Foundations of Physics, Vol. 13, No. 1, 1983

The Wigner Distribution Function—50th Birthday

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Received June 1, 1982

We discuss the profound influence which the Wigner distribution function has had in many areas of physics during its fifty years of existence.

1. INTRODUCTION

Among the many great achievements in theoretical physics associated with the name of Eugene Paul Wigner, one must surely include his seminal work on distribution functions. His initial paper on this subject[(1)] appeared in 1932 when he was thirty years old and thus perhaps it is appropriate that we celebrate the 50th anniversary of this paper in conjunction with Eugene Wigner's own 80th birthday.

This paper was written shortly after Wigner arrived in the U.S. in 1930 and, to my knowledge,[(2)] it is the first paper which he published in English and also his first paper in the American *Physical Review*. It bears the rather innocuous title "On the Quantum Correction for Thermodynamic Equilibrium," which belies the profound influence it has had on many aspects of physics. The notable feature is the range of its impact: from the insight it has provided to investigators of fundamental problems in theoretical physics to its usefulness as a calculational tool in such diverse areas as statistical mechanics, condensed-matter, gravitational-wave detection, optics, nuclear physics, and communication theory. Since the present author is currently participating in the preparation of a detailed review[(3)] on the Wigner distribution function (WDF) and the many ramifications associated with it, our present approach will simply be

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0015-9018/83/0100-0083$03.00/0

confined to a "broad-brush" attempt to identify *some* of the major benchmarks associated with progress in this area over the last fifty years. Thus, in particular, we will *not* discuss the plethora of applications nor the fascinating connection between the WDF and distribution functions in quantum optics, which—among other things—will be discussed at length in our forthcoming review.

In Section 2, we will discuss the Wigner distribution function and its properties as well as mentioning extensions which treat spin, statistics, and relativity. Section 3 will be concerned with some of the many other distribution functions which the WDF has essentially spawned. In Section 4 we consider the relevance of the WDF to some fundamental questions in quantum mechanics, such as the relationship between classical and quantum mechanics, and whether or not the "correct" relativistic equation for elementary particles should be inherently nonlocal. Finally we point out that some basic questions which have been considered by other techniques are more naturally treated—and with a concomitant increase in physical insight—by WDF techniques. By way of example, we consider the so-called two-photon coherent states. Another example is concerned with a new interpretation which was proposed for the scalar product in Hilbert space, with an attendant suggestion of a generalized notion of measurement.

2. THE WIGNER DISTRIBUTION FUNCTION (WDF)

It is widely accepted that one of the most powerful and elegant approaches to classical mechanics is via the concept of *phase-space*. Thus, for example, a particle may be described by a classical phase-space distribution function $P_{cl}(q, p)$ where q and p denote its position and momentum, respectively (restricting our discussion to one dimension since generalization to three dimensions is straightforward). Since $P_{cl}(q, p)$ is the probability that a particle simultaneously has position q and a momentum p, it follows that the average of any function of q and p, $A(q, p)$ say, may be written

$$\langle A \rangle = \int_{\infty}^{\infty} dq \int_{\infty}^{\infty} dp\, A(q, p)\, P_{cl}(q, p) \tag{1}$$

In quantum mechanics, by contrast, the uncertainty principle states that we cannot simultaneously know the position and momentum of a particle. Instead, one deals with a wave-function $\psi(q)$, and its Fourier transform $\phi(p)$, where $|\psi(q)|^2$ gives the probability of finding the particle at position q and $|\phi(p)|^2$ gives the probability that the particle has momentum p. In 1927, soon after the introduction of quantum mechanics, it was realized by both Landau and von Neumann that an even more fundamental quantity than the wavefunction is the density matrix $\hat{\rho}$ (we will designate all operators by $\hat{}$). By use of $\hat{\rho}$ one can write the average of a function of the position and momentum operators, $\hat{A}(\hat{q}, \hat{p})$ say, as

$$\langle \hat{A} \rangle = \mathrm{Tr}(\hat{A}\hat{\rho}) \tag{2}$$

Part of Wigner's great contribution was to shown that $\langle \hat{A} \rangle$ could also be written in a form analogous to the classical expression given in Eq. (1). Specifically

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, A(q, p)\, P(q, p) \tag{3}$$

where P is the Wigner distribution function (WDF) and is given by

$$P(q, p) = (\pi\hbar)^{-1} \int_{-\infty}^{\infty} dy \langle q - y | \hat{\rho} | q + y \rangle\, e^{2ipy/\hbar} \tag{4}$$

if the system is in a mixed state represented by a density matrix $\hat{\rho}$, and by

$$P(q, p) = (\pi\hbar)^{-1} \int_{-\infty}^{\infty} dy\, \psi^*(q + y)\, \psi(q - y)\, e^{2ipy/\hbar} \tag{5}$$

if the system is in a pure state $\psi(q)$. Also, $A(q, p)$ is a classical function which is derived from the operator $\hat{A}(\hat{q}, \hat{p})$ by a precisely defined correspondence rule.

Thus we have the remarkable result, given by Eq. (3), that the quantum (ensemble) expectation value can be replaced by a classical phase-space integration. In the classical limit, $P(q, p)$ is the phase-space distribution function which gives the probability that the coordinates and momenta have the values q and p. In general, $P(q, p)$ depends on $\hbar$ and may assume negative values,[1,4] which accounts for the frequent description of this quantity as a quasi-classical distribution function. Its usefulness stems from the fact that it provides a framework for *an exact reformation of nonrelativistic quantum mechanics in terms of classical concepts.*

Among the many appealing features of the Wigner distribution function (WDF) is that, as distinct from the Schrödinger equation, the limit $\hbar \to 0$ leads to classical mechanics and the development of various results in powers of $\hbar$ is a relatively straightforward procedure. In the words of Balescu,[5]"... the identity of structure between classical and quantum mechanics, when expressed in terms of Wigner functions, is an extremely remarkable feature. It will help us, particularly in nonequilibrium theory, in constructing a quite general and unified formalism, which can be translated at will into classical or quantum mechanics by simply inserting the corresponding definitions of the symbols."

For example, if one wishes to consider the statistical mechanics of a gas of particles the basic starting point is the calculation of the partition function. In the case of quantum statistics this involves a complicated sum over all the states. However, if one makes use of the WDF then the *sum of states* may be transformed into *an integral in phase-space*, which considerably simplifies the problem. However, this integral is more difficult to evaluate than the corresponding classical one ($\hbar = 0$), so one usually proceeds by carrying out an expansion in powers of $\hbar$, the so-called Wigner[1]–Kirkwood[6] expansion.

Wigner[1,4] presented a specific form for $P(p, q)$, while recognizing that other possibilities exist, depending on the conditions which are imposed on P. The P chosen by Wigner has the following properties:[4,7,8]

(a) it is a Hermitian, that is bilinear, form of the wave-function ψ. Hence it is real for all q and p. The Hermitian operator is, of course a function of q and p,

(b) if integrated over p, it gives the proper probabilities for the different values of q, and similarly with $p \leftrightarrow q$,

(c) that the correspondence between P and the wave function ψ is a Galilei invariant, i.e., invariant with respect to displacements and nonrelativistic transitions, to moving coordinate systems,

(d) it is invariant with respect to time reflections,

(e) the transition probability between two states ψ and ϕ is given, in terms of the corresponding distribution functions, P_ψ and P_ϕ say, as follows:

$$\left| \int \psi(x)^* \, \phi(x) \, dx \right|^2 = 2\pi\hbar \iint P_\psi(q, p) \, P_\phi(q, p) \, dq \, dp \tag{6}$$

(f) in the force-free case the equation of motion is the classical one. Also it is clear from Eq. (6) that P has to be able to assume negative values.

We turn now to a consideration of the manifestations of Bose and Fermi statistics on the WDF. The initial work in this area was carried out by Uhlenbeck and Gropper,[9] who calculated the partition function, and then the equation of state, of a nonideal Bose–Einstein or Fermi–Dirac gas. Their results were later verified by Kirkwood.[6] A very clear treatment of exchange effects has recently been presented by Alastuey and Jancovici,[10] with special emphasis on two-dimensional condensed-matter systems.

Perhaps the most elegant method of taking statistics into account is with the use of second quantization. This approach was pioneered by Stratonovich and Klimontovich,[11] and an explicit expression for the WDF in second quantized form was written down by Brittin and Chappell.[12,13]

Stratonovich[11] also considered distribution functions in the space of orientations of the spin, dealing first with spin $\frac{1}{2}$ and then the case of arbitrary spin.

For a relativistic generalization of the WDF we refer to the work of Cooper and Sharp[14] in their discussion of the determination of the hydrodynamic and transport properties of a pion field produced in a high-energy collison, Carruthers and Zachariasen[15] in their relativistic quantum transport approach to multiparticle production, and also the work of Balescu,[16] Van Weert,[17] and Hakim.[18] It has been considered anew by de Groot *et al.*[19] in their extensive exposition of relativistic kinetic theory.

3. OTHER DISTRIBUTION FUNCTIONS

Moyal[20] has shown that Wigner's method of going from quantum to classical concepts is the inverse of Weyl's rule[21] for obtaining operators from classical quantities viz.

$$e^{(i/\hbar)(\sigma q+\tau p)} \rightarrow e^{(i/\hbar)(\sigma \hat{q}+\tau \hat{p})} \tag{7}$$

Since the latter association is only one of many such associations[22] it became clear that corresponding to each is a distribution function. Some investigators were and are particularly interested in distribution functions which are everywhere nonnegative[23] on the premise that such functions are more closely parallel to their classical counterparts. However, there are others who feel that such nonnegative characteristics should not be the primary goal, particularly as other desirable properties are lost in the process.

Perhaps the most widely known of such so-called positive distribution functions (more strictly, nonegative functions) are introduced by Husimi[24] and recently considered anew by Cartwright.[25] Husimi obtained a nonnegative distribution function, P_s say, by smoothing the WDF with a Gaussian, for which $\Delta q\, \Delta p$ is no smaller than the $\hbar/2$ of the minimum uncertainty wave-packet. However, it has been shown[8,26] that P_s does not possess property (f) referred to in Section 2. In other words, the time dependence of P_s, in contrast to that of P, contains a correction term of $\hbar^2$ which as we remarked in Ref. 8—would also appear in the time derivative of the classical distribution function if this were "smoothed."

As we already remarked, Wigner in his initial paper recognized that there are a virtual infinity of choices for a distribution function but he made a choice which was guided by the desired properties outlined above. For various reasons—some of which will be discussed below—other investigators prefer some alternative properties and thus other distribution functions. However, all viable choices lead to the same observational results and from the point of view of computational simplicity the WDF is clearly to be preferred, except for some problems in quantum optics for which a more natural choice is the P distribution.[27,28]

4. RELATION TO SOME FUNDAMENTAL QUESTIONS IN QUANTUM MECHANICS

Since the WDF formalism is equivalent to quantum mechanics one expects that it should throw additional light on the relationship between classical and quantum mechanics. This is in fact turning out to be the case.

For example, Berry[29] has demonstrated that the WDF provides an attractive way to link the classical and quantum descriptions of integrable states. His work on one-dimensional tori was extended to two dimensions by Ozorio de Almeida and Hannay,[30] who also pointed out that the local behavior of the semiclassical Wigner function is governed by Thom's catastrophe theory.[31,32]

Furthermore, Berry and Balazs[33] have considered the evolution of semiclassical quantum states in phase-space by using the WKB expression for the wave-function to construct an approximation for Wigner's function in phase-space. Their approach is similar to that of Heller[34] who developed a new approach to semiclassical dynamics.

The question of the semiclassical approximation of the WDF was taken up again by Balazs,[35] who pointed out that, whereas Wigner's pq space (which he refers to as W space) becomes the classical phase space when $\hbar \to 0$, in general the W space has a different invariance structure than phase-space. In particular, it admits an affine metric so that linear combinations of p and q are constant under linear transformations in the W space. In the $\hbar \to 0$ limit these transformations are replaced by canonical transformations.

In a different vein, Wigner's work has motivated others to study the—"very suggestive connection between quantum mechanics and some kind of phase-space similar to that of classical physics."[36] For example, Bohm and Wiley[36] were anxious to avoid the possibility of negative probabilities in phase-space, and they proceeded by expressing the laws of classical and quantum mechanics in terms of different algebras operating on the same phase space but, as they stress, "this phase-space is not derived from classical physics." As a result, they deduced that in this phase-space the quantum mechanical motion is inherently *nonlocal.* They also remark that there is a close relationship between their viewpoint and the Prigogine group,[37] the difference being that the latter group proceed by extending from the classical to the quantum domain whereas Bohm and Wiley start from quantum theory and then develop classical theory as a limiting case.

Another generalization of classical phase-space has been proposed by Prugovečki[38] and developed in detail by him and his coworkers.[39] This is the so-called fuzzy or stochastic phase-space, which consists of stochastic points, i.e., points which are not sharp but instead are spread out to an extent that is in keeping with the uncertainty principle. As a result one can obtain a positive distribution function, a particular example[39] being the Husimi function.[24]

In addition, Prugovečki[40] extended the concept of stochastic phase-space to the realm of relativistic quantum mechanics in a manner that is in conformity with the results of Newton and Wigner[41] concerning the localizability of relativistic particles, viz. that even free relativistic particles cannot be localized in arbitrarily small regions. He also showed that a *nonlocal* type of dynamics could be formulated and covariant models of extended spin 0 and $\frac{1}{2}$ particles were constructed which were free of some well-known problems associated with the relativistic quantum mechanics of point particles.

However, it should be emphasized that the use of stochastic phase-space has not led to any new observable results. This is not altogether surprising since the WDF is the essential starting point. Another unsatisfactory aspects is the arbitrariness associated with smoothing. Thus, as we've already noted, one can obtain a positive distribution function by smoothing with a Gaussian with an infinity of possibly variances,[25] the only restriction being that $\nabla q\, \nabla p \geqslant \hbar/2$. The smoothing function associated with the choice of the equality sign is identical to the Glauber coherent state,[27] whereas, if the inequality sign is used, then the operative function is the two-photon coherent states (TCP) introduced by Stoler[42] and Yuen[43,44] and considered in detail by them and many others.[45–47] [2]

[2] Yuen originally referred to these states as "generalized coherent states" (Ref. 43) but later (Ref. 44) adopted the name "two-photon coherent states," when he realized that the former nomenclature was associated with the generalized states introduced by Titulaer and Glauber [*Phys. Rev.* **145**, 1041 (1966)]. The latter satisfy the full coherence conditions of Glauber but are not minimum—uncertainty states. Yuen's states are not coherent and, in general, are not minimum uncertainty states. Actually Stoler (Ref. 42) appears to have been the first to consider such states and he noticed that the unitary transformation which generates these states from the coherent states also effects a scale transformation of q and p by reciprocal scale factors. This accounts for the name "squeeze states" which has been used to describe these states by "quantum nondemolition" investigators (Ref. 47).

Finally, before leaving the realm of basic questions, we would like to point out that that many investigations using other techniques can often be treated more naturally, and with more physical insight, by the use of WDF techniques. In others cases it is often a useful complementary tool. For example, let us consider the TCP. The starting point is Glauber's coherent state $|\alpha\rangle$, which are eigenstates of the annihilation operator $\hat{a}$. Yuen then carried out a Bogoliubov–Valatin transformation:

$$\hat{b} \equiv \mu\hat{a} + \nu\hat{a}^{+} \tag{8}$$

with $|\mu|^2 - |\nu|^2 = 1$ to ensure that $[\hat{b}, \hat{b}^{+}] = 1$. The eigenstates of $\hat{b}$ are the two-photon coherent states. Making use of a theorem of von Neumann,[48] led Yuen to remark that the linear canonical transformation given by Eq. (8) can be represented as a unitary transformation viz. $\hat{b} = U\hat{a}U^{+}$. What we wish to remark here is that the same results may be obtained in Wigner phase-space language by simply carrying out phase-space canonical transformations.[46] Thus, for example, the WDF for the TCP can be immediately obtained from the WDF for the Glauber coherent states [Eq. (4) of Ref. 8] by letting

$$q' \to q_b \equiv Aq' + Bp' \tag{9a}$$

$$p' \to p_b \equiv Cq' + Dp' \tag{9b}$$

with the coefficients chosen so that $AD - BC = 1$ and also so that q_b, p_b are related to the complex eigenvalues of $\hat{b}$ in the same way that q', p' are related to the complex eigenvalues of $\hat{a}$.

As a final example, we refer to some recent work which we carried out,[49] in which we demonstrated that the new interpretation given by Aharonov, Albert, and Au[50] to the scalar product of two states in Hilbert space—with an attendant suggestion of a generalized notion of measurement—is essentially equivalent to Wigner's phase-space representation of quantum mechanics.

To summarize, we have presented a *smörgåsbord* of topics dealing with the WDF which should give a flavor of its pervasive influence in the realm of theoretical physics. Since I feel that its greatest impact is probably in the future, I hope that I will be able to write about this subject twenty years hence and join with Eugene Wigner in the celebration of his hundredth birthday. For now, I would like to acknowledge my debt to him for teaching me about distribution functions, among other topics, and conclude by saying: Happy Eightieth Birthday, Eugene.

ACKNOWLEDGMENT

This research was partially supported by the U. S. Department of Energy, Division of Materials Science, under Contract No. DE-AS05-79ER10459.

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