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CONSTRUCTIVE APPROXIMATION

Constrained L_p Approximation

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Abstract. In this paper, we solve a class of constrained optimization problems that lead to algorithms for the construction of convex interpolants to convex data.

1. Introduction

Given points $0 \le t_1 < \ldots < t_n \le 1$ in $[0, 1]$ and a function $\hat{g} \in L_p^k[0, 1]$, the Sobolev space of functions with k-th derivatives in $L_p[0, 1]$, the "best" interpolant is defined to be $s \in L_p^k$ satisfying

$$(1.1)\quad \begin{cases} \text{(i)} & \|s^{(k)}\|_p = \inf\{\|f^{(k)}\|_p : f(t_i) = \hat{g}(t_i), \quad i = 1, \ldots, n\}, \\ \text{(ii)} & s(t_i) = \hat{g}(t_i), \quad i = 1, \ldots, n. \end{cases}$$

Assuming that $1 < p \le \infty$, a best interpolant exists (and is unique if $n \ge k$ and $1 < p < \infty$). For a nice discussion of these results the reader is referred to [1]. In particular, it is shown in [1] that for $1 < p < \infty$

$$(1.2)\qquad s^{(k)} = |\phi|^{q-1} \operatorname{sgn} \phi,$$

where

$$\phi = \sum_{j=1}^{n-k} \alpha_j M_{j,k}.$$

The $M_{j,k}$ are the B-splines determined by the knots $t_j, \ldots, t_{j+k}$, and q is the conjugate index of p.

In particular, when $p = 2$ $(q = 2)$ the best interpolant s is the famous (and infamous) natural spline of order $2k$. One of the unfortunate features of these functions is that they can have inflection points not suggested by the data. This undersirable property could be eliminated for $k = 2$ and controlled for $k > 2$ by considering the following

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problem: Let G be the set of functions in L_p^k whose k-th derivative is nonnegative on Ω_1 and nonpositive on Ω_2 ($\Omega_1 \cap \Omega_2 = \emptyset$). We seek s satisfying

$$(1.3)\quad \begin{cases} \text{(i)} & s(t_i) = \hat{g}(t_i) \qquad i = 1, \ldots, n, \\ \text{(ii)} & s \in G, \\ \text{(iii)} & \|s^{(k)}\|_p = \min\{\|f^{(k)}\|_p : f(t_i) = g(t_i),\ i = 1, \ldots, n \text{ and } f \in G\}. \end{cases}$$

In Sections 3 and 4 we characterize the solution to this problem for $1 < p < \infty$. For example, when $\hat{g}^{(k)} > 0$ and $\Omega_1 = [0, 1]$ the solution is characterized by

$$(1.4)\quad \begin{cases} s^{(k)} = (\phi)_+^{q-1}, \qquad \text{where} \quad \phi = \sum_{j=1}^{n-k} \alpha_j M_{j,k}, \\ \int (s^{(k)}) M_{j,k} = d_j, \qquad j = 1, \ldots, n-k, \end{cases}$$

where d_j is an appropriate multiple of $\hat{g}[t_j, \ldots, t_{j+k}]$, the divided difference of $\hat{g}$ at $t_j, \ldots, t_{j+k}$. Thus we have exchanged an infinite convex programming problem for a more tractable finite nonlinear programming problem.

In Section 2, we develop the theory for the fundamental L_p problem, $1 < p < \infty$: Given $\hat{g} \in L_p[0, 1]$, $\psi_1, \ldots, \psi_n \in L_q[0, 1]$ with $\hat{g} \geq 0$, find $g \geq 0$ such that

$$(1.5)\quad \begin{cases} \text{(i)} & (g, \psi_i) = (\hat{g}, \psi_i), \qquad i = 1, \ldots, n \\ \text{(ii)} & \|g\|_p = \min\{\|f\|_p : (f, \psi_i) = (\hat{g}, \psi_i) \\ & \qquad\qquad \text{for } i = 1, \ldots, n \text{ and } f \geq 0\}. \end{cases}$$

In Section 3 we apply these results to best interpolation. Finally, in Section 4 we indicate some easy extensions and state our most general throrems.

We close this introduction with some historical information. Favard [4] considered (1.1) for $p = \infty$. Karlin [7] was the first to prove there is a perfect spline solution in this case. Then de Boor [1] interpreted and extended Favard's work for $1 < p \leq \infty$. Chui, Smith, and Ward [2] contributed to the $p = 1$ problem. The constrained interpolation problem has been studied by several authors; we mention here only Copley and Schumaker [3] and Hornung [5]. In particular, Hornung [5] was able to characterize the solution to (1.3) when $\hat{g}^{(k)} > 0$, $k = 2$, and $p = 2$. He also obtained a partial characterization for $k > 2$, $p = 2$. Finally, our results were suggested by conversations with and by the work of G. Iliev and W. Pollul [6]. In particular, they first formulated the solution as we have presented it here. In addition, they have treated the problem (1.3) with $p = \infty$, $k = 2$, proving that the problem of convex interpolation with minimal L_∞-norm of the second derivative has a quadratic spline solution characterized by the existence of a core interval on which the second derivative is the positive part of a perfect spline and that all solutions agree on core intervals.

To provide additional motivation for the contents of this paper, we consider the finite dimensional constrained optimization problem (here $1 < p < \infty$):

$$\min \sum_{i=1}^{n} |x_i|^p$$

subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \theta$. We assume the data are compatible and there is a solution $\mathbf{x}^*$. Then $\mathbf{x}^*$ is characterized by the Kuhn-Tucker conditions

$$p\mathbf{x}^{*p-1} + A^T\lambda - \beta = \theta,$$

where $\beta \geq 0$ and $\beta^T\mathbf{x}^* = \theta$.

Thus for $\lambda^* := -\frac{1}{p}\lambda$ we have

$$\mathbf{x}^* = (A^T\lambda^*)_+^{1/(p-1)},$$

where of course $(\mathbf{z})_+(i) = \mathbf{z}(i)$ if $\mathbf{z}(i) \geq 0$ and $(\mathbf{z})_+(i) = 0$ otherwise. Thus the finite dimensional problem of minimizing the p-norm of $\mathbf{x}$ subject to the conditions $\mathbf{x} \geq \theta$ and $\mathbf{r}_i^T \cdot \mathbf{x} = b_i$, $i = 1, \ldots, k$, has a solution $\mathbf{x}^*$ of the form

$$\mathbf{x}^* = \left(\sum_{j=1}^{k} \alpha_j \mathbf{r}_j\right)_+^{q-1},$$

where the α_j are determined from the interpolation conditions and $p + q = pq$.

This paper attempts to extend these observations to the analogous setting in L_p.

2. Approximation by Nonnegative Elements

Let X be a measure space with a σ-finite measure μ. For $1 < p < \infty$, $L_p(\mu)$ will denote the Banach space of p-th power integrable real-valued functions on X. It is well known that $[L_p(\mu)]^* = L_q(\mu)$, where q is the conjugate index to p satisfying $p + q = pq$. In this section we will develop an explicit characterizaton of the solution to the following minimal norm problem. Fix p, $1 < p < \infty$, let

(2.1) $$C = \{g \in L_p(\mu) \colon g \geq 0\}$$

and suppose $\{\psi_i\}_{i=1}^n$ are linearly independent functions in $L_q(\mu)$. Given $\mathbf{d} \in R^n$, set

(2.2) $$D = \left\{g \in L_p(\mu) \colon \int_X g\psi_i \, d\mu = d_i, \quad i = 1, \ldots, n\right\}.$$

Note that C and D are nonempty closed subsets of $L_p(\mu)$. If $K := C \cap D$, we see that K is a closed convex subset of $L_p(\mu)$ and if K is not empty there is an $f \in K$ of minimal $L_p(\mu)$ norm. Furthermore, this f is implicitly characterized by the inequalities

(2.3) $$\int_X (g - f) f^{p/q} \, d\mu \geq 0 \qquad \text{for all} \qquad g \in K.$$

As a rule, the characterization in (2.3) is not very helpful if one wants to calculate f.

For many problems Proposition 2.1 below provides a calculable solution to the following problem: Find $f \in K$ such that

(2.4) $$\|f\|_p = \inf\{\|g\|_p \colon g \in K\}.$$

Proposition 2.1. *Suppose that K is not empty and that the $\{\psi_i\}_{i=1}^n$ are linearly independent over $\{x \in X \colon \hat{g}(x) > 0\}$ for some $\hat{g} \in K$. Then there are real numbers*

$\{\alpha_i^*\}_{i=1}^n$ *such that the solution, f, to* (2.4) *satisfies*

$$f = \left(\sum_{j=1}^n \alpha_j^* \psi_j\right)_+^{q-1}. \tag{2.5}$$

Furthermore, since $f \in D$ the $\{\alpha_i^*\}_{i=1}^n$ must satisfy the n interpolation conditions

$$\int_X \left(\sum_{j=1}^n \alpha_j^* \psi_j\right)_+^{q-1} \psi_i \, d\mu = d_i, \qquad i = 1, \ldots, n. \tag{2.6}$$

Proof. Recall that for a real-valued function s we set

$$(s)_+(x) := \begin{cases} s(x), & \text{if } s(x) \geq 0 \\ 0, & \text{if } s(x) < 0. \end{cases}$$

Similarly, $(s)_- := (-s)_+$ and hence $s = (s)_+ - (s)_-$ decomposes s into its positive and negative parts. We first show that if coefficients $\{\alpha_i\}_{i=1}^n$ exist such that (2.6) holds, then the solution f to (2.4) satisfies (2.5). To this end, set $s = \Sigma_{j=1}^n \alpha_j \psi_j$ and assume that for $i = 1, \ldots, n$,

$$\int_X (s)_+^{p-1} \psi_i \, d\mu = d_i.$$

Then for any $g \in K$ we have

$$\begin{aligned} \int_X [g - (s)_+^{q-1}](s)_+^{(q-1)p/q} \, d\mu &= \int_X [g - (s)_+^{q-1}](s)_+ \, d\mu \\ &= \int_X [g - (s)_+^{q-1}][s + (s)_-] \, d\mu = \int_X [g - (s)_+^{q-1}](s)_- \, d\mu \\ &= \int_X g(s)_- \, d\mu \geq 0. \end{aligned} \tag{2.7}$$

Thus we see that $(s)_+^{q-1}$ must be the unique element of minimal norm using (2.3).

It follows that we need only show that (2.6) has a solution. This we do using an argument motivated by a duality principle. Consider the problem

$$\inf\left\{\int_X \left(\sum_{j=1}^n \alpha_j \psi_j\right)_+^q d\mu : \sum_{j=1}^n \alpha_j d_j = 1\right\}. \tag{2.8}$$

If this problem has a solution, it is a critical point of the Lagrangian

$$\int_X \left(\sum_{j=1}^n \alpha_j \psi_j\right)_+^q d\mu + \lambda\left(1 - \sum_{j=1}^n \alpha_j d_j\right), \tag{2.9}$$

and hence at a solution $(\hat{\alpha}, \hat{\lambda})$ we have

$$\begin{cases} 0 = \displaystyle\int_X q\left(\sum_{j=1}^n \hat{\alpha}_j \psi_j\right)_+^{q-1} \psi_i \, d\mu - \hat{\lambda} d_i, & i = 1, \ldots, n, \\ 0 = 1 - \displaystyle\sum_{j=1}^n \hat{\alpha}_j d_j. \end{cases} \tag{2.10}$$

Now multiplying the first n equations by $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ and summing yields

$$\hat{\lambda} = \int_X q\left(\sum_{j=1}^n \hat{\alpha}_j\psi_j\right)_+^{q-1}\left(\sum_{j=1}^n \hat{\alpha}_j\psi_j\right) d\mu \geq 0.$$

If $\hat{\lambda} > 0$ then we see from (2.10) that

$$d_i = \int_X \left(\sum_{j=1}^n \alpha_j^*\psi_j\right)_+^{q-1} \psi_i \, d\mu \tag{2.11}$$

when $\alpha_j^* = \hat{\alpha}_j(q/\hat{\lambda})^{1/q-1}$. If $\hat{\lambda} = 0$ we would have

$$\int_X \left(\sum_{j=1}^n \hat{\alpha}_j\psi_j\right)_+^{q-1} d\mu = 0$$

and hence $\Sigma_{j=1}^n \hat{\alpha}_j\psi_j \leq 0$ a.e. (μ). But we also have

$$\int_X \left(\sum_{j=1}^n \hat{\alpha}_j\psi_j\right)\hat{g}\, d\mu = \sum_{j=1}^n \hat{\alpha}_j d_j = 1,$$

which is not possible, since $\hat{g} \geq 0$. Thus $\hat{\lambda}$ must be strictly positive.

We see that if (2.8) has a solution, then from (2.11) we have an interpolant of the desired form and hence a solution for our minimal norm problem. It remains to prove that the infimum is attained in (2.8). Let $\{\alpha^k\}_{k=1}^\infty$ be the coefficients of a minimizing sequence. If $\{\|\alpha^k\|_\infty\}$ has a bounded subsequence, then clearly (2.8) has a solution. Suppose $\|\alpha^k\|_\infty \to \infty$. Then dividing the objective function by $\|\alpha^k\|_\infty^q$ and the constraint by $\|\alpha^k\|_\infty$, we conclude that there exists an $\hat{\alpha} := (\hat{\alpha}_1, \ldots, \hat{\alpha}_n)$ such that

$$\max\{|\alpha_j| : j = 1, \ldots, n\} = 1,$$

$$\sum_{j=1}^n \hat{\alpha}_j d_j = 0,$$

and

$$\int_X \left(\sum_{j=1}^n \hat{\alpha}_j\psi_j\right)_+^q d\mu = 0.$$

As before, we conclude that $\Sigma_{j=1}^n \hat{\alpha}_j\psi_j \leq 0$ a.e. (μ). Now we have hypothesized a $\hat{g} \in K$ such that the $\{\psi_j\}$ are linearly independent over the support of $\hat{g}$. Hence

$$0 = \sum_{j=1}^n \hat{\alpha}_j d_j = \int_X \left(\sum_{j=1}^n \hat{\alpha}_j\psi_j\right)\hat{g}\, d\mu < 0.$$

This contradiction means that the coefficients of a minimizing sequence must be bounded and hence that the infimum in (2.8) is attained. ■

Next, we characterize the solution to (2.4) without any linear independence hypothesis, provided that $L_p(\mu)$ is separable, which we will assume for the duration. First let us begin with an observation concerning the supports of the functions in K.

Proposition 2.2. *Let $L_p(\mu)$ be separable and let K be nonempty and $\mathbf{d} \neq \theta$. Let $\{g_n\}_{n=1}^{\infty}$ be a dense subset of K and set*

$$\hat{g} := \left(\sum_{n=1}^{\infty} 2^{-n} g_n / \|g_n\|\right) / \sum_{n=1}^{\infty} 2^{-n} / \|g_n\|.$$

Then $\hat{g} \in K$ and if $\Omega := \{x: \hat{g}(x) > 0\}$ we have $g = g\chi_\Omega$ for all $g \in K$.

Proof. It is easy to verify that $\hat{g} \in K$. Let $g \in K$ and $g_{n_k} \to g$. Then

$$\int_{\Omega^c} |g|^p \leq \lim_{n\to\infty} \int_X |g - g_{n_k}|^p \, d\mu = 0.$$

Thus $g = 0$ a.e. (μ) on the complement of Ω and hence as an element of $L_p(\mu)$, $g = g\chi_\Omega$. We remark that in a measure theoretic sense, Ω is the smallest set for which this is true. ■

This brings us to the main theorem.

Theorem 2.3. *Let K be nonempty and separable. We assume that $\mathbf{d} \neq \theta$. Let f be the element in K of minimal $L_p(\mu)$ norm. Then f is characterized by $f \in D$ and*

$$f = \left(\sum_{j=1}^{n} \alpha_j \psi_j\right)_+^{q-1} \chi_\Omega, \tag{2.12}$$

where Ω is as in Proposition 2.2.

Proof. All the hard work has already been done in Proposition 2.2. We just note the minimal changes needed here. If there is an $f \in D$ of the form (2.12), then for any $g \in K$ we have

$$\int_X (g - f) f^{p/q} \, d\mu = \int_X (g - f)\left(\sum_{j=1}^{n} \alpha_j \psi_j\right)_+ d\mu \geq 0,$$

where the first equality follows from the definition of Ω and the inequality follows just as in (2.7). Thus we need to prove that we can solve the equations

$$\int_\Omega \left(\sum_{j=1}^{n} \alpha_j \psi_j\right)_+^{q-1} \psi_i \, d\mu = d_i, \qquad i = 1, \ldots, n.$$

If the $\{\psi_i\}_{j=1}^{n}$ are linearly independent on Ω then we go directly to the problem [see (2.8)]

$$\min\left\{\int_\Omega \left(\sum_{j=1}^{n} \alpha_j \psi_j\right)_+^{q} : \sum_{j=1}^{n} \alpha_j d_j = 1\right\}.$$

On the other hand, if the $\{\psi_i\}_{i=1}^{n}$ are linearly dependent on Ω then we choose a maximal linearly independent subset that we will call $\{\psi_i\}_{i=1}^{m}$, $m < n$. Again considering

$$\min\left\{\int_\Omega\left(\sum_{j=1}^m \alpha_j\psi_j\right)_+^q d\mu:\ \sum_{j=1}^m \alpha_j d_j = 1\right\},$$

we first obtain coefficients $\hat{\alpha}_j$ such that

$$\int_\Omega\left(\sum_{j=1}^m \hat{\alpha}_j\psi_j\right)_+^{q-1}\psi_i\, d\mu = d_i,\qquad i = 1, \ldots, m.$$

Using linear dependence one finally concludes

$$\int_\Omega\left(\sum_{j=1}^m \hat{\alpha}_j\psi_j\right)_+^{q-1}\psi_i\, d\mu = d_i,\qquad i = 1, \ldots, n,$$

which of course guarantees a solution of type (2.12). ■

We can refine this result as follows. Note first that if $\{\psi_i\}_{i=1}^n$ are linearly independent on Ω then in fact in the argument above one may use X instead of Ω. Secondly, if the $\{\psi_i\}_{i=1}^n$ are dependent on Ω and if $\{\psi_i\}_{i=1}^m$ form a maximal independent subset as above, then for $r = m+1, \ldots, n$ set

$$s_r := \psi_r - \sum_{j=1}^m \beta_j^r\psi_j,$$

where the β_j^r's are chosen such that $s_r(x) = 0$ for $x\in\Omega$. Now let Γ be defined by

$$\Gamma = \bigcap_{r=m+1}^n \{x\in X:\ s_r(x) = 0\}.$$

Note that $\Gamma \supset \Omega$, and a little thought shows that one may replace Ω with Γ in Theorem 2.3. We summarize these remarks in Corollary 2.4.

Corollary 2.4. *Let f be the element in K of minimal $L_p(\mu)$ norm. Then f has the representation*

$$f = \left(\sum_{j=1}^n \alpha_j\psi_j\right)_+^{q-1}\chi_\Gamma,$$

where $\Gamma = X$ if the $\{\psi_i\}_{i=1}^n$ are linearly independent over Ω and Γ is as above otherwise.

We close this section by remarking that the introduction of the characteristic function is essential. This can be seen by considering the functions

$$\psi_i(x) = (1 - |x-i|)_+,\qquad i = 1, 2$$

as elements of $L_2[0, 3]$ (Lebesgue measure), and data vector $\mathbf{d} = (1, 0)$; that is,

$$D = \left\{g:\ \int_0^3 \psi_1 g = 1,\qquad \int_0^3 \psi_2 g = 0\right\}.$$

It is easy to see that $D \cap C = K \neq \emptyset$ and that the element of minimal norm is

$$f = 3\psi_1 \chi_{[0,1]}.$$

3. Constrained Spline Interpolation

In this section we characterize the solution to the following problem: Given p, $1 < p < \infty$, $n > k$, $0 \leq t_1 < \ldots < t_n \leq 1$, and $h \in L_p^k[0, 1]$ satisfying $h^{(k)} \geq 0$, find $s \in L_p^k[0, 1]$ such that $s(t_i)$, $s^{(k)} \geq 0$, and

$$\|s^{(k)}\|_p = \inf \{\|g^{(k)}\|_p : g(t_i) = h(t_i),\ i = 1, \ldots, n \text{ and } g^{(k)} \geq 0\}. \tag{3.1}$$

It is well documented [1] that this problem can be cast in terms of the k-th derivative using B-splines. Thus (3.1) is equivalent to

$$\min \left\{ \|g\|_p : g \geq 0 \text{ and } \int_0^1 gM_{i,k} = d_i \text{ for } i = 1, \ldots, n-k \right\}, \tag{3.2}$$

where $M_{i,k}$ is the B-spline that represents the divided difference functional $f \rightarrow f[t_i, \ldots, t_{i+k}]$ in $L_p^k[0, 1]$ and $d_i = h[t_i, \ldots, t_{i+k}]$. It is well- known that $\{M_{i,k}\}_{i=1}^{n-k}$ is a B-spline basis for the C^{k-2} order k splines supported on $[t_1, t_n]$. Of course, the solution f to (3.2) is just the k-th derivative, $s^{(k)}$, of the solution s to (3.1). Now applying the results of the previous section we conclude

Theorem 3.1. *The unique solution s to problem* (3.1) *is characterized by*

$$(3.3)\quad \begin{cases} \text{(i)} & s^{(k)} = \left(\sum_{j=1}^{n-k} \alpha_j M_{j,k} \right)_+^{q-1} \chi_\Gamma, \\ \text{(ii)} & \int_0^1 s^{(k)} M_{i,k} = h[t_i, \ldots, t_{i+k}], \qquad i = 1, \ldots, n-k, \\ \text{(iii)} & \Gamma = [0, 1] \setminus \bigcup_{j=1}^{n-k} \{(t_j, t_{j+k});\ d_j = 0\}. \end{cases}$$

We remark that if $p = 2$ then $q - 1 = 1$ and we see that the solution s is a piecewise polynomial of order $2k$ and continuity at least C^{k-1}. If all the d_i are positive then $s \in C^k$. Of course, one could allow repeated interpolation as long as the k-th derivative is not prescribed. This would only entail listing the knots according to their multiplicity; we leave this to the interested reader.

4. Extensions and Refinements

In this section, we state the most general theorems concerning constrained interpolation as we have presented it here. We first define another cone. Returning to the general setting of Section 2, let X be the disjoint union of three measureable sets, say

$$X = \Omega_1 \cup \Omega_2 \cup \Omega_3.$$

Let C be the cone of functions nonnegative on Ω_1, nonpositive on Ω_2, and unconstrained on Ω_3; that is,

$$C := \{g \in L_p(\mu)\colon g(x) \geq 0 \text{ on } \Omega_1 \text{ and } g(x) \leq 0 \text{ on } \Omega_2\}.$$

As before, let $\{\psi_i\}_{i=1}^n \subset L_q(\mu)$, and let D be as in (2.2). We assume that $K \cap D \neq \emptyset$ and, to avoid trivialities, that $\theta \notin K \cap D$.

As in Proposition 2.2, let Ω be the smallest measurable set such that

$$g = g\chi_\Omega$$

for all $g \in K$ (clearly $\Omega \supset \Omega_3$). The analogue to Theorem 2.3 is now apparent.

Theorem 4.1. *Let K and Ω be as above. Suppose that f is the element of minimal $L_p(\mu)$ norm in K. Then f is characterized by*

$$(4.1)\quad \begin{cases} \text{(i)} \quad f = [(s)_+^{q-1}\chi_{\Omega_1} - (s)_-^{q-1}\chi_{\Omega_2}]\chi_\Omega + |s|^{q-1} \operatorname{sgn}(s)\chi_{\Omega_3}, \\ \text{(ii)} \quad s = \sum_{j=1}^{n} \alpha_j\psi_j, \\ \text{(iii)} \quad f \in D. \end{cases}$$

The proof follows the same path used in Section 2, so we leave it to the reader to complete the argument.

We close this section with an application to spline interpolation. Let $h \in L_p^k[0, 1]$, $n > k$, $0 \leq t_1 < \ldots < t_n \leq 1$. We want to find the $s \in L_p^k[0, 1]$ that interpolates h at $t_1, \ldots, t_n$, satisfies the side conditions

$$(4.2)\quad \begin{cases} \text{i)} \quad s^{(k)}(t) \geq 0 \text{ for } t \in \Omega_1 \\ \text{ii)} \quad s^{(k)}(t) \leq 0 \text{ for } t \in \Omega_2 \end{cases}$$

with $\Omega_1 \cap \Omega_2 = \emptyset$, and

$$\|s^{(k)}\|_p = \inf\{\|g^{(k)}\|_p\colon g(t_i) = h(t_i),\ i = 1, \ldots, n \text{ and } g^{(k)}\chi_{\Omega_1} - g^{(k)}\chi_{\Omega_2} \geq 0\}.$$

Of course, we assume that $h^{(k)}$ satisfies (4.2) as well. This guarantees that the minimization problem has a solution. Setting $\Omega_3 = (\Omega_1 \cup \Omega_2)^c$, we obtain

Theorem 4.2. *If $h[t_i, \ldots, t_{i+k}] \neq 0$ for $i = 1, \ldots, n-k$, then the solution, s, to the above minimum norm problem is characterized by*

(i) $s^{(k)} = (r)_+^{q-1}\chi_{\Omega_1} - (r)_-^{q-1}\chi_{\Omega_2} + |r|^{q-1} \operatorname{sgn} r\chi_{\Omega_3}$,

(ii) $r = \sum_{j=1}^{n-k} \alpha_j M_{j,k}$, and

(iii) $s(t_i) = h(t_i)$, $i = 1, \ldots, n$.

If $h[t_i, \ldots, t_{i+k}] = 0$ *for some* i, *then it might be necessary to introduce the set* Γ *as described in Theorem* 4.1 *and Corollary* 2.4.

This theorem is an immediate consequence of Theorem 4.1, provided we show that $\Gamma = [0, 1]$. This is easy to verify, since the data $h[t_i, \ldots, t_{i+k}] \neq 0$ for $i = 1, \ldots, n-k$.

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