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COVERING THE PLANE BY CONVEX DISCS

By

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In the 'thirties German location theorists, W. Christaller, A. Lösch and others (see [12]), proposed the following problem. On a uniformly populated domain we want to distribute a great but given number of producers, each of which supplies those points of the plane with a certain kind of goods which lie nearer to it than to any other producer. The transportation cost is supposed to be an increasing function of the distance. Find the arrangement of the producers which yields the minimum of the total cost of transportation.

It has been conjectured that the best arrangement is given by the vertices of an equilateral triangular lattice. The correctness of this conjecture immediately follows from the following theorem of L. FEJES TÓTH [7], who came essentially to the same problem independently from the authors mentioned above by purely geometrical considerations.

THEOREM 1. *Let F be the part of a convex hexagon H covered by a finite number of congruent circles of total area T. Then*

$$F \leqq \bar{F},$$

where $\bar{F}$ is the part of a regular hexagon of area H covered by a concentric circle of area T.

Here and in what follows we denote the area of a domain X simply by X or sometimes, to avoid confusion, by $|X|$. By a hexagon we mean a polygon with at most six sides.

We start with a simple new proof of Theorem 1 which will enable us to give far-reaching generalizations.

Let A and B be two convex domains. The quantity $|A \dot{-} B| + |B \dot{-} A|$ is called area deviation of A and B. Here $X \dot{-} Y$ denotes the set of those points of X which are not contained in Y. In what follows a central part will play the weighted area deviation, shortly the *deviation of B from A,* defined by

$$a(A,B) = p\,|A \dot{-} B| + q\,|B \dot{-} A|,$$

where p and q are fixed positive numbers such that $p + q = 1$. (Note that for $p \neq q$ the deviation is not symmetric in A and B.)

Let A be a strictly convex domain. It is easily seen that the n-gon having the minimal deviation from A has the property that each side is divided by the boundary of A in three segments in the ratio $p : 2q : p$. It follows that if A is a circle then the n-gon of least deviation is regular and concentric with A.

Let a_n be the minimum of the deviation of an n-gon from a circle of unit area. We claim that the sequence $a_3, a_4, \dots$ is convex:

$$a_{n-1} + a_{n+1} \geqq 2a_n, \qquad n = 4, 5, \dots .$$

Let C be a unit circle (of radius 1) centered at O. Let P be an n-gon of minimal deviation from C. Let L and M be an endpoint and the midpoint of a side of P, respectively. Let the boundary of C intersect the segment LM in N (Fig. 1). We

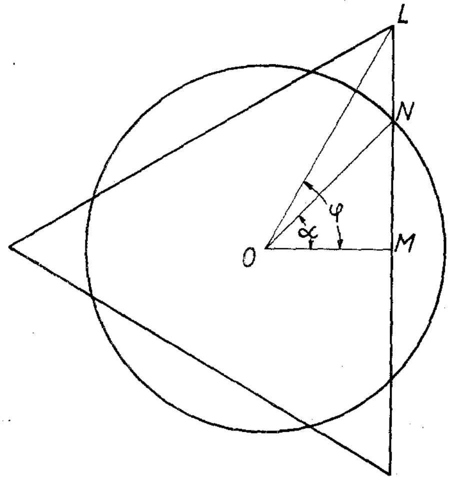

Fig. 1

have $LN : NM = p : q$, whence

$$\tan \alpha = q \tan \varphi$$

and

$$a_n = \frac{n}{\pi}\left[p(\alpha - \sin\alpha\cos\alpha) + q\left(\frac{p}{q}\sin\alpha\cos\alpha - \varphi + \alpha\right)\right] = \frac{n}{\pi}(\alpha - q\varphi) = \frac{n}{\pi}\alpha - q,$$

where $\varphi = \sphericalangle LOM = \pi/n$ and $\alpha = \sphericalangle NOM$. To show the convexity of a_n, we refer to the fact [8] that if $f(x)$ is a convex function of $x > 0$ then so is the function $xf(1/x)$. Thus it suffices to show that the function $\alpha = \alpha(\varphi)$ defined for $0 \leqq \varphi \leqq \pi/3$ by $\tan\alpha = q \tan\varphi$ is convex. But this can be verified simply by an elementary computation.

After these preliminaries the proof of Theorem 1 will make no difficulties. We may suppose that T lies between the area of the incircle and circumcircle of $\bar{H}$. For otherwise the theorem is trivial. Again we may suppose that the circles $K_1, \dots, K_m$ considered in Theorem 1 are of unit area. Let D_i be the Dirichlet cell of K_i with respect to H, i.e. the set of those points of H whose power with respect to K_i is less than the power with respect to any other circle. We may assume that $D_i > 0$. D_i is a convex polygon of number of sides, say, s_i. As a simple consequence of Euler's formula, we have

$$s_1 + \ldots + s_m \leqq 6m .$$

Since $F = \bigcup_{i=1}^{m} (K_i \cap D_i)$, we have, on the one hand,

$$|F| = \sum_{i=1}^{m} (|K_i| - |K_i \dot{-} D_i|),$$

on the other hand,

$$|F| = \sum_{i=1}^{m} (|D_i| - |D_i \dot{-} K_i|).$$

Hence

$$F = pF + qF = p \sum_{i=1}^{m} K_i + q \sum_{i=1}^{m} D_i - \sum_{i=1}^{m} a(K_i, D_i).$$

Thus, in view of $a(K_i,D_i) \geqq a_{s_i}$ and the convexity of the sequence $a_3, a_4, \ldots$,

$$F = pT + qH - \sum_{i=1}^{m} a(K_i, D_i) \leqq pT + qH - \sum_{i=1}^{m} a_{s_i} \leqq pT + qH - ma_6 .$$

Note that ma_6 is the minimum of the deviation of a hexagon from the circle U of area T mentioned in the theorem. Thus, for a certain regular hexagon V we have $a(U,V) = ma_6$. Since, by supposition, T lies between the area of the incircle and circumcircle of $\bar{H}$, we can choose p and q so that $V \equiv \bar{H}$. Now

$$pT + qH - ma_0 = pU + q\bar{H} - [p(U - \bar{F}) + q(\bar{H} - \bar{F})] = F .$$

This completes the proof of Theorem 1.

Our following theorem states that Theorem 1 continues to hold for circles with not too different areas.

We define a function $w(x)$ for $x > 0$ as follows. For $\sqrt{27}/2\pi < x < \sqrt{12}/\pi$ we choose the weights p and q so that the quotient of the areas of a circle and the hexagon having the least deviation from this circle should be $1/x$. In this case we define $w(x)$ by

$$w(x) = \frac{a_6 - a_7}{a_5 - a_6} .$$

For $x \leqq \sqrt{27}/2\pi$ and $x \geqq \sqrt{12}/\pi$ let $w(x)$ be zero. In $(\sqrt{27}/2\pi, \sqrt{12}/\pi)$ $w(x)$ can be represented by

$$w(x) = \frac{6 \arctan q \tan \frac{\pi}{6} - 7 \arctan q \tan \frac{\pi}{7}}{5 \arctan q \tan \frac{\pi}{5} - 6 \arctan q \tan \frac{\pi}{6}},$$

where

$$q=\sqrt{3\left(\frac{\sqrt{12}}{\pi x}-1\right)}\,.$$

Observe that for any $x>0$ we have

$$w(x)<\frac{7\sin\frac{2\pi}{7}-6\sin\frac{2\pi}{6}}{6\sin\frac{2\pi}{6}-5\sin\frac{2\pi}{5}}\sim 0{,}6275\,.$$

THEOREM 2. *Theorem* 1 *holds, instead of congruent circles, for circles such that the quotient of the areas of any two is at least* $w(H/T)$.

This theorem contains earlier results of BÖRÖCZKY (see [7], p. 194) and BLIND [3] which concern the limiting cases $T/H=\pi/\sqrt{12}$ (packing) and $T/H=2\pi/\sqrt{27}$ (covering).

In order to prove Theorem 2 we may suppose that $\pi/\sqrt{12}<T/H<2\pi/\sqrt{27}$. Let the circles considered in the theorem be $K_1,\dots,K_m$. Then, using the notations of the proof of Theorem 1, we have

$$F=pT+qH-\sum_{i=1}^{m}a(K_i,D_i)\leqq pT+qH-\sum_{i=1}^{m}K_ia_{s_i}\,.$$

We choose p and q so that the quotient of the areas of a circle and the hexagon having the least deviation from this circle should be T/H.

Suppose that for some i we have $s_i>6$. Then, by $s_1+\dots+s_m\leqq 6m$, there is a j such that $s_j<6$. Therefore, in view of the supposition $K_j/K_i\geqq w(H/T)$ and the convexity of the sequence $a_3,a_4,\dots$, we have

$$\frac{K_j}{K_i}(a_{s_j}-a_{s_j+1})\geqq a_{s_i-1}-a_{s_i}\,,$$

i.e.

$$K_ia_{s_i}+K_ja_{s_j}\geqq K_ia_{s_i-1}+K_ja_{s_j+1}\,.$$

Replace, in the sum $\sum_{i=1}^{m}K_ia_{s_i}$, s_i by s_i-1 and s_j by s_j+1, and repeat this process until no s_i is greater than 6. Finally, replace all s_i's less than 6 by 6. Then, by $a_3>a_4>\dots$ and the above inequality, we have, for the original s_i's,

$$\sum_{i=1}^{m}K_ia_{s_i}\geqq a_6\sum_{i=1}^{m}K_i=Ta_6\,.$$

Consequently

$$F\leqq pT+qH-Ta_6\,.$$

But, by the choice of p and q, the right side equals $\bar{F}$.

In what follows we shall give a generalization of Theorem 2, considering, instead of circles, affine images of an arbitrary convex disc. We say that two convex discs A and B *cross* each other if neither $A \dot{-} B$ nor $B \dot{-} A$ is connected.

THEOREM 3. *Let K be a convex disc of unit area. Consider a finite number of affine images of K of equal area not crossing each other. Let F be the part of a convex hexagon H covered by the discs. In order to give an upper bound for F, we define $f(h)$ as the maximum of the area of that part of K which can be covered by a hexagon of area h. Then*

$$F \geqq T\,F(H/T),$$

where $F(h)$ is the least concave function not less than $f(h)$.

If K has central symmetry and $F(H/T)=f(H/T)$, then our bound can be arbitrarily approximated by a great number of congruent discs arranged in a conveniently chosen lattice. However, it is likely that there are centro-symmetric K's such that $f(h)$ is not concave. But even in this case the above bound cannot be replaced by a smaller one. We shall see later that if for a centro-symmetric K $F(H/T) > f(H/T)$ then the bound can be arbitrarily approximated by a great number of congruent discs one part of which is arranged in one lattice, the rest in another.

Theorem 3 contains important known results about packing and covering a hexagon by convex discs (see e. g. [1, 2, 4, 7, 9, 10, 13]).

In order to formulate the more general Theorem 4 we introduce the analogue of the function $w(x)$, considered above in connection with a circle, for an arbitrary convex disc.

Let K be a convex disc of unit area. Let u be the greatest number such that $f(u) = u$. Let U be the smallest number such that $f(U) = 1$. We claim that, with the possible exception of $h = U$, $F(h)$ has a derivative.

To see this we may assume that $h < U$. Consider a hexagon of area h covering a part of K of area $f(h)$. This hexagon has a side of length, say, s having a part of length qs in the interior of K for some $q > 0$. Small parallel displacements of this side show that

$$\overline{\lim_{k\to+0}}\frac{f(h+k)-f(h)}{k} \leqq q \qquad \text{and} \qquad \overline{\lim_{k\to-0}}\frac{f(h+k)-f(h)}{k} \geqq q\,.$$

This proves the assertion.

Let $a(K,P_n)$ be the deviation of a convex n-gon P_n from K for some fixed values of p and q. Let

$$a_n = \min_{P_n} a(K, P_n),$$

where P_n ranges over all convex n-gons. For $x \in (u,U)$ we take the deviations a_i with the weight $q = F'(x)$ and define $w(x)$ by

$$w(x) = \frac{a_6 - a_7}{a_5 - a_6},$$

with the convention that in the case when $a_6 - a_7 = a_5 - a_6 = 0$ we write $w(x) = 0$. For $x \notin (u,U)$ let $w(x)$ be 0.

THEOREM 4. *Theorem 3 continues to hold if we consider, instead of discs of equal area, discs of area such that the quotient of any two is not less than* $w(H/T)$.

The proof rests on the following

LEMMA. *Let a_n be the minimum of the deviation of a variable n-gon from a given convex disc for any prescribed values of p and q. Then*

$$a_{n+1} + a_{n-1} \geqq 2a_n, \qquad n = 4, 5, \ldots .$$

In the case when $p = q$ the lemma has been proved by EGGLESTON [5]. The general case can be settled by similar methods. We intend to give a detailed proof in another paper. Note that the lemma implies $w(x) \leqq 1$.

Using this lemma, the proof of Theorem 4 is similar to that of Theorem 2, except the construction of the "Dirichlet cells".

Let $K_1, \ldots, K_m$ be convex discs no two of which cross each other. Assume that each disc has an inner point common with H which is not contained in any other disc. This means that no disc can be removed without diminishing the part F of H covered by the discs. Then one can construct to each K_i a convex polygon D_i of number of sides, say, s_i such that

1. each D_i is contained in H,
2. no two D_i's have inner points in common,
3. $F = \bigcup\limits_{i=1}^{m} (D_i \cap K_i)$,
4. $s_1 + \ldots + s_m \leqq 6m$.

The construction of the polygons D_i proceeds as follows. First we replace each K_i by $K_i \cap H$. Then we reduce the discs to disjoint convex discs without diminishing the part F of H covered by the original discs. Finally we blow up the reduced discs as far as possible under the condition that they remain convex, disjoint and contained in H. As to the details we refer to [1, 10, 11].

Applying this construction to the discs considered in Theorem 4, we have

$$F \leqq pT + qH - \sum_{i=1}^{m} a(K_i, D_i).$$

Obviously, we may suppose that $u < H/T < U$. Choosing $q = F'(H/T)$, we obtain, in the same way as in the proof of Theorem 2,

$$F \leqq pT + qH - Ta_6.$$

Let g be the graph of the function $F(h)$. Let t be the tangent of g of slope q. Let $(h_1, f(h_1))$ be a point of t. Then, for any $h > 0$, we have

$$f(h) \leqq f(h_1) + q(h - h_1).$$

Therefore, if a is the deviation (weighted with p and q) of a hexagon of area h from K, then

$$a \geqq p + qh - f(h) \geqq p + qh_1 - f(h_1).$$

It follows that a hexagon of area h_1 covering a part of K of area $f(h_1)$ has a least possible deviation from K, i.e.

$$a_6 = p + qh_1 - f(h_1).$$

Thus

$$F \leqq T\left(p + q\frac{H}{T} - a_6\right) \leqq T\left\{f(h_1) + q\left(\frac{H}{T} - h_1\right)\right\} = TF\left(\frac{H}{T}\right).$$

This completes the proof of Theorem 4.

Let H be a fixed convex hexagon, K an arbitrarily given centro-symmetric convex disc and T a positive number. We claim that there are congruent, homothetic replicas of K of total area T such that the area F of the part of H covered by them is arbitrarily close to $TF(H/T)$. To see this, we refer to the fact [6] that to any centro-symmetric K and any p and q there is a centro-symmetric hexagon having a minimal deviation from K.

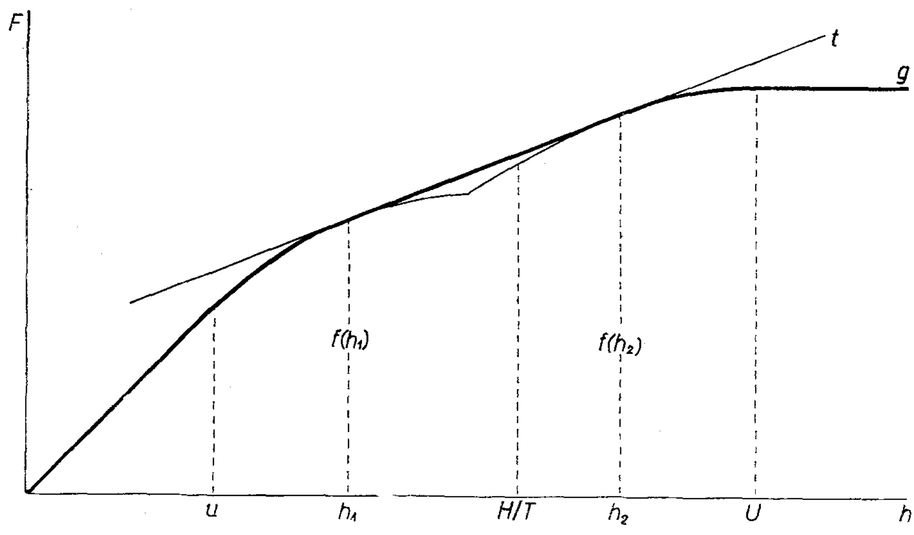

Fig. 2

Supposing again that $K = 1$, there are two centro-symmetric hexagons h_1 and h_2 such that (Fig. 2) $0 < h_1 \leqq H/T \leqq h_2$ and, for $q = F'(H/T)$,

$$a_6 = a(K, h_i) = p + qh_i - f(h_i), \qquad i = 1, 2.$$

The hexagons h_1 and h_2 may coincide. Suitably chosen translates of h_i form a tessellation. Respective translates of K form a lattice Λ_i of density $1/h_i$.

Divide H, say by a straight line, in two parts H_1 and H_2 so that

$$\frac{H_1}{h_1}+\frac{H_2}{h_2}=T.$$

Let $\mu\Lambda_i$ be a lattice arising from Λ_i by a similitude of ration μ. Consider those discs of $\mu\Lambda_i$ which are contained in H_i. Provided that μ is very small, the total area of these discs is approximately H_i/h_i. On the other hand, the area of the part covered by these discs is approximately $f(h_i)H_i/h_i$. Thus we have constructed congruent and homothetic copies of K contained in H whose total area is arbitrarily close to T and which cover a part of H of area arbitrarily close to

$$f(h_1)\frac{H_1}{h_1}+f(h_2)\frac{H_2}{h_2}.$$

But a simple computation shows that the last sum equals $TF(H/T)$.

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