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ON HERMITE-FEJÉR INTERPOLATION SEQUENCES

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Let $w(x) \in \mathfrak{L}$ be a non-negative weight function with support in $[-1, 1]$, $\int_{-1}^{1} w(x)dx > 0$. We consider the orthogonal polynomials $\{p_n(w; x)\}$ belonging to this weight; let the roots of $p_n(w; x)$ be

$$1 > x_{1n} > x_{2n} > \ldots > x_{nn} > -1 .$$

For an arbitrary $f \in C[-1, 1]$ we construct the uniquely defined polynomial in x, $\mathcal{H}_n(w; f; x)$, of degree at most $2n-1$, satisfying

$$(1) \qquad \mathcal{H}_n(w; f; x_{kn}) = f(x_{kn}), \qquad \mathcal{H}_n'(w; f; x_{kn}) = 0 \qquad (k = 1, 2, \ldots, n).$$

The sequence (1) of interpolatory polynomials was introduced and investigated first by L. FEJÉR [1], [2], and called by him "step parabolas". They form the most important case of the Hermite–Fejér interpolation sequences. The general case is obtained if we prescribe the values $\mathcal{H}_n'(w; f; x_{kn})$ in some other preassigned way (see L. FEJÉR [4], [5], [6]). In his first papers L. FEJÉR treated the case $w_0(x) \equiv 1$, i.e. if x_{kn} are the roots of the Legendre polynomials. For this case L. FEJÉR proved that for an arbitrary $f \in C[-1, 1]$ the sequence $\mathcal{H}_n(w_0; f; x)$ tends to $f(x)$ uniformly in every closed part of $(-1, 1)$, but near the end point ± 1 he observed a remarkable anomaly: the sequences $\{\mathcal{H}_n(w; f; 1)\}$ and $\{\mathcal{H}_n(w; f; -1)\}$ tend to $\frac{1}{2}\int_{-1}^{1} f(x)dx$ which is – in general – different from $f(1)$ resp. $f(-1)$.

In his later paper L. FEJÉR [3] proved that for $w_1(x) = (1-x^2)^{-1/2}$, i.e. for the zeros of the Chebychev polynomials of the first kind, $\mathcal{H}_n(w_1; f; x)$ tends uniformly in $x \in [-1, 1]$ to $f(x)$ for every $f \in C[-1, 1]$. Following the investigations of L. FEJÉR, G. SZEGŐ [8] gave a complete treatment for the zeros of Jacobi polynomials $P_n^{(\alpha, \beta)}$, orthogonal to $w_{\alpha, \beta}(x) = (1-x)^\alpha(1+x)^\beta$ $(\alpha > -1, \beta > -1)$. He proved that $\mathcal{H}_n(w_{\alpha, \beta}; f; x) \to f(x)$ uniformly in $[-1, 1]$ for every $f \in C[-1, 1]$ if and only if $\alpha < 0$ and $\beta < 0$; for $\alpha \geqq 0$ there is no uniform convergence in the neighbourhood of $x = 1$, and for $\beta \geqq 0$ in the neighbourhood of $x = -1$, resp., though there is uniform convergence inside of $[-1, 1]$. (See also SZEGŐ [9], § 14.6.) In the present paper we are going to prove a more general negative result. Our method is an extension of the idea of L. FEJÉR [1], and furnishes an alternative proof of G. SZEGŐ's results concerning non-uniformity of convergence.

THEOREM. *For a fixed but arbitrary* $\xi \in [-1, 1]$ *let* $W_\xi(x) = (x - \xi)w(x)$ *be of bounded variation in* $[-1, 1]$; *then there exists an* $f \in C[-1, 1]$ *so that* $\mathcal{H}_n(w; f; x)$ *does not converge uniformly to* $f(x)$ *in* $[-1, 1]$ *for* $n \to \infty$.

REMARKS. a) Taking $\xi = -1$, we see that $\mathcal{H}_n(w_{\alpha,\beta}; f)$ is not uniformly converging if $\alpha \geqq 0$, $\beta > -1$ and taking $\xi = 1$, we see that it is not uniformly converging if $\alpha > -1$, $\beta \geqq 0$ and at least one f; this is SZEGŐ's result.

b) The author proved in his paper [7], that if $w(x)$ is positive in $[-1, 1]$ and satisfies there uniformly the Dini–Lipschitz condition

$$w(x_2) - w(x_1) = o\left(\log^{-1} \frac{1}{|x_2 - x_1|}\right),$$

then $\mathcal{H}_n(w; f; x)$ converges to an arbitrary $f \in C[-1, 1]$ uniformly in every closed part of $(-1, 1)$. Our present Theorem shows that this is no more valid uniformly for the closed interval $[-1, 1]$. (At least if w is of bounded variation.)

PROOF OF THE THEOREM. By (1), $\mathcal{H}_n'(w; f; x)$ vanishes at all zeros of $p_n(w; x)$, so that

$$\mathcal{H}_n'(w; f; x) = p_n(w; x) r_{n-2}(x). \tag{2}$$

Since $\mathcal{H}_n'$ is a polynomial of degree at most $2n - 2$, r_{n-2} is a polynomial of degree at most $n - 2$. Now $p_n(w; x)$ is orthogonal to every polynomial of degree at most $n - 1$, so that

$$\int_{-1}^{1} \mathcal{H}_n'(w; f; x) W_\xi(x) dx = \int_{-1}^{1} p_n(w; x) r_{n-2}(x)(x - \xi) w(x) dx = 0. \tag{3}$$

As a consequence of (3) we have

$$\mathcal{H}_n(w; f; 1) W_\xi(1) - \mathcal{H}_n(w; f; -1) W_\xi(-1) = \int_{-1}^{1} \mathcal{H}_n(w; f; x) dW_\xi(x). \tag{4}$$

This relation is a generalization of L. FEJÉR's [2] equation

$$\mathcal{H}_n(w_0; f; 1) = \frac{1}{2} \int_{-1}^{1} \mathcal{H}_n(w_0; f; x) dx. \tag{5}$$

In fact, taking $\xi = -1$, $w(x) = w_0(x) \equiv 1$, (5) turns out to be a special case of (4).

Concluding our proof, we distinguish the two cases when

$$\| \mathcal{H}_n(w; f; x) \| \leqq K \| f \| \tag{6}$$

holds with some constant K independent of n[1] and the case when (6) does not hold.

[1] Here $\| \, . \, \|$ denotes the usual supremum norm of $C[-1, 1]$.

In the second case the existence of a not uniformly converging sequence $\{\mathcal{H}_n(w;f)\}$ follows from H. LEBESGUE's well known theorem. Let us now assume that (6) holds. In this case we obtain that for every $f \in C[-1, 1]$ satisfying

$$\lim_{n\to\infty} ||f - \mathcal{H}_n(w;f)|| = 0 \tag{7}$$

we have

$$f(1)W_\xi(1) - f(-1)W_\xi(-1) = \int_{-1}^{1} f(x)dW_\xi(x) \tag{8}$$

as a consequence of (4), i.e. (8) is a necessary condition for (7).

Now (8) is not satisfied for every $f \in C[-1, 1]$, unless $W_\xi(x) \equiv 0$, i.e. $w(x) \equiv 0$, and this case is excluded by $\int_{-1}^{1} w(x)dx > 0$. So there exist functions $f \in C[-1, 1]$ not satisfying (8) and consequently, not satisfying (7). Q.e.d.

We call the reader's attention to the following open

PROBLEM. Are there any weight functions $w(x)$ of bounded variation, for which (7) holds for all functions $f \in C[-1, 1]$ satisfying (8) for $\xi = 1$ and $\xi = -1$?[2]

Even the special case $w(x) = w_0(x) \equiv 1$ seems to be not settled. We formulate this case (on behalf of its interest) separately. Let the x_{kn} be the zeros of the n-th Legendre polynomial $P_n(x)$. Is it true that the step parabolas defined by (1) converge uniformly in x to $f \in C[-1, 1]$, provided that

$$f(1) = f(-1) = \frac{1}{2}\int_{-1}^{1} f(x)dx\ ?$$

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[2] It is easy to see that if (8) holds for $\zeta = 1$ and $\zeta = -1$, it holds for every value of ζ.
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