Foundations of Physics, Vol. 15, No. 4, 1985

Logarithmic Ambiguities in the Description of Spatial Infinity[1]

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Received October 3, 1984

Logarithmic ambiguities in the choice of asymptotically Cartesian coordinates at spatial infinity are discussed. It is shown that they do not affect the definitions of energy-momentum and angular momentum at i°. Thus, from a physical viewpoint, the ambiguities are "pure gauge." A prescription is given for fixed this gauge freedom for the class of space-times in which the leading-order part of the Weyl tensor satisfies a certain reflection symmetry. This class admits, in all (relatively boosted) rest frames at infinity, a one-parameter family of asymptotically distinct 3-surfaces (generalized 3-planes) on which the trace of the extrinsic curvature falls off faster than usual.

1. INTRODUCTION

As early as 1961, Peter Bergmann[(1)] pointed out that there exist logarithmic ambiguities in the choice of the asymptotically Cartesian coordinates (or, alternatively, in the choice of the flat metric to which the physical one is supposed to approach) at spatial infinity. These arise as follows. Let us call a 4-metric g_{ab} asymptotically flat at spatial infinity if there exists a flat metric η_{ab}, with its Cartesian chart $x^{\mathbf{a}}$ ($\mathbf{a} = 1, 2, 3, 4$) such that[3] the components $g^{\mathbf{ab}}$ of g_{ab} in this chart are of the form

$$g_{\mathbf{ab}} = \eta_{\mathbf{ab}} + R_{\mathbf{ab}} \tag{1}$$

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[3] Our conventions are the following. Unless otherwise stated, all manifolds and fields are assumed to be C^∞ for simplicity. Space-time metrics have signature $-+++$. Latin indices $(a, b,...)$ are abstract space-time indices à la Penrose. Bold-faced indices $(\mathbf{a}, \mathbf{b},...)$ denote components of tensor fields in a chart.

0015-9018/85/0400-0419$04.50/0

where the remainder R_{ab} is $O(1/\rho)$, where $\rho^2 = \eta_{ab} x^a x^b$ (i.e., limit of ρR_{ab} exists as ρ^2 tends to $+\infty$). Now, given such a chart x^a, one can introduce another chart $\bar{x}^a$ by

$$x^a = \bar{x}^a + C^a \ln \bar{\rho} \tag{2}$$

where C^a are constants. It is straightforward to verify that g_{ab} satisfies the above asymptotic flatness conditions w.r.t. x^a if and only if it does so w.r.t. $\bar{x}^a$. Thus, there is a four-parameter family of logarithmic ambiguities in the definition of asymptotically Cartesian charts (or, equivalently, flat metrics η_{ab}). These are in addition to, and worse than, the so-called supertranslation ambiguities

$$x^a \to x^a + f^a(\theta, \varphi, \chi) \tag{3}$$

where θ, φ, and χ are the hyperbolic angular coordinates on surfaces $\rho = \text{const}$ in the asymptotic regime. While the supertranslations received some attention in the literature in the early sixties, the logarithmic translations (2) seem to have been pretty much ignored until recently. This is not because their role was well understood. Rather, my feeling is that Peter was simply ahead of this time: Tools needed for a systematic analysis of these ambiguities were not available in the sixties and ignoring them did not seem to lead to any obvious problem.

More recently, Penrose's[(2)] conformal techniques have been used to explore the structure of spatial infinity, which is represented, in this approach, by a boundary point i° (which serves as the vertex of the null cone $\mathscr{I}$ which represents null infinity).[(3,4)] This approach is manifestly 4-dimensional, i.e., does not require the splitting of space-time into space and time. Consequently, several issues such as the structure of the asymptotic symmetry group, the role of supertranslations, and the definition of angular momentum are clarified. However, as was pointed out by Beig and Schmidt,[(5)] the approach is *not* immune to the logarithmic ambiguities. These make their appearance in a new disguise: give a completion at i° of the physical space-time, there exists a 4-parameter family of *inequivalent*, logarithmically related completions[4]! At first sight, this would appear to be a disaster, for one would expect inequivalent completions to yield inequivalent answers for physical quantities such as the 4-momentum and angular momentum. It turns out, however, that this is *not* the case. Logarithmic ambiguities do not affect any of the asymptotic physical fields;

[4] Actually, the asymptotic conditions used in Ref. 5 are somewhat stronger than those of Ref. 4, even when restricted to first order. Consequently, contrary to the conjecture in Ref. 5, the ambiguities at i° persist even in the absence of a Killing field.

only certain potentials are reshuffled. Thus, the ambiguities are rather like a gauge freedom. A natural question then is whether there is a canonical choice of gauge, i.e., if one can single out a "logarithmic frame" which is better than the others.

The purpose of this paper is to show that this is possible for a wide class of space-times. This "gauge fixing" procedure is quite similar to the one used for fixing the supertranslation freedom. It turns out that in the resulting preferred completion, the issue of existence of maximal slices simplifies: Among the space-like manifolds which are well behaved at i°, one can select, in any asymptotic rest frame, a one-parameter family of preferred ones whose trace of the extrinsic curvature (w.r.t. the physical metric) falls off faster than $1/\rho^2$. This is the necessary asymptotic ingredient required in Bartnik's[6] theorem on the existence of extremal slices in asymptotically flat space-times.

Section 2 shows how the logarithmic ambiguities arise in the conformal completion at i° and outlines why they do not affect the energy-momentum and angular momentum. Section 3 discusses the effect of the logarithmic translations on the trace of the extrinsic curvature of 3-slices. Section 4 uses the results of the previous two sections to show how one can eliminate the ambiguities in space-times in which the leading-order part of the Weyl curvature satisfies a certain reflection symmetry.

2. LOGARITHMIC TRANSLATIONS

To make the paper self-contained, we first recall the structure available at i°.

Definition 1. A space-time (M, g_{ab}) will be said to be asymptotically flat at spatial infinity if there exists a space-time $(\hat{M}, \hat{g}_{ab})$ which is C^∞ everywhere except at a point i°, where $\hat{M}$ is $C^{>1}$ and $\hat{g}_{ab}$ is $C^{>0}$, together with am imbedding of M into $\hat{M}$, such that:

(i) $\bar{J}(i^\circ) = \hat{M} - M$;

(ii) There exists a function Ω on $\hat{M}$ which is C^2 at i°, C^∞ elsewhere such that, on M, $\hat{g}_{ab} = \Omega^2 g_{ab}$, and, at i°, $\Omega = 0$, $\nabla_a \Omega = 0$ and $\nabla_a \nabla_b \Omega = 2\hat{g}_{ab}$; and

(iii) The Ricci tensor of the physical metric g_{ab} admits regular direction-dependent limit at i°.

The role of these conditions is as follows: (i) implies that i° is space-like related to all points of M; it is the point at spatial infinity; (ii) implies that the conformal factor Ω "falls off as $1/\rho^2$"; while (iii) ensures that the stress-energy of the matter sources "falls off as $1/\rho^4$" in the physical space-time. The awkward differentiability conditions at i° are necessary because, if the total mass of the space-time is nonzero, the conformal curvature of $\hat{g}_{ab}$ diverges there. The condition ensures that the rescaled metric $\hat{g}_{ab}$ is continuous at i° and that its Christoffel symbols admit limits there which, however, can depend on the direction of approach. If translated in the physical space language, this implies that g_{ab} satisfies the asymptotic flatness condition (1). (For details, see Ref. 4 and Appendix 1 in Ref. 7.)

The differentiability conditions imply that the Riemann tensor $\hat{R}_{abcd}$ of $\hat{g}_{ab}$ is such that $\Omega^{1/2}\hat{R}_{abcd}$ admits regular direction-dependent limit, $\underline{R}_{abcd}(\eta)$, where η^a is the unit tangent to the curve of approach to i°. Consider the hyperboloid $\mathscr{D}$ of unit space-like vectors η^a in the tangent space at i°. The "Weyl part" of $\underline{R}_{abcd}$ is coded in two symmetric traceless tensor fields $\underline{E}_{ab}$ and $\underline{B}_{ab}$ on the hyperboloid $\mathscr{D}$. The "Ricci-part" of $\underline{R}_{abcd}$ provides potentials for these electric and magnetic parts of the asymptotic Weyl curvature. The electric part, $\underline{E}_{ab}$, admits a natural scalar potential, $\underline{E}$:

$$\underline{E}_{ab} = (-1/4)(\underline{D}_a \underline{D}_b \underline{E} + \underline{E}\underline{h}_{ab}) \tag{4}$$

while the magnetic part, $\underline{B}_{ab}$, admits a natural tensor potential $\underline{K}_{ab}$:

$$\underline{B}_{ab} = (-1/4)(\underline{\varepsilon}_{mnb} \underline{D}^m \underline{K}^n_a) \tag{5}$$

where $\underline{h}_{ab}$, $\underline{D}$, abd $\underline{\varepsilon}_{abc}$ are, respectively, the natural metric, the derivative operator, and the alternating tensor on $\mathscr{D}$. The 4-momentum is a vector at i° constructed from $\underline{E}_{ab}$. To define angular momentum, one has to eliminate the supertranslation freedom. This can be achieved if $\underline{B}_{ab}$ vanished on $\mathscr{D}$. Then, one can set the potential $\underline{K}_{ab}$ equal to zero, a condition that is preserved by translations but by no other supertranslations. The asymptotic symmetry group then reduces to the Poincaré group, and the angular momentum can be defined unambiguously. The condition $\underline{B}_{ab} = 0$ is satisfied by a large class of space-times. It turns out to play a key role also in the construction of asymptotic twistor spaces at spatial infinity. (For details, see Refs. 4 and 8.)

Let us now turn to the logarithmic ambiguities. Let (M, g_{ab}) satisfy Definition 1. Let $\hat{x}^a$ be a chart near i° which defines the $C^{>1}$ structure of $\hat{M}$ there. Thus, the components, $\hat{g}_{\mathbf{ab}}$ of $\hat{g}_{ab}$ in this chart are $C^{>0}$ functions at i°. We shall now construct a new conformal completion (actually, a four-parameter family of them) of (M, g_{ab}). Define a chart $\bar{x}^a$ near i° via

$$\hat{x}^{\mathbf{a}} = \bar{x}^{\mathbf{a}}(1 + 2\bar{x}_{\mathbf{b}} C^{\mathbf{b}} \ln \bar{\rho}) - \bar{\rho}^2 C^{\mathbf{a}} \ln \bar{\rho} \tag{6}$$

where, as before, $\bar{\rho}^2 = \bar{\eta}_{\mathbf{ab}} \bar{x}^{\mathbf{a}} \bar{x}^{\mathbf{b}} = -\bar{x}_0^2 + \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2$, and $C^{\mathbf{a}}$ are arbitrary constants. It is easy to verify that $\partial \bar{x}^{\mathbf{a}} / \partial \bar{x}^{\mathbf{b}}$ admits a continuous limit to i°. Thus, both charts define the same C^1 structure there. Now, if $\partial^2 \hat{x}^{\mathbf{a}} / \partial \bar{x}^{\mathbf{b}}\, \partial \bar{x}^{\mathbf{c}}$ admitted a regular direction-dependent limit at i°, both charts would endow the same $C^{>1}$ manifold structure there. However, the limit *diverges* logarithmically, whence $\hat{M}$ and $\bar{M}$ are distinct as $C^{>1}$ manifolds near i°. As a particular consequence, the Christoffel symbols of $\hat{g}_{ab}$ in the $\bar{x}$-chart diverge logarithmically. Consider, however, the metric

$$\bar{g}_{ab} = \bar{\Omega}^2 g_{ab} = \omega^2 \hat{g}_{ab}$$

with (7)

$$\omega = 1 - 2\hat{x}_{\mathbf{a}} C^{\mathbf{a}} \ln \hat{\rho}$$

It is easy to verify that g_{ab} *is* $C^{>0}$ in the x-chart. This comes about because the relative conformal factor, ω, is C^0 at i° but not $C^{>0}$ there and just compensates for the pathological behavior of $\hat{g}_{ab}$ in the $\bar{x}$-chart. Thus, if $(\hat{M}, \hat{g}_{ab})$ is a completion satisfying the definition, so is $(\bar{M}, \bar{g}_{ab})$. The metric $\hat{g}_{ab}$ is $C^{>0}$ on the $\hat{x}$-chart but not in the $\bar{x}$-chart, while $\bar{g}_{ab}$ is $C^{>0}$ in the $\bar{x}$-chart but not in the $\hat{x}$-chart.[5]

Let us now consider the two completions $(\hat{M}, \hat{g}_{ab})$ and $(\bar{M}, \bar{g}_{ab})$. One can naturally identify the point i° of $\hat{M}$ with that of $\bar{M}$. Since $\hat{M}$ and $\bar{M}$ are identical as C^1-manifolds, they have the same tangent space at i°. Hence, it is meaningful to compare the limiting values of various fields in $\hat{M}$ and $\bar{M}$. An explicit calculation yields

$$\begin{aligned} \bar{\underline{E}}_{ab} &= \underline{E}_{ab}, \qquad & \bar{\underline{B}}_{ab} &= \underline{B}_{ab} \\ \bar{\underline{E}} &= \underline{E} + 4\underline{C}^a \underline{\eta}_a, \qquad & \bar{\underline{K}}_{ab} &= \underline{K}_{ab} \end{aligned} \tag{8}$$

(Here, $\underline{C}^a$ is the vector at i° with components $C^{\mathbf{a}}$ (in either charts) and $\underline{\eta}^a$ is the position vector, in the tangent space of i°, of points on $\mathscr{D}$. Thus $\underline{C}^a \underline{\eta}_a$ is a function on $\mathscr{D}$.) As a consequence of these transformation properties, it follow that: (i) the 4-momentum vector at i° obtained using $(\hat{M}, g_{ab})$ is the same as that obtained using $(\bar{M}, \bar{g}_{ab})$; (ii) if $\underline{B}_{ab}$ vanishes on $\mathscr{D}$ in one completion, it does so in the other; and (iii) If $\underline{K}_{ab}$ vanishes on $\mathscr{D}$ in one completion, it does so in the other so that the Poincaré groups selected in the two cases are identical. Finally, by using the conformal invariance of the Weyl tensor in M, it is easy to show that the two completion yield the same angular momentum.

Thus, the logarithmic ambiguities lead us to a curious situation. Given one conformal completion satisfying the asymptotic flatness conditions at

[5] The existence of such inequivalent completions in the case of Minkowski space was first pointed out to me by R. Beig (private Communication, 1981).

i°, one can obtain a four-parameter family of *inequivalent* ones, which, nonetheless, yield the same physical quantities. The freedom in the choice of one completion over the others is thus a "gauge freedom."

Remarks.

(i) On the face of it, logarithmic translations (6) and (7) appear to be quite complicated. However, they are obtained from the physical space transformations (2) by a standard procedure which enables one to pass back and foth between the physical space description and the conformally completed picture. (The procedure is outlined, for example, in Appendix 1 of Ref. 7 and Appendix 2 of Ref. 8.)

(ii) Beig and Schmidt[(5)] have conjectured that the freedom to perform the logarithmic translations exists only if the space-time admits a Killing field, while we find that the freedom always exists. This apparent contradiction arises because Ref. 5 uses stronger boundary conditions. There, the requirement (in the lowest order, $m=1$ in the terminology of Definition 2.1 in Ref. 5) is that the components of the physical metric g_{ab} in an asymptotically Cartesian chart (of the flat metric to which g_{ab} approaches) should behave as

$$g^{\mathbf{ab}} = \eta^{\mathbf{ab}} + \frac{1^{\mathbf{ab}}(\theta, \varphi, \chi)}{\rho} + {}^{2}f_{ab} \tag{9}$$

where $l^{\mathbf{ab}}$ are smooth functions of (hyperboloid) angles and where $|{}^{2}f^{\mathbf{ab}}| \leqslant (\text{constant})/\rho^{2}$. For a generic metric satisfying (9), a logarithmic translation changes ${}^{2}f_{ab}$ by terms of the type $(\ln \rho)/\rho^{2}$, thereby violating the requirement on $|{}^{2}f_{ab}|$. The $C^{>0}$ differentiability, on the other hand, requires only $\lim \rho\, {}^{2}f_{ab} = 0$ [or that ${}^{2}f_{ab}$ be $o(1/\rho)$ rather than $O(1/\rho^{2})$], and this requirement *is* preserved by logarithmic translations. (iii) Note that the coefficients $C^{\mathbf{a}}$ in the logarithmic translations *have to be constants*; they cannot be functions of the angular variables. (If they are, $\bar{g}_{ab}$ will not be $C^{>0}$ in the $\bar{x}$-chart, or, alternatively, condition (1) will not be preserved.) Thus, the logarithmic translations are complementary to "pure supertranslations" (3) where the coefficients are angle dependent. Indeed, under supertranslations (3), the fields on $\mathscr{D}$ transform in just the way which is complimentary to (8):

$$\underline{E}_{ab} \to \underline{E}_{ab}; \qquad \underline{B}_{ab} \to \underline{B}_{ab}$$
$$\underline{E} \to \underline{E}; \qquad \underline{K}_{ab} \to \underline{K}_{ab} + \underline{D}_{a}\underline{D}_{b}\underline{f} + \underline{f}\underline{h}_{ab}$$

so that $\underline{K}_{ab}$ is unchanged if and only if the coefficients $\underline{f}$ are of the type $\underline{f} = \underline{C}^{a}\underline{\eta}_{a}$ on $\mathscr{D}$.

3. TRACE OF THE EXTRINSIC CURVATURE

Logarithmic translations are intimately intertwined with the fall-off properties of the trace of the extrinsic curvature of asymptotically flat 3-surfaces. This issue has been well understood in the 3 + 1 frameworks, and the situation is summarized in this section.

Fix a 3-dimensional space-like subspace ${}^3T$ of the tangent space at i° and consider space-like submanifolds $\hat{\Sigma}$ of $\hat{M}$ which are $C^{>1}$ at i° and tangential to ${}^3T$ there. All these $\hat{\Sigma}$ have the same metric $\hat{q}_{ab}$ at i°; they agree to first order asymptotically. However, they can differ to second order, and this difference shows up in the limiting value of the extrinsic curvature $\hat{\Pi}_{ab}$ induced on $\hat{\Sigma}$ by the rescaled metric $\hat{g}_{ab}$. These values were analysed in some detail in Appendix 1 of Ref. 3, and it was shown that the difference between the limits of $\hat{\Pi}_{ab}$ associated with any two of these surfaces is characterized by a function $\psi(\theta, \varphi)$ on the 2-sphere cross section C_T of $\mathscr{D}$ defined by its intersection with ${}^3T$. Thus, there are "as many" $C^{>1}$ submanifolds $\hat{\Sigma}$, which agree at i° to first order but not to second as there are functions on a 2-sphere. In the physical space language, these submanifolds are related by a time-like supertranslation, $t \to t + \psi(\theta, \varphi)$. [In the i°-framework, supertranslations are represented by functions on the hyperboloid $\mathscr{D}$. The transformation $t \to t + \psi$ corresponds to an i°-supertranslation whose value on C_T is zero and whose normal derivative there is $\psi(\theta, \varphi)$.]

Let us now consider the intrinsic metric q_{ab} and the extrinsic curvature Π_{ab} induced on $\Sigma = \hat{\Sigma} - i^\circ$ by the physical metric g_{ab}. Appendix 1 of Ref. 7 shows that (q_{ab}, Π_{ab}) satisfy the Arnowitt–Deser–Misner[8] asymptotic flatness conditions. In particular, the extrinsic curvature, and hence also its trace, fall off as $1/\rho^2$ on Σ. Thus, one knows that if $\hat{\Sigma}$ is $C^{>1}$ at i°, Π on Σ is such that

$$\underline{K}(\theta, \varphi) := \lim \rho^2 \Pi = \lim \Omega^{-1} \Pi \tag{10}$$

is a smooth function on C_T. What is the effect of the supertranslation on the value of $\underline{K}(\theta, \varphi)$? A simple calculation either in the conformal picture (see, e.g., Appendix 2 of Ref. 10), or, more directly, in the physical space picture yields

$$\underline{K}(\theta, \varphi) \to \underline{K}'(\theta, \varphi) = \underline{K}(\theta, \varphi) + {}^2D^2\psi \tag{11}$$

where ${}^2D^2$ is the Laplacian on the 2-sphere cross-section C_T of $\mathscr{D}$. Consequently, if $\underline{K}(\theta, \varphi)$ "has no $l = 0$ part," i.e., if $\oint \underline{K}\, d^2S = 0$, by making a suitable supertranslation, one can find a new surface $\hat{\Sigma}'$, also $C^{>1}$ at i° and tangential to ${}^3T$ there, for which $\underline{K}'(\theta, \varphi)$ vanishes. Thus, on Σ', the trace

Π' of the extrinsic curvature would fall faster than $1/\rho^2$. Furthermore, since the value of $\underline{K}(\theta, \varphi)$ is unaffected by a translation, $t \to t +$ const, there would exist precisely a one-parameter family of asymptotically distinct Σ' on which Π' falls faster than $1/\rho^2$.

What happens, however, if $\oint \underline{K}(\theta, \varphi)\, d^2S$ does not vanish? Then, in the completion $(\hat{M}, \hat{g}_{ab})$, one cannot find *any* $\hat{\Sigma}'$ which is $C^{>1}$ at i°, tangential to ^3T there, on which Π' falls faster than $1/\rho^2$. However, it has long been noted (see, e.g., Ref. 11) that, in the physical space, one can make a logarithmic transformation,

$$t \to t + C \ln \rho = \bar{t} \tag{12}$$

say, to get rid of the $l = 0$ part of $\underline{K}(\theta, \varphi)$. For, under (12), we have

$$\underline{K}(\theta, \varphi) \to \underline{K}(\theta, \rho) + C \tag{13}$$

Thus, if Σ is defined by $t = \text{const}$, one can find a logarithmically translated $\bar{\Sigma}$ (defined by $\bar{t} = \text{const}$) for which $\bar{K}(\theta, \varphi)$ will have no $l = 0$ part. Then, by a subsequent supertranslation of $\bar{\Sigma}$ we can obtain a one-parameter family of surfaces on which $\bar{K}(\theta, \varphi)$ vanishes identically.

The problem, from the i° viewpoint, is that the logarithmically translated surface $\bar{\Sigma}$ is not regular (i.e., $C^{>1}$) at i°. From the $3+1$, physical space viewpoint, this failure of $\hat{\Sigma}$ is irrelevant. However, the problem now is that it is not clear that one can simultaneously make the required logarithmic translations in all (relatively boosted) rest frames, i.e., for all choices of ^3T at i°. A logarithmic translation needed to make $\oint \underline{K}(\theta, \varphi)\, d^2S$ vanish in one frame may destroy the vanishing of this quantity in another. Indeed, it appears that for the class of space-times considered so far, one would not be able to make $\oint \underline{K}(\theta, \varphi)\, d^2S$ vanish for all ^3T. An additional restriction on the class of space-times seems to be necessary. The i° framework, being 4-dimensional, is ideally suited to analyze this issue, for it is easy to treat boosts in this framework. The next section is devoted to this analysis.

Remarks.

(i) Let (M, g_{ab}) be a Minkowski space and let $(\hat{M}, \hat{g}_{ab})$ be a completion in which $\hat{g}_{ab}$ is again flat in a neighborhood of i°. Then, given any $C^{>1}$ submanifold $\hat{\Sigma}$ of $\hat{M}$, one does have the equality $\oint \underline{K}(\theta, \varphi)\, d^2S = 0$. Hence, by a suitable supertranslation, one can find a $\hat{\Sigma}$, regular at i°, such that, on $\Sigma' = \hat{\Sigma}' - i^\circ$, the trace Π of the extrinsic curvature falls off faster than $1/\rho^2$. These surfaces are, to both first and second asymptotic orders, indistinguishable from 3-planes in Minkowski space.

(ii) In a general space-time satisfying certain interior regularity conditions, Bartnik[6] has shown that, given a Σ on which the trace of the extrinsic curvature falls as $1/\rho^3$, there exists an asymptotically parallel surface on which the trace vanishes identically, i.e., which is extremal. Thus, the surfaces on which the stronger fall-off is satisfied are indistringuishable, to first and second order, from extremal surfaces. This generalizes the situation in Minkowski space discussed above.

(iii) Translations, $t \to t + \text{const}$, leave the asymptotic values, $\underline{K}(\theta, \varphi)$ of $\rho^2 \Pi$ invariant. Consequently, $\underline{K}(\theta, \varphi)$ is not a good label to distinguish surfaces which agree to first order but not to second; there exists a one-parameter family of such surfaces all of which have the same value of $\underline{K}(\theta, \varphi)$. On the other hand, at i°, the trace $\underline{\hat{\Pi}}$ of the extrinsic curvature $\underline{\hat{\Pi}}_{ab}$ of these surfaces w.r.t. $\hat{g}_{ab}$ transforms, under supertranslations $t \to t + \psi(\theta, \varphi)$, by

$$\underline{\hat{\Pi}} \to \underline{\hat{\Pi}} + (1/2)\, {}^2D^2\psi + 3\psi$$

Since the operator $({}^2D^2 + 6)$ *is* invertible on a 2-sphere, it follows that $\underline{\hat{\Pi}}$ is a good label for distinguishing surfaces which are all tangential to a given ${}^3T$ at i° but which differ to second order.

4. ELIMINATION OF LOGARITHMIC AMBIGUITIES

To treat the issue of vanishing of $\oint \underline{K}(\theta, \varphi)\, d^2S$ in all (relatively boosted) rest frames simultaneously, one can attempt to express this quantity using 4-dimensional fields. This is indeed possible. A somewhat long but straightforward calculation yields, on setting the potential $\underline{K}_{ab}$ equal to zero on $\mathscr{D}$,

$$\oint_{C_T} \underline{K}(\theta, \varphi)\, d^2S = (-1/4) \oint_{C_T} \underline{t}^a \underline{D}_a \underline{E}\, d^2S \tag{14}$$

where $\underline{t}^a$ is the unit (time-like) normal, within $\mathscr{D}$ to the 2-sphere cross-section C_T. The usefulness of (14) lies in the fact that it holds for any cross-section of $\mathscr{D}$ obtained by its intersection with a space-like 3-plane in ${}^3T$ in the tangent space at i°, passing through i°. Such cross-sections will be called *planer*. Thus, (14) enables one to express the $l = 0$ part of the limiting value of $\rho^2 \Pi$ on *any* space-like 3-surface $\hat{\Sigma}$ which is $C^{>1}$ at i° in terms of the integral of the normal derivative of $\underline{E}$ within $\mathscr{D}$, on the 2-sphere cross-section of $\mathscr{D}$ defined by $\hat{\Sigma}$. $\underline{E}$ is, of course, a function on $\mathscr{D}$, fixed by the choice of the conformal completion, independently of the choice of any cross-section. The question now is whether one can make $\oint \underline{t}^a \underline{D}_a \underline{E}\, d^2S$ vanish on *every* planer section, $\underline{t}^a$ being the unit normal to the section under consideration. Since $\underline{E}$ transforms under a logarithmic translation by (8), the answer depends not only the choice of the physical space-time but also on that of its completion.

Let (M, g_{ab}) be such that its Weyl tensor $C_{abc}{}^{d}$ satisfies, at i°, the following *reflection symmetry*:

$$\underline{C}_{abc}{}^{d}(\eta) = \underline{C}_{abc}{}^{d}(-\eta) \tag{15}$$

where $\underline{C}_{abc}{}^{d}(\eta)$ is the limit to i° of $\Omega^{1/2} C_{abc}{}^{d}$ along a curve whose unit tangent at i° is η. This property holds in one completion if it holds in any logarithmically (or supertranslationally) related one. We shall now show that such a space-time admits a completion in which $\oint \underline{t}^a \underline{D}_a \underline{E}\, d^2S$ vanishes on *any* planer section and that this requirement exhausts the logarithmic freedom completely.

Equation (15) implies that $\underline{E}_{ab}$ is also reflection symmetric on $\mathscr{D}$: $\underline{E}ab(\eta) = \underline{E}_{ab}(-\eta)$. Now, the mapping $\eta^a \to -\eta^a$ is an isometry of $(\mathscr{D}, \underline{h}_{ab})$, whence it preserves the derivative operator $\underline{D}$. Since by (4),

$$\underline{E}_{ab}(\eta) = \underline{D}_a \underline{D}_b E(\eta) + \underline{E}(\eta) \underline{h}_{ab}(\eta)$$

applying the mapping $\eta^a \to -\eta^a$, we have

$$\underline{E}_{ab}(\eta) = \underline{D}_a \underline{D}_b \underline{E}(-\eta) + \underline{E}(-\eta)\, \underline{h}_{ab}(\eta)$$

whence it follows that

$$\underline{D}_a \underline{D}_b(\underline{E}(\eta) - \underline{E}(-\eta)) + (\underline{E}(\eta) - \underline{E}(-\eta))\underline{h}_{ab} = 0 \tag{16}$$

Now, it is known[(4)] that every solution to this equation on $\mathscr{D}$ is of the form

$$\underline{E}(\eta) - \underline{E}(-\eta) = \underline{K}^a \eta_a \tag{11}$$

for some fixed vector $\underline{K}^a$ at i°. Let us now make a logarithmic transformation (6), (7), i.e., choose a new completion $(\bar{M}, \bar{g}_{ab})$. Then, by (8), we have

$$\bar{\underline{E}}(\eta) = \underline{E}(\eta) + 4\underline{C}^a \eta_a$$

Hence, if we choose $\underline{C}^a = (1/8)\underline{K}^a$, we find that $\bar{\underline{E}}(\eta)$ is reflection invariant:

$$\begin{aligned} \bar{\underline{E}}(\eta) - \bar{\underline{E}}(-\eta) &= (\underline{E}(\eta) + 4\underline{C}^a \eta_a) - (E(-\eta) - 4\underline{C}^a \eta_a) \\ &= \underline{K}^a \eta_a - 8\underline{C}^a \eta_a = 0 \end{aligned} \tag{18}$$

It is obvious that any further logarithmic translation will destroy the reflection symmetry of $\underline{E}$. Thus, if $\underline{C}_{abc}{}^{d}(\eta) = \underline{C}_{abc}{}^{d}(-\eta)$, there exists a unique "logarithmic frame" in which $\underline{E}$ itself is reflection invariant.

Let us now consider the integral $\oint \underline{t}^a \underline{D}_a E \, d^2S$ on planer sections. Under the reflection, $\eta^a \to -\eta^a$, each planer section is mapped to itself, $\underline{t}^a$ is mapped to $-\underline{t}^a$, and $\underline{D}_a \underline{E}(\eta)$ is mapped to $\underline{D}_a \underline{E}(-\eta)$. Hence, in the preferred completion in which $\underline{E}(\eta)$ is reflection invariant, $\underline{t}^a \underline{D}_a \underline{E}$ simply changes sign under reflection, whence $\oint \underline{t}^a \underline{D}_a \underline{E} \, d^2S$ vanishes on any planer section of $\mathscr{D}$.

From (14) it now follows that, in this completion, the limiting value of $\rho^2 \Pi$ on *any* $C^{>1}$ submanifold $\hat{\Sigma}$ passing through i° has no $l=0$ part. Hence, by a suitable supertranslation, one can find a one-parameter family of $C^{>1}$ submanifolds Σ' on which $K'(\theta, \varphi) = \lim \rho^2 \Pi'$ vanishes. These surfaces serve as the starting point in Bartnik's[(6)] analysis of extremal slices.[6]

Remarks.

(i) Note that the requirement of reflection symmetry of $\underline{C}_{abc}{}^d$ is a property of the physical space-time and not of a specific completion thereof. The requirement does *not* demand that the Weyl tensor itself be reflection symmetric; only the leading part—the coefficient of $1/\rho^3$-which contains only the 4-momentum information is required to be reflection symmetric.

(ii) Let (M, g_{ab}) be stationary and satisfy Definition 1 of Section 2. Then, it follows from results in Ref. 12 that $\underline{B}_{ab}$ is zero on $\mathscr{D}$ and that $\underline{E}_{ab}$ is reflection symmetric there. Thus, stationary space-times automatically satisfy the reflection symmetry of $\underline{C}_{abc}{}^d$.

(iii) We have only shown that the symmetry requirement is sufficient for $\oint \underline{t}^a \underline{D}_a \underline{E} \, d^2S$ to vanish on any planer section. Whether the condition is necessary is unclear. The assumption that the integral should vanish on any planer section leads to an infinite number of integral constraint equations on $\underline{E}$. If one could show that these, together with the field equation, $\underline{D}^a \underline{D}_a \underline{E} + 3\underline{E} = 0$, satisfied by $\underline{E}$ on $\mathscr{D}$, imply that $\underline{E}$ must be reflection symmetric, the necessity would be established since the reflection symmetry of $\underline{E}$ implies that of $\underline{E}_{ab}$.

(iv) Fix any planer section C_T of $\mathscr{D}$. Then, it is trivial to verify that by making a suitable logarithmic translation (8), one can remove the $l=0$ part of $r^a D_a E$ on this cross-section *even when the Weyl tensor does not satisfy the reflection symmetry*. This is in agreement with the results in Ref. 11 quoted in Section 3. However, if the Weyl tensor $\underline{C}_{abc}{}^d$ at i° is not reflection symmetric, the 4-parameter freedom in the logarithmic trans-

[6] Some of these results were briefly reported at the Oregon conference (Ref. 10) in October 1983. However, the discussion there was too sketchy and the reflection symmetry conditions were not mentioned.

lations does not appear to suffice to make $\oint \underset{\sim}{t}^a \underset{\sim}{D}_a \underset{\sim}{E}\, d^2S$ vanish on *every* planar section.

(v) In the physical space language, the results of our analysis may be summarized as follows. The assumption (1) of asymptotic flatness always permits logarithmic translations. One may choose the asymptotic coordinates suitably[6] to express the metric as

$$g_{ab}\, dx^a\, dx^b = \left(1 + \frac{\sigma}{\rho}\right)^2 d\rho^2 + \rho^2 \left(h_{ij} + \frac{1}{\rho}\, h'_{ij}\right) d\Phi^i\, d\Phi^j + o\left(\frac{1}{\rho}\right)$$

where Φ^i, $i = 1, 2, 3$, are the angular coordinates on the $\rho = \text{const}$ surfaces in the asymptotic regime, σ, h_{ij}, and h'_{ij} are smooth functions of Φ^i, and h_{ij} are the components of the unit hyperboloid metric in the Φ^i chart. Then, provided the magnetic part of the asymptotic Weyl curvature vanishes, the reflection symmetry of $\underset{\sim}{C}_{abc}{}^d$ implies that one can always require that σ be reflection symmetric. This requirement eliminates the freedom of making the logarithmic translations (2). Let us now suppose that we have chosen the asymptotic coordinates such that σ is reflection symmetric. Then, given any $t = \text{const}$ surface, the limit of $\rho^2 \Pi$ on that surface has no $l = 0$ part. Consequently, given any asymptotic rest frame, one can find 3-surfaces which are asymptotically well behaved in the Cartesian chart (i.e., are supertranslates of planes in the chart) on which the trace Π of the extrinsic curvature falls faster than $1/\rho^2$.

ACKNOWLEDGMENT

This work is partially supported by NSF grant No. PHY83-10041. It is a pleasure to thank Robert Bartnik, Robert Beig, and Bernd Schmidt for discussion.

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