-analogue of Szegö's [14, 16] formula for the ultraspherical polynomials:

$$C_n(x;\beta|q)=\frac{4\sin\theta}{W_\beta(x|q)}\sum_{k=0}^{\infty}b(k,n;\beta)\sin(n+2k+1)\theta, \tag{1.9}$$

$0 < \theta < \pi$, where

$$(1.10)\qquad b(k,n;\beta) = \frac{(\beta,\beta q;q)_\infty}{(q,\beta^2;q)_\infty}\frac{(\beta^2;q)_n(q\beta^{-1};q)_k(q;q)_{n+k}}{(q;q)_n(q;q)_k(\beta q;q)_{n+k}}\beta^k.$$

The purpose of this paper is to show that there is a representation, closely related to (1.9), of $C_n(x;\beta|q)$ as a q-integral:

$$(1.11)\qquad C_n(x;\beta|q) = \frac{2i\sin\theta}{(1-q)W_\beta(x|q)}\frac{(\beta,\beta;q)_\infty}{(q,\beta^2;q)_\infty}\frac{(\beta^2;q)_n}{(q;q)_n}\cdot\int_{e^{i\theta}}^{e^{-i\theta}}\frac{(que^{i\theta},que^{-i\theta};q)_\infty}{(\beta ue^{i\theta},\beta ue^{-i\theta};q)_\infty}u^n d_qu,$$

where $0 < \theta < \pi$ and the q-integral on the right side is defined by

$$(1.12)\qquad \int_0^a f(u)\,d_qu = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n,$$
$$\int_a^b f(u)\,d_qu = \int_0^b f(u)\,d_qu - \int_0^a f(u)\,d_qu.$$

Note that (1.11) resembles the Dirichlet–Mehler formula [15, Section 4.82] for ultraspherical polynomials. Expressing the integral in (1.11) as an infinite series and simplifying the coefficients, we obtain

(1.13)

$$C_n(x;\beta|q) = \frac{(\beta;q)_\infty}{(\beta^2;q)_\infty}W_\beta^{-1}(x|q)\frac{(\beta^2;q)_n}{(q;q)_n}\cdot\left\{\frac{(e^{-2i\theta};q)_\infty}{(\beta e^{-2i\theta};q)_\infty}e^{-in\theta}\cdot{}_2\phi_1\left[\begin{matrix}\beta e^{-2i\theta}, & \beta \\ & qe^{-2i\theta}\end{matrix};\ q,q^{n+1}\right]\right.$$
$$\left.+\frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty}e^{in\theta}\cdot{}_2\phi_1\left[\begin{matrix}\beta e^{2i\theta}, & \beta \\ & qe^{2i\theta}\end{matrix};\ q,q^{n+1}\right]\right\}.$$

Using Heine's transformation formula [9]

$$(1.14)\qquad {}_2\phi_1\left[\begin{matrix}a, & b \\ & c\end{matrix};\ q,z\right] = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left[\begin{matrix}c/b, & z \\ & az\end{matrix};\ q,b\right]$$

one can easily show that (1.13) reduces to (1.9). Conversely, one can say that the integral representation (1.11) follows directly from (1.9) via (1.14). In fact, this is how we discovered (1.11) in the first place. However, as we shall show in Section 3, (1.11) also follows from the generating function (1.1), almost as directly as does (1.4). In the same section we shall also show how easily one can obtain other generating functions for $C_n(x;\beta|q)$ by using some special cases of Bailey's transformation formula [4], giving a very well-poised ${}_8\phi_7$ series in terms of two balanced ${}_4\phi_3$ series that we shall express as q-integrals in Section 2. As an application of (1.11) we shall give in Section 4 a very simple derivation of the Poisson kernel for the q-ultraspherical polynomials that Gasper and Rahman obtained in [7]. As another, and perhaps more important, application, we shall show in Section 5 that (1.13), and hence (1.11), is the most appropriate form for an asymptotic expansion of $C_n(x;\beta|q)$ when $|x|<1$ and $0<q<1$.

2. Bailey's Transformation Formula as a q-Integral

One of the most useful formulas in the theory of basic hypergeometric series was given by Bailey [4] in the form:

$$(2.1)\quad {}_8\phi_7\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ f, \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq/f \end{matrix};\ q,\ a^2q^2/bcdef\right]$$

$$= \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty}$$

$$\cdot {}_4\phi_3\left[\begin{matrix} d,\ e,\ f,\ aq/bc \\ aq/b,\ aq/c,\ def/a \end{matrix};\ q, q\right]$$

$$+\frac{(aq, d, e, f, aq/bc, a^2q^2/bdef, a^2q^2/cdef; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, a^2q^2/bcdef, def/aq; q)_\infty}$$

$$\cdot {}_4\phi_3\left[\begin{matrix} aq/de,\ aq/df,\ aq/ef,\ a^2q^2/bcdef \\ aq^2/def,\ a^2q^2/bdef,\ a^2q^2/cdef \end{matrix};\ q, q\right],$$

provided the series are convergent, which requires that $|q|<1$ and that either the ${}_8\phi_7$ series on the left terminates or $|a^2q^2/bcdef|<1$. The important property that validates this transformation is that the ${}_8\phi_7$ series is very well-poised and the two ${}_4\phi_3$ series are balanced. The basic hypergeometric series (1.8) is said to be balanced if $z=q$ and $qa_1a_2\dots a_{r+1}=b_1b_2\dots b_r$; it is called well-poised if $a_2b_1=a_3b_2=\cdots=a_{r+1}b_r=qa_1$, and very well-poised if, in addition, $b_1=\sqrt{a_1}$, $b_2=-\sqrt{a_1}$.

The important step for our purpose is to recognize that we can express (2.1) as a q-integral (see Al-Salam and Verma [1]; also Gasper and Rahman [8]):

(2.2)

$$\int_a^b \frac{(qu/a, qu/b, cu, du; q)_\infty}{(eu, fu, gu, hu; q)_\infty}\, d_qu$$

$$= \frac{b(1-q)(q, bq/a, a/b, cd/eh, cd/fh, cd/gh, bc, bd; q)_\infty}{(ae, af, ag, be, bf, bg, bh, bcd/h; q)_\infty}$$

$$\cdot {}_8\phi_7\left[\begin{matrix} bcd/hq,\ q\sqrt{\ \ },\ -q\sqrt{\ \ },\ be,\ bf,\ bg,\ c/h,\ d/h \\ \sqrt{\ \ },\ -\sqrt{\ \ },\ cd/eh,\ cd/fh,\ cd/gh,\ bc,\ bd \end{matrix};\ q,\ cd/befg\right],$$

provided

$$cd = abefgh. \tag{2.3}$$

The open square roots in (2.2) are over bcd/hq.

Note that the parameters $a, b, \ldots$ in (2.2) are not the same as those in (2.1), but it is not difficult to see that (2.2) is equivalent to (2.1) by virtue of (1.12). Condition (2.3) is simply an expression of the fact that the two ${}_4\phi_3$ series are balanced. Some special cases of (2.2) are of particular interest to us. First, let us set $h = d$. Then the ${}_8\phi_7$ series has a parameter value 1 in the numerator and hence attains the value 1, so that (2.2) reduces to Sears' summation formula [12]

$$\int_a^b \frac{(qu/a, qu/b, cu; q)_\infty}{(eu, fu, gu; q)_\infty} d_q u = b(1-q) \frac{(q, bq/a, a/b, c/e, c/f, c/g; q)_\infty}{(ae, af, ag, be, bf, bg; q)_\infty}, \tag{2.4}$$

with $c = abefg$.

Next, let us replace d and g by λd and λg, respectively, and let $\lambda \to 0$. Then (2.2) gives

$$\int_a^b \frac{(qu/a, qu/b, cu; q)_\infty}{(eu, fu, hu; q)_\infty} d_q u = \frac{b(1-q)(q, bq/a, a/b, bc, abef; q)_\infty}{(ae, af, be, bf, bh; q)_\infty} {}_3\phi_2\left[\begin{matrix} be, & bf, & c/h \\ & abef, & bc \end{matrix}; q, ah\right], \tag{2.5}$$

provided $|ah| < 1$ and $|q| < 1$, if the ${}_3\phi_2$ series on the right side does not terminate. If we now set $c = 0$ in (2.5) we get

$$\int_a^b \frac{(qu/a, qu/b; q)_\infty}{(eu, fu, hu; q)_\infty} d_q u = b(1-q) \frac{(q, bq/a, a/b, abef; q)_\infty}{(ae, af, be, bf, bh; q)_\infty} {}_2\phi_1\left[\begin{matrix} be, & bf \\ & abef \end{matrix}; q, ah\right], \tag{2.6}$$

and setting $f = 0$ in (2.5) gives

$$\int_a^b \frac{(qu/a, qu/b, cu; q)_\infty}{(eu, hu; q)_\infty} d_q u = \frac{b(1-q)(q, bq/a, a/b, bc; q)_\infty}{(ae, be, bh; q)_\infty} {}_2\phi_1\left[\begin{matrix} be, & c/h \\ & bc \end{matrix}; q, ah\right]. \tag{2.7}$$

Transformation formulas (2.5), (2.6), and (2.7) are not new; they are contained in Sears' work [13].

3. Proof of (1.11) and Some Generating Functions of $C_n(x;\beta|q)$

Using (2.4), we find that for $0<\theta<\pi$,

$$(3.1)\quad \int_{e^{i\theta}}^{e^{-i\theta}} \frac{(que^{-i\theta}, que^{i\theta}, \beta^2 tu; q)_\infty}{(\beta ue^{-i\theta}, \beta ue^{i\theta}, tu; q)_\infty} d_q u$$

$$= \frac{(1-q)e^{-i\theta}(q, qe^{-2i\theta}, e^{2i\theta}, \beta^2; q)_\infty}{(\beta, \beta e^{2i\theta}, \beta e^{-2i\theta}, \beta; q)_\infty} \frac{(\beta te^{i\theta}, \beta te^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty}.$$

So for $|t|<1$, we may use (1.3) to get $(\beta^2 tu;q)_\infty/(tu;q)_\infty = \Sigma_{n=0}^\infty[(\beta^2;q)_n/(q;q)_n](tu)^n$, which immediately leads to (1.11).

If we now multiply (1.11) by $t^n(\lambda;q)_n/(\beta^2;q)_n$, where $|t|<1$ and λ is arbitrary; sum over n; and use (1.3), we obtain a somewhat more general generating function for $C_n(x;\beta|q)$:

$$(3.2)\quad \sum_{n=0}^\infty C_n(x;\beta|q)\frac{(\lambda;q)_n}{(\beta^2;q)_n} t^n$$

$$= \frac{2i\sin\theta}{(1-q)W_\beta(x|q)} \frac{(\beta,\beta;q)_\infty}{(q,\beta^2;q)_\infty} \int_{e^{i\theta}}^{e^{-i\theta}} \frac{(que^{i\theta}, que^{-i\theta}, \lambda tu; q)_\infty}{(\beta ue^{i\theta}, \beta ue^{-i\theta}, tu; q)_\infty} d_q u.$$

Obviously (1.1) is a special case of (3.2) by virtue of (2.4) when we set $\lambda=\beta^2$. Setting $\lambda=0$ and using (2.6) we obtain

$$(3.3)\quad \sum_{n=0}^\infty C_n(x;\beta|q)\frac{t^n}{(\beta^2;q)_n}$$

$$= \frac{1}{(te^{-i\theta};q)_\infty}\, {}_2\phi_1\left[\begin{matrix}\beta, & \beta e^{-2i\theta} \\ & \beta^2\end{matrix}; q, te^{i\theta}\right].$$

Also, if we replace t by $-\beta t/\lambda$ in (3.2) and take the limit as $\lambda\to\infty$ we obtain another generating function:

$$(3.4)\quad \sum_{n=0}^\infty \frac{C_n(x;\beta|q)}{(\beta^2;q)_n} q^{\binom{n}{2}} (\beta t)^n$$

$$= \frac{(\beta, -\beta te^{i\theta}; q)_\infty}{(\beta^2;q)_\infty}\, {}_2\phi_1\left[\begin{matrix}\beta, & -te^{-i\theta} \\ & -\beta te^{i\theta}\end{matrix}; q, \beta\right], \qquad \text{by (2.7)}$$

$$= (-te^{-i\theta};q)_\infty\, {}_2\phi_1\left[\begin{matrix}\beta, & \beta e^{2i\theta} \\ & \beta^2\end{matrix}; q, -te^{-i\theta}\right], \qquad \text{by (1.14)}.$$

This was found previously by Askey and Ismail [3].

4. Poisson Kernel for q-Ultraspherical Polynomials

The Poisson kernel for $C_n(x;\beta|q)$ is defined by

$$(4.1)\quad K_t(x,y;\beta|q)=\sum_{n=0}^{\infty}\frac{(q;q)_n}{(\beta^2;q)_n}\frac{1-\beta q^n}{1-\beta}C_n(x;\beta|q)C_n(y;\beta|q)t^n.$$

Gasper and Rahman [7] used Rogers' linearization formula [11] and a transformation formula connecting a ${}_2\phi_1$ with a ${}_4\phi_3$ to prove that

(4.2)

$$K_t(x,y;\beta|q)$$
$$=\frac{(t^2,\beta;q)_\infty}{(\beta qt^2,\beta^2;q)_\infty}\left|\frac{(\beta te^{i\theta+i\phi},\beta tqe^{i\theta-i\phi};q)_\infty}{(te^{i\theta+i\phi},te^{i\theta-i\phi};q)_\infty}\right|^2$$
$$\cdot{}_8\phi_7\left[\begin{matrix}\beta t^2,\ q(\beta t^2)^{1/2},\ -q(\beta t^2)^{1/2},\ \beta,\ qte^{i\theta+i\phi},\ qte^{-i\theta-i\phi},\ te^{-i\theta-i\phi},\ te^{i\phi-i\theta}\\ (\beta t^2)^{1/2},\ -(\beta t^2)^{1/2},\ qt^2,\ \beta te^{-i\theta-i\phi},\ \beta te^{i\theta+i\phi},\beta qte^{i\phi-i\theta},\ \beta qte^{i\theta-i\phi}\end{matrix};q,\beta\right]$$

where $|t|<1$, $|q|<1$, $|\beta|<1$, $x=\cos\theta$, and $y=\cos\phi$. It was also shown in [7] that

(4.3)

$$L_t(x,y;\beta|q)=\sum_{n=0}^{\infty}\frac{(q;q)_n}{(\beta^2;q)_n}C_n(x;\beta|q)C_n(y;\beta|q)t^n$$
$$=\frac{(t^2,\beta;q)_\infty}{(\beta t^2,\beta^2;q)_\infty}\left|\frac{(\beta te^{i\theta+i\phi},\beta te^{i\theta-i\phi};q)_\infty}{(te^{i\theta+i\phi},te^{i\theta-i\phi};q)_\infty}\right|^2$$
$$\cdot{}_8\phi_7\left[\begin{matrix}\beta t^2/q,\ q(\beta t^2/q)^{1/2},\ -q(\beta t^2/q)^{1/2},\ \beta,\ te^{i\theta+i\phi},\ te^{-i\theta-i\phi},\ te^{i\theta-i\phi},\ te^{i\phi-i\theta}\\ (\beta t^2/q)^{1/2},\ -(\beta t^2/q)^{1/2},\ t^2,\ \beta te^{-i\theta-i\phi},\ \beta te^{i\theta+i\phi},\ \beta te^{i\phi-i\theta},\ \beta te^{i\theta-i\phi}\end{matrix};q,\beta\right].$$

We will show that these results follow from the integral representation (1.11) in a very straightforward manner. First, multiply (1.11) by $(q;q)_nC_n(y;\beta|q)t^n/(\beta^2;q)_n$ and sum over n. This gives

(4.4)

$$L_t(x,y;\beta|q)=\frac{2i\sin\theta}{(1-q)W_\beta(x|q)}\frac{(\beta,\beta;q)_\infty}{(q,\beta^2;q)_\infty}\int_{e^{i\theta}}^{e^{-i\theta}}d_qu\frac{(que^{i\theta},que^{-i\theta};q)_\infty}{(\beta ue^{i\theta},\beta ue^{-i\theta};q)_\infty}$$
$$\cdot\sum_{n=0}^{\infty}C_n(y;\beta|q)(ut)^n$$
$$=\frac{2i\sin\theta}{(1-q)W_\beta(x|q)}\frac{(\beta,\beta;q)_\infty}{(q,\beta^2;q)_\infty}$$
$$\cdot\int_{e^{i\theta}}^{e^{-i\theta}}d_qu\frac{(que^{i\theta},que^{-i\theta},\beta ute^{i\phi},\beta ute^{-i\phi};q)_\infty}{(\beta ue^{i\theta},\beta ue^{-i\theta},ute^{i\phi},ute^{-i\phi};q)_\infty}$$

by (1.1). But the q-integral on the right side has the same balanced property as that in (2.2), and so choosing $\beta te^{i\phi}/r = c$, $\beta te^{-i\phi}/r = d$, $h = \beta e^{-i\theta}/r$, $e = \beta e^{i\theta}/r$, and $f = te^{-i\phi}/r$, one can see that use of (2.2) in (4.4) directly leads to (4.3).

To prove (4.2) one needs to do a little more work. Since

(4.5)

$$\sum_{n=0}^{\infty} C_n(y;\beta|q)\frac{1-\beta q^n}{1-\beta}(ut)^n$$
$$= (1-\beta)^{-1}\left\{\frac{(\beta ute^{i\phi}, \beta ute^{-i\phi};q)_\infty}{(ute^{i\phi}, ute^{-i\phi};q)_\infty} - \frac{\beta(\beta qute^{i\phi}, \beta qute^{-i\phi};q)_\infty}{(qute^{i\phi}, qute^{-i\phi};q)_\infty}\right\}$$

we have, by using (2.2),

(4.6)

$$K_t(x,y;q)$$
$$= \frac{(\beta q, t^2;q)_\infty}{(\beta t^2, \beta^2;q)_\infty}\left\{\left|\frac{(\beta te^{i\theta+i\phi}, \beta te^{i\theta-i\phi};q)_\infty}{(te^{i\theta+i\phi}, te^{i\theta-i\phi};q)_\infty}\right|^2 \times\right.$$
$${}_8\phi_7\left[\begin{matrix}\beta t^2/q,\ q(\beta t^2/q)^{1/2},\ -q(\beta t^2/q)^{1/2},\ \beta,\ te^{i\theta+i\phi},\ te^{-i\theta-i\phi},\ te^{i\theta-i\phi},\ te^{i\phi-i\theta} \\ (\beta t^2/q)^{1/2},\ -(\beta t^2/q)^{1/2},\ t^2,\ \beta te^{-i\theta-i\phi},\ \beta te^{i\theta+i\phi},\ \beta te^{i\phi-i\theta},\ \beta te^{i\theta-i\phi}\end{matrix};q,\beta\right]$$
$$-\beta\frac{(1-\beta t^2)(1-\beta qt^2)}{(1-t^2)(1-qt^2)}\left|\frac{(\beta tqe^{i\theta+i\phi}, \beta tqe^{i\theta-i\phi};q)_\infty}{(tqe^{i\theta+i\phi}, tqe^{i\theta-i\phi};q)_\infty}\right|^2 \times$$
$$\left.{}_8\phi_7\left[\begin{matrix}q\beta t^2,\ q(q\beta t^2)^{1/2},\ -q(q\beta t^2)^{1/2},\ \beta,\ qte^{i\theta+i\phi},\ qte^{-i\theta-i\phi},\ qte^{i\theta-i\phi},\ qte^{i\phi-i\theta} \\ (q\beta t^2)^{1/2},\ -(q\beta t^2)^{1/2},\ q^2t^2,\ \beta tqe^{-i\theta-i\phi},\ \beta tqe^{i\theta+i\phi},\ \beta tqe^{i\phi-i\theta},\ \beta tqe^{i\theta-i\phi}\end{matrix};q,\beta\right]\right\}$$

Combining the $(n+1)$th term of the first series with the nth term of the second series for $n = 0, 1, 2, \ldots,$ it is easy to show that (4.6) leads to (4.2).

5. The Asymptotic Formulas

For $|x| < 1$, $|q| < 1$, and large n it is clear from (1.13) that the leading term in the asymptotic expansion of $C_n(x;\beta|q)$ is given by

$$(5.1)\quad C_n(x;\beta|q) \approx \frac{(\beta;q)_\infty}{(q;q)_\infty}\left\{\frac{(\beta e^{2i\theta};q)_\infty}{(e^{2i\theta};q)_\infty}e^{-in\theta} + \frac{(\beta e^{-2i\theta};q)_\infty}{(e^{-2i\theta};q)_\infty}e^{in\theta}\right\}$$
$$= 2\frac{(\beta;q)_\infty}{(q;q)_\infty}|A(e^{i\theta})|\cos(n\theta-\alpha),$$

where

$$(5.2)\qquad A(z) = \frac{(\beta z^2;q)_\infty}{(\text{[illegible]};q)_\infty} \quad\text{and}\quad \alpha = \arg A(e^{i\theta}).$$

This, of course, is the same as Eq. (3.11) in [2], which Askey and Ismail found by using Darboux's method. Equation (1.13), however, gives more than the leading term. For $0 < \theta < \pi$ and any fixed positive integer N, it is easy to deduce from (1.13) that for large n

(5.3)

$$C_n(x;\beta|q) \simeq 2\frac{(\beta;q)_\infty}{(q;q)_\infty}|A(e^{i\theta})|\cos(n\theta-\alpha) + 4\frac{(\beta;q)_\infty}{(q;q)_\infty}\sin\theta\, W_\beta^{-1}(x|q)\left\{\sum_{k=1}^{\infty}\frac{(\beta;q)_k}{(q;q)_k}|B_k(e^{i\theta})| \cdot\cos\left[(n+1)\theta+\phi_k-\frac{\pi}{2}\right]q^{nk+k} + O\left(q^{N(n+1)}\right)\right\},$$

where

$$B_k(z) = \frac{\left(q^{k+1}z^2;q\right)_\infty}{\left(\beta q^k z^2;q\right)_\infty},\qquad k=1,2,\ldots, \tag{5.4}$$

and

$$\phi_k = \arg B_k(e^{i\theta}). \tag{5.5}$$

If $x \notin [-1,1]$, Eq. (1.13) can be written as

(5.6)

$$C_n(x;\beta|q) = \frac{(\beta;q)_\infty}{(\beta^2;q)_\infty}\frac{(\beta^2;q)_\infty}{(q;q)_\infty}W_\beta^{-1}(x|q) \cdot\left\{\frac{(z^2;q)_\infty}{(\beta z^2;q)_\infty}z^n {}_2\phi_1\left[\begin{matrix}\beta z^2, & \beta \\ & qz^2\end{matrix};\ q,q^{n+1}\right] + \frac{(z^{-2};q)_\infty}{(\beta z^{-2};q)_\infty}z^{-n}{}_2\phi_1 \cdot\left[\begin{matrix}\beta z^{-2}, & \beta \\ & qz^{-2}\end{matrix};\ q,q^{n+1}\right]\right\},$$

where $x = \frac{1}{2}(z+z^{-1})$ and $W_\beta(x|q) = (z^2, z^{-2};q)_\infty/(\beta z^2, \beta z^{-2};q)_\infty$.

Writing $1-2xz+z^2 = (1-zz_1)(1-zz_2)$, where $|z_1| < |z_2|$ with $|z_1| < 1$ and $|z_2| > 1$, we find that

$$C_n(x;\beta|q) \simeq \frac{(\beta;q)_\infty}{(q;q)_\infty}\frac{\left(\beta z_1^2;q\right)_\infty}{\left(z_1^2;q\right)_\infty}z_2^n\left[1+\sum_{k=1}^{N}\frac{\left(\beta,\beta z_2^2;q\right)_k}{\left(q,qz_2^2;q\right)_k}q^{nk+k}\right] + O\left(z_2^n q^{N(n+1)}\right),\qquad N=1,2,\ldots. \tag{5.7}$$

The leading term in (5.7) is the same as (3.5) of [2].

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