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# Partial Order Behaviour and Structure of Petri Nets[1]

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**Abstract.** This paper argues that partial order semantics can be used profitably in the proofs of some nontrivial results in Petri net theory. We show that most of Commoner's and Hack's structure theory of free choice nets can be phrased and proved in terms of partial order behaviour. The new proofs are considerably shorter (and, arguably, more lucid) than the old ones; they also generalise the results from (safe) free choice nets to (bounded) extended free choice nets.

## 1. Introduction

Partial orders are increasingly found useful to describe the behaviour of concurrent systems [BeF88, Pra86]. However, partial orders have not yet been found of much use in proving behavioural properties of concurrent systems; rather, the prevalent way of proving such properties is still by means of arbitrarily interleaved execution sequences.

This paper argues that for a particular concurrent systems model – Petri nets – partial orders may very well be used for proofs. We intend to show that some proofs can be shortened by an order of magnitude. To make this point some of the more intricate results in Petri net theory have been selected:

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Commoner's and Hack's results on free choice nets, e.g. the liveness results, the duality results and the coverability results [Hac72, Com72]. We will give short proofs of these results, at the same time generalising them and eliminating some errors in previous publications (see, e.g., [Hac74, Döp83]). Furthermore some new results are presented.

At first sight partial order semantics seem to be more complicated than interleaved sequential semantics. In fact, some expense to define notations relating to partially ordered processes is necessary. Furthermore it is known that, in some sense, partial order semantics and arbitrary interleaving semantics are equivalent (see e.g. [BeD87]). Hence the claims made above might be surprising.

The following example may help motivating the potential profitability of using partial orders in behavioural analysis and proofs. Consider a system consisting of a producer, a consumer and a potentially unbounded buffer. Further, consider the problem of proving its unboundedness.

Each sequential run of this system determines a sequence of states of the buffer. Each such state can be described as an integer which is equal to the number of items currently inside the buffer. For instance, strict alternation of producing an item and consuming an item yields a state sequence 010101. . . , while for other sequential runs the numbers may grow arbitrarily. Reasoning about the behaviour of the buffer therefore requires the study of all such sequential runs.

A partial order behaviour of the same system takes only the actual dependencies into account. In particular, the existence of some item inside the buffer depends on its having been produced by the producer and, in turn causes its eventual consumption (if it is ever consumed). Since there are no assumptions about any additional dependencies between producer and consumer, different items inside the buffer are not ordered. Hence they can all coexist. This directly leads to the desired result without the need to consider an exploding number of sequential runs.

More formally and more generally we can prove, using the same method, that a system – given as a Petri net – which is live, bounded and connected is also strongly connected. Hence, since the producer/consumer system is not strongly connected, it is not bounded.

The paper is organised as follows. In Section 2 we define the necessary basic notions consistently by means of partial orders, and we give the proof of some elementary results using partial orders. In Section 3 we prove Commoner's liveness criterion using a basic technical lemma on partial orders. In Section 4 we turn to the coverability of extended free choice systems by certain structures, originally due to Hack. A result concerning interrelations between certain structures and system runs is presented, which has nice consequences for the analysis of mutual dependencies between system events. We show in Section 5 how our proof method can be applied to generalise Hack's duality result on special free choice nets. As a consequence we get the coverability by structures dual to the ones mentioned above. Section 6 presents a result conjectured by Thiagarajan and Merceron [LuM84]. Setion 7 gives conclusions.

Recently, there has been considerable renewed interest in the theory of free choice nets (see for example [ThV84, BeT87, EsS89]). We do not claim novelty for most of the results in the present paper, but we do claim some originality for the organisation and the style of their proofs.

## 2. Basic Concepts and General Results

### 2.1. Elementary Definitions

**Definition 2.1.** *Nets.* A net is a triple $N=(S, T, F)$ with $S \cap T = \emptyset$ and $F \subseteq ((S \times T) \cup (T \times S))$. $S$ is the set of places, $T$ is the set of transitions and $F$ is the interconnection relation between them. $N$ is finite iff $|S \cup T| \in \mathbf{N} = \{0, 1, 2, \ldots\}$.

For $x \in S \cup T$, the pre-set ${}^\bullet x$ is defined as ${}^\bullet x = \{y \in S \cup T \mid (y, x) \in F\}$ and the post-set $x^\bullet$ is defined as $x^\bullet = \{y \in S \cup T \mid (x, y) \in F\}$.

For $X \subseteq S \cup T$, ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$ and $X^\bullet = \bigcup_{x \in X} x^\bullet$.

We will require $T \subseteq dom(F) \cap cod(F)$, i.e., for all transitions $t$ neither the pre-set nor the post-set is empty. (This is no severe restriction; every transition $t$ with empty pre-set or empty post-set can be provided with an appropriately marked side place $s$ with ${}^\bullet s = s^\bullet = \{t\}$ without altering essential system properties.)

$N$ is weakly connected (or just connected) iff $(S \cup T) \times (S \cup T) = (F \cup F^{-1})^*$, $N$ is strongly connective iff $(S \cup T) \times (S \cup T) = F^*$ (where $\rho^*$ denotes the reflexive and transitive closure of a relation $\rho$).

For each set of transitions $T_1 \subseteq T$ the net

$$N_1 = ({}^\bullet T_1 \cup T_1^\bullet, T_1, F \cap ((S \times T_1) \cup (T_1 \times S)))$$

is called the subnet generated by $T_1$.

Nets are used as models of concurrent systems in the following way: places represent local states, transitions represent actions, and a transition $t$ has the set ${}^\bullet t$ of inputs and the set $t^\bullet$ of outputs. The states of a system are described by markings:

**Definition 2.2.** *Markings.* A marking of a net $N=(S, T, F)$ is defined as a function $M: S \to \mathbf{N}$.

For $s \in S$, $M(s)$ denotes the number of tokens on $s$.

For $X \subseteq S$ we define $M(X) = \sum_{s \in X} M(s)$.

$X$ is called unmarked at $M$ iff $M(X) = 0$.

A marked net $\Sigma = (S, T, F, M_0)$ is a net $N=(S, T, F)$ with an initial marking $M_0$. We transfer the properties of $N$ to $\Sigma$. Thus $\Sigma$ will be called finite (weakly connected, etc.) iff $N$ is finite (weakly connected, etc.).

The (partial order) behaviour of systems is described by processes [GoR83]. Processes are based on a special class of nets called occurrence nets:

**Definition 2.3.** *Occurrence nets.* An occurrence net $N=(B, E, K)$ is an acyclic net without branched places, i.e., $K^+ \cap (K^{-1})^+ = \emptyset$ (acyclicity) and $\forall b \in B: |{}^\bullet b| \leq 1 \wedge |b^\bullet| \leq 1$ (no branching of places), where $\rho^+$ denotes the irreflexive transitive closure of a relation $\rho$. Elements of $E$ are called events and elements of $B$ are called conditions; $K$ is called causal dependency relation.

A subset $c \subseteq (B \cup E)$ of an occurrence net $N=(B, E, K)$ is called a co-set of $N$ iff any two elements in $c$ are unordered (taking $< = K^+$ as the ordering), i.e., $K^+ \cap (c \times c) = \emptyset$.

A cut $c \subseteq (B \cup E)$ is a maximal co-set. A cut $c$ is called $B$-cut iff $c \subseteq B$.

For an element $x \in (B \cup E)$ of an occurrence net $N=(B, E, K)$, $\downarrow x$ denotes the set of elements $\{y \in B \cup E \mid (y, x) \in K^*\}$, i.e., the set of elements below $x$, and $\uparrow x$ denotes the set of elements $\{y \in B \cup E \mid (x, y) \in K^*\}$, i.e., the set of elements above $x$.

For $X \subseteq B \cup E$, $\downarrow X = \bigcup_{x \in X} \downarrow x$ and $\uparrow X = \bigcup_{x \in X} \uparrow x$. The set $\min(X)$ is defined as $\{x \in X \mid {}^\bullet x \cap X = \varnothing\}$. Similarly, the set $\max(X)$ is defined as $\{x \in X \mid x^\bullet \cap X = \varnothing\}$.

The sets $\min(N)$ and $\max(N)$ are defined as $\min(B \cup E)$ and $\max(B \cup E)$, respectively.

**Definition 2.4.** *Processes.* A process $\pi = (N, p) = (B, E, K, p)$ of a marked net $\Sigma = (S, T, F, M_0)$ consists of an occurrence net $N = (B, E, K)$ together with a labelling $p: B \cup E \to S \cup T$ which satisfy appropriate properties such that $\pi$ can be interpreted as a concurrent run of $\Sigma$. We distinguish initial processes which start with the initial marking $M_0$ and general processes which may start with any successor marking of $M_0$, to be defined below. If a marking $M$ of $(S, T, F)$ and a $B$-cut $c$ of $N$ satisfy $\forall s \in S: M(s) = |p^{-1}(s) \cap c|$ then $M$ is denoted by $M_c$ and is said to correspond to $c$ (where $p^{-1}(s)$ denotes the set of elements of $B$ which are mapped to $s$ by $p$). To be an initial process of $\Sigma$, $\pi$ must satisfy the following properties:

(i) $p(B) \subseteq S$, $p(E) \subseteq T$ (conditions represent the appearances of tokens on places while events represent the occurrences of transitions).

(ii) $\forall x \in B \cup E$: $|\downarrow x| \in \mathbf{N}$ (this implies that $\min(N)$ is a cut).

(iii) $\forall e \in E$:
$p({}^\bullet e) = {}^\bullet p(e)$, $|p({}^\bullet e)| = |{}^\bullet p(e)|$ and $p(e^\bullet) = (p(e))^\bullet$, $|p(e^\bullet)| = |(p(e))^\bullet|$
(transition environments are respected).

(iv) $M_{\min(N)} = M_0$ (i.e., $\min(N)$ corresponds to the initial marking $M_0$. Notice that because of our requirement that $T \subseteq dom(F) \cap cod(F)$ for all nets, $\min(N)$ is a $B$-cut. If $N$ is finite, $\max(N)$ is a $B$-cut, too).

We shall say that a transition (a place, respectively) $x \in S \cup T$ occurs in $\pi$ iff there is $y \in B \cup E$ with $p(y) = x$.

For each finite initial process $\pi = (N, p)$ of $\Sigma$, $M_{\max(N)}$ is called a successor marking of $M_0$. The set of all successor markings of $M_0$ is called the set of reachable markings and is denoted by $[M_0\rangle$. $\pi$ is a process of $\Sigma$ iff $\pi$ is an initial process of $(S, T, F, M)$ for some $M \in [M_0\rangle$.

Two processes $\pi_1 = (N_1, p_1)$ and $\pi_2 = (N_2, p_2)$ of $\Sigma$ are concatenable if $\pi_1$ is finite and $M_{\max(N_1)} = M_{\min(N_2)}$. Then $\pi_1$ and $\pi_2$ can be concatenated in the obvious way (by identifying elements of $\max(N_1)$ and $\min(N_2)$ which are mapped to the same place), resulting in a process $\pi = \pi_1\pi_2$. (Actually, concatenation is not unique, so that a set of processes may result from the concatenation of $\pi_1$ and $\pi_2$; however, we shall always be interested in an arbitrary member of this set, since all members behave similarly with respect to the properties we will be considering. Hence pretending uniqueness, we are allowed to write $\pi = \pi_1\pi_2$.)

A finite process $\pi$ with $\min(\pi) \neq \max(\pi)$ and $M_{\min(\pi)} = M_{\max(\pi)}$ is called a reproduction process.

We transfer properties of $N$ to a process $\pi = (N, p)$, i.e., $\pi$ is finite iff $N$ is finite etc. and we write $\min(\pi)$ ($\max(\pi)$) instead of $\min(N)$ ($\max(N)$, respectively).

Property 2.4(iii) codes the transition rule of Petri nets in terms of processes. For the consistency of the notions defined in 2.4 with the notions usually employed in Petri nets, see [BeD87].

From 2.4(iii) it follows immediately that $(x, y) \in K^*$ implies $(p(x), p(y)) \in F^*$ ($K$, $F$ and $p$ being defined as above).

A transition $t$ is enabled by a marking $M$ (denoted by $M[t\rangle$) iff all places of ${}^\bullet t$ are marked by $M$. The connection of this notion to processes is as follows: if $c$ is a $B$-cut which contains ${}^\bullet e$ for an event $e$, then $M_c$ enables $p(e)$.

Next we define two behavioural properties of systems. Boundedness implies a restriction on (the number of tokens on) places. Liveness implies the absence of partial deadlocks.

**Definition 2.5.** *Boundedness, liveness.* A place $s \in S$ of a marked net $\Sigma = (S, T, F, M_0)$ is $n$-bounded (for $n \in \mathbf{N}$) iff $\forall M \in [M_0\rangle: M(s) \leq n$; $s$ is bounded iff there is a number $n \in \mathbf{N}$ such that $s$ is $n$-bounded.

$\Sigma$ is $n$-bounded iff all $s \in S$ are $n$-bounded. $\Sigma$ is bounded iff all $s \in S$ are bounded. $\Sigma$ is safe iff all $s \in S$ are 1-bounded.

$\Sigma$ is live iff for all $M \in [M_0\rangle$ there is a process $\pi = (B, E, K, p)$ of $\Sigma$ with $M_{\min(\pi)} = M$ such that $p(E) = T$, i.e., all transitions of $T$ occur in $\pi$.

*Basic assumptions:* From now on, whenever we speak of a net $(S, T, F)$ or a marked net $(S, T, F, M_0)$, we will assume that $S \cup T$ is a finite set, that $(S, T, F)$ is (weakly) connected, and that, as already required for transitions, for all places neither the pre-set nor the post-set is empty, i.e., $S \subseteq dom(F) \cap cod(F)$. This implies in particular that there are no isolated places, i.e., places $s$ with ${}^\bullet s = \emptyset = s^\bullet$.

These assumptions are not required to hold for processes $(B, E, K, p)$.

**Definition 2.6.** *Reverse-dual net.* If $N = (S, T, F)$ is a net then the reverse-dual of $N$ is defined as the net $N^{-d} = (T, S, F^{-1})$.

It can easily be verified that $N^{-d}$ is in fact a net. Our assumptions and restrictions on nets amount to weak connectedness and the equality $S \cup T = dom(F) = cod(F)$. They are respected by $N$ if and only if the same is true for $N^{-d}$.

## 2.2. General Results

An important property of processes is that each finite $B$-cut $c$ corresponds to a reachable marking. Here we only sketch the proof; it can be found, for instance, in [BeF88].

**Lemma 2.7.** *Let $\pi = (B, E, K, p)$ be a process of a marked net $\Sigma$.*

(i) *If $c_0 \subseteq B$ is a finite co-set of $\pi$ then $c_0$ can be extended to a finite $B$-cut $c \supseteq c_0$ of $\pi$.*

(ii) *If $c$ is a finite $B$-cut of $\pi$ then $M_c$ is a reachable marking of $\Sigma$.*

*Proof.* (i) Let $c_0 \subseteq B$ be a finite co-set of $\pi$. Define $d = \downarrow({}^\bullet c_0)$. Then $\max(d^\bullet)$ is a co-set containing $c_0$. Define $c = \max(\min(\pi) \cup d^\bullet)$. Then $c$ is a finite $B$-cut containing $c_0$.

(ii) Define $B' = B \cap \downarrow c$, $E' = E \cap \downarrow c'$, $K' = K \cap ((B' \times E') \cup (E' \times B'))$ and $p' = p_{|B' \cup E'}$.

It is easy to verify, using property 2.4(iv), that $\pi' = (B', E', K', p')$ is a finite process of $\Sigma$ with $\min(\pi') = \min(\pi)$ and $\max(\pi') = c$. The proposition follows since $M_{\min(\pi)}$ is a reachable marking and the process leading to $M_{\min(\pi)}$ can be concatenated with $\pi'$.

**Corollary 2.8.** *Let $\Sigma = (S, T, F, M_0)$ be a bounded finite marked net. Then no process $\pi$ of $\Sigma$ contains an infinite B-cut.*

*Proof.* Suppose there is a process of $\Sigma$ containing an infinite $B$-cut $c$. Any finite subset of $c$ can be covered by a reachable marking according to lemma 2.7. Since $c$ contains arbitarily large finite subsets but the set $S$ is finite, the result follows.

The following lemmata state that a bounded system which has the ability to run forever has to be repetitive, in the sense that there are reproduction processes. In the case of liveness there are, furthermore, reproduction processes which use the entire system.

**Lemma 2.9.** *Let $\Sigma = (S, T, F, M_0)$ be a live and bounded marked net. Then there exists a reproduction process $\pi = (B, E, K, p)$ of $\Sigma$ with $p(E) = T$.*

*Proof.* Since $\Sigma$ is live there is an infinite sequence $(\pi_i = (B_i, E_i, K_i, p_i))$ $(i = 0, 1, 2, \ldots)$ of finite processes of $\Sigma$ such that $\forall i \in \mathbf{N}$: $M_{\max(\pi_i)} = M_{\min(\pi_{i+1})} \wedge p_i(E_i) = T$.

Since $\Sigma$ is bounded there is only a finite number of markings, say $m$, reachable in $\Sigma$. So $|\{M_{\min(\pi_0)}, M_{\min(\pi_1)}, \ldots, M_{\min(\pi_m)}\}| \le m$ and we can find indices $i$ and $j$ with $i < j$ such that $M_{\min(\pi_i)} = M_{\min(\pi_j)}$. Then $\pi = \pi_i \cdots \pi_{j-1}$ is a reproduction process of $\Sigma$.

**Lemma 2.10.** *Let $\Sigma = (S, T, F, M_0)$ be a bounded marked net and let $\pi = (B, E, K, p)$ be an infinite process of $\Sigma$. Then there is a reproduction process $\bar{\pi} = (\bar{B}, \bar{E}, \bar{K}, \bar{p})$ of $\Sigma$ with $\bar{p}(\bar{E}) \subseteq p(E)$.*

*Proof.* $\pi$ can be decomposed as an infinite sequence $(\pi_i = (B_i, E_i, K_i, p_i))$, $i = 0, 1, 2, \ldots$ of finite processes of $\Sigma$ such that $\forall i \in \mathbf{N}$: $M_{\max(\pi_i)} = M_{\min(\pi_{i+1})}$ and $\pi = \pi_0 \pi_1 \pi_2 \cdots$

The proof then proceeds as in Lemma 2.9.

The last result of this subsection states that live and bounded marked nets are strongly connected. We split the proof and formulate a lemma first, which will also be used later in a different context. It states that the part of a marked net used by a reproduction process is a collection of strongly connected components.

**Lemma 2.11.** *Let $\pi' = (B', E', K', p')$ be a reproduction process of a bounded market net $\Sigma = (S, T, F, M_0)$. Let $(S', T', F')$ be the subnet of $N$ generated by $p(E')$. Then $F'^{-1} \subseteq F'^*$.*

*Proof.* Two cases can be distinguished:

(i) $(t, s) \in F'^{-1} \cap (T' \times S')$. Let $\pi = (B, E, K, p) = \pi'\pi'\pi' \cdots$ be an infinite concatenation of $\pi'$. Then $t$ occurs infinitely often in $\pi$. For every $e \in p^{-1}(t)$ there is a condition $b \in {}^\bullet e \cap p^{-1}(s)$. Since $\Sigma$ is bounded, $\pi$ contains no infinite $B$-cut (Corollary 2.8), whence we can find $b_1$, $b_2 \in {}^\bullet p^{-1}(t) \cap p^{-1}(s)$ with $(b_1, b_2) \in K^*$. As conditions are not branched we get with $b_1^\bullet = \{e_1\}$: $p(e_1) = t$ and $(e_1, b_2) \in K^*$. Thus, $(p(e_1), p(b_2)) = (t, s) \in F'^*$.

(ii) $(s, t) \in F'^{-1} \cap (S' \times T')$. Analogously.

**Theorem 2.12.** *Let $\Sigma = (N, M_0)$ with $N = (S, T, F)$ be a live and bounded marked net. Then $N$ is strongly connected.*

*Proof.* By Lemma 2.9 there is a reproduction process $\pi = (B, E, K, p)$ of $\Sigma$ with $p(E) = T$. Lemma 2.11 can be applied with $S = S'$, $T = T'$ and $F = F'$. Hence $F^{-1} \subseteq F^*$ and thus $(F \cup F^{-1})^* = F^*$. This achieves the proof, since by our basic assumption, $N$ is weakly connected.

### 2.3. Invariants

**Definition 2.13.** *T-invariants, S-invariants.* Let $N = (S, T, F)$ be a net. A mapping $J: T \to \mathbf{N}$ is called a semipositive T-invariant (or just T-invariant) iff $J(T) \neq \{0\}$ and

$$\forall s \in S: \sum_{t \in {}^\bullet s} J(t) = \sum_{t \in s^\bullet} J(t).$$

The set $T_J = \{t \in T \mid J(T) > 0\}$ is called the support of $J$.

A T-invariant $J$ is minimal iff its support is minimal (there is no T-invariant whose support is a proper subset of $T_J$) and furthermore $J \neq k \cdot J'$ for all T-invariants $J'$ and integers $k > 1$.

A (semipositive) S-invariant $I: S \to \mathbf{N}$ of $N$ is a T-invariant of the net $N^{-d} = (T, S, F^{-1})$. Minimality, support etc. of $I$ are defined as above.

Later we will need to transfer properties of minimal T-invariants to general (semipositive) T-invariants. It is well known (see, for example, [MeR80]) that T-invariants can be composed additively from minimal ones:

**Lemma 2.14.** *Let $N = (S, T, F)$ be a net and let $J: T \to \mathbf{N}$ be a T-invariant of $N$. Then there exist minimal T-invariants $J_1, J_2, \ldots, J_k$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbf{Q}^+$, such that*

$$\forall t \in T: J(t) = \sum_{i=1}^{k} \lambda_i \cdot J_i(t)$$

*where $\mathbf{Q}^+$ denotes the set of nonnegative rational numbers.*

**Lemma 2.15.** *Let $\pi = (N, p)$ be a reproduction process of a marked net $\Sigma = (S, T, F, M_0)$. Then $J: T \to \mathbf{N}$ with $J(t) = |p^{-1}(t)|$ for all $t \in T$ is a T-invariant.*

*Proof.* Obvious since $\pi$ takes as many tokens from each place as it puts there.

## 3. Free Choice Systems and Liveness

For the class of free choice systems liveness, as well as liveness together with safeness, can be characterized by means of structural properties of the underlying nets [Com72, Hac72]. So for these systems, behavioural properties can be proved or disproved without investigating all reachable markings or all processes. The free choice property is structural, i.e., a property of the underlying net. A net is free choice if a certain substructure does not appear in it; in a free choice net there is never an arc from a forward branched place ($s$ with $|s^\bullet| > 1$) to a backward branched transition ($t$ with $|{}^\bullet t| > 1$). Throughout what follows we will define and use the class of extended free choice nets which properly includes the class of free choice nets and shares many of its properties.

**Definition 3.1.** A net $N=(S, T, F)$ is called extended free choice net (EFC net) iff $\forall s_1, s_2 \in S: s_1^\bullet \cap s_2^\bullet \neq \emptyset \Rightarrow s_1^\bullet = s_2^\bullet$.

A marked net $\Sigma=(N, M_0)$ is called EFC system iff $N$ is an EFC net.

The (extended) free choice property ensures that for any pair of transitions in the post-set of a place, every marking enables one of them if and only if it enables the other one. Liveness of free choice systems can be characterised structurally by means of the dt-property, to be defined next.

**Definition 3.2.** Let $\Sigma=(S, T, F, M_0)$ be a marked net with $N=(S, T, F)$.

(i) A set $D \subseteq S$ is called d-set (for 'deadlock') iff ${}^\bullet D \subseteq D^\bullet$.

(ii) A set $Q \subseteq S$ is called t-set (for 'trap') iff $Q^\bullet \subseteq {}^\bullet Q$.

(iii) $\Sigma$ satisfies the dt-property (short for deadlock/trap property) iff every nonempty d-set $D$ contains a t-set $Q$ which is marked at $M_0$, i.e., satisfies $M_0(Q) \neq 0$.

Since the union of d-sets (t-sets) is again a d-set (t-set, respectively), any set of places contains a unique maximal d-set and a unique maximal t-set.

Before giving the proofs of the main results we will state a lemma which unifies a recurring argument in those proofs.

**Definition 3.3.** Let $N=(S, T, F)$ be a net and $X \subseteq S$ a set of places. An allocation of $X$ is a function $\alpha: X \to T$ with $\forall s \in X: \alpha(s) \in s^\bullet$. An allocation $\alpha$ of $X$ is cycle-free iff there is no nonempty set of places $X' \subseteq X$ such that $X' \subseteq (\alpha(X'))^\bullet$.

Since, by our general assumptions, $s^\bullet \neq \emptyset$ for all places $s$, for each set of places $X$ there is at least one allocation of $X$.

**Lemma 3.4.** *Let $\Sigma=(S, T, F, M_0)$ be a live EFC system. Let $X \subseteq S$, $X \neq \emptyset$ and let $\alpha: X \to T$ be an allocation. Define $Y=\alpha(X)$.*

(a) *There exists a process $\pi=(B, E, K, p)$ of $\Sigma$ with the following properties:*
$M_{\min(\pi)}=M_0$ *(i.e., $\pi$ is an initial process);*
$p^{-1}(X^\bullet \backslash Y)=\emptyset$ *(i.e., tokens on places of $X$ are only removed by transitions of $Y$);*
$|p^{-1}(Y)| \notin \mathbf{N}$ *(i.e., at least one transition of $Y$ occurs infinitely often).*

(b) *If, additionally, $\Sigma$ is bounded and $Y^\bullet \subseteq X$, then there is a process $\pi=\pi_a\pi_b$ of $\Sigma$ such that with $\pi_a=(B_a, E_a, K_a, p_a)$ and $\pi_b=(B_b, E_b, K_b, p_b)$ the following holds:*
$M_{\min(\pi_a)}=M_0$;
$p_a(E_a) \cap X^\bullet=\emptyset$;
$p_b(E_b)=Y$;
*$E_b$ is infinite.*

*Proof.* (a) Inductively define a sequence of processes as follows:

$j=0, 2, 4, \ldots$: $\pi_j$ is a minimal process with $M_0=M_{\min(\pi_0)}$ and $M_{\min(\pi_j)}=M_{\max(\pi_{j-1})}$ for $j>0$, such that $M_{\max(\pi_j)}$ enables a transition $t \in Y$. Such a process exists by the liveness of $\Sigma$. By the minimality of $\pi_j$, no $t \in Y$ occurs in $\pi_j$, and by the EFC property, no $t \in X^\bullet \backslash Y$ occurs in $\pi_j$ either.

$j=1,3,5,\ldots$: $\pi_j$ is a maximal process with $M_{\min(\pi_j)}=M_{\max(\pi_{j-1})}$ such that only transitions of $Y$ occur in $\pi_j$; in case $\pi_j$ is infinite, the construction stops and $\pi=\pi_0\pi_1\cdots\pi_j$ satisfies the requirements.

Because each $\pi_j$ ($j=1,3,5,\ldots$) contains at least one event, $\pi=\pi_0\pi_1\pi_2\pi_3\ldots$ is an infinite process which satisfies the requirements.

(b) Define a sequence $\pi_0, \pi_1, \pi_2, \ldots$ of processes as in (a) and let $\pi=(B,E,K,p)=\pi_0\pi_1\pi_2\ldots$

For $j=2,4,6,\ldots$ there is a condition $b$ in $\pi_j$ with $p(b)\in {}^\bullet Y$ and $p({}^\bullet b)\nsubseteq Y$ by the defining rules.

We show that the set $B'=\{b\in B \mid p(b)\in {}^\bullet Y \wedge p({}^\bullet b)\nsubseteq Y\}$ is a co-set.

First, $Y^\bullet\subseteq X$ implies $(p^{-1}(Y))^\bullet\subseteq p^{-1}(X)$. Since no transition $t\in X^\bullet\backslash Y$ occurs in $\pi$, we also have $(p^{-1}(X))^\bullet\subseteq p^{-1}(Y)$. Hence $\uparrow p^{-1}(Y)\subseteq p^{-1}(X)\cup p^{-1}(Y)$.

Let $b, b'\in B'$, $b\neq b'$. Then $\uparrow b\subseteq\{b\}\cup\uparrow p^{-1}(Y)\subseteq\{b\}\cup p^{-1}(X)\cup p^{-1}(Y)$. Since ${}^\bullet b'\nsubseteq p^{-1}(Y)$ we get $b'\notin\uparrow b$. Similarly, $b\notin\uparrow b'$, and thus $B'$ is a co-set.

Since $\Sigma$ is bounded there is no infinite co-set in $\pi$ (Corollary 2.8) and hence there is an odd $j$ such that $\pi_j$ is infinite.

Let $\pi=\pi_a\pi_b$ such that $\pi_a$ is a maximal process with $p_a(E_a)\cap Y=\varnothing$. Since $E_a\subseteq E_1\cup\cdots\cup E_{j-1}$, $E_a$ is finite and $E_b$ is infinite.

With $\uparrow p^{-1}(Y)\subseteq p^{-1}(X)\cup p^{-1}(Y)$ and $(\min(\pi_b))^\bullet\subseteq p^{-1}(Y)$ we finally get $p_b(E_b)=Y$.

We turn to Commoner's criterion which characterises liveness in EFC systems [Com72].

**Lemma 3.5.** *Let $N=(S,T,F)$ be a net, $X\subseteq S$ any set of places and $Q\subseteq X$ the maximal t-set in $X$. Then there exists a cycle-free allocation $\alpha$ of $X\backslash Q$ with $\alpha(X\backslash Q)\subseteq T\backslash{}^\bullet Q$.*

*Proof.* By induction on $|X|$ (we have $|Q|\le|X|\le|S|$).

*Base*: $|Q|=|X|$.

Then $\alpha=\varnothing$ satisfies the requirements.

*Step*: $|Q|<|X|\le|S|$.

Define the sets of places

$$X_1'=\{s\in X \mid \neg(s^\bullet\subseteq{}^\bullet X)\}$$
$$X_1=X\backslash X_1'.$$

Because of $Q\neq X$ and by the maximality of $Q$, $X$ is not a t-set and hence, $X_1'\neq\varnothing$ and $X_1\neq X$. By the definition of $X_1$, $Q\subseteq X_1$.

For every $s\in X_1'$, choose $\alpha'(s)\in s^\bullet$ arbitrarily such that $(\alpha'(s))^\bullet\cap X=\varnothing$.

Since $|X_1|<|X|$, the induction hypothesis can be applied to $X_1$, yielding a cycle-free allocation $\alpha''$: $X_1\backslash Q\to T$ with $\alpha''(X_1\backslash Q)\subseteq T\backslash{}^\bullet Q$.

Put $\alpha=\alpha'\cup\alpha''$. Because $(\alpha(X_1'))^\bullet\subseteq S\backslash X\subseteq S\backslash X_1$, $\alpha$ is cycle-free and satisfies $\alpha(X\backslash Q)\subseteq T\backslash{}^\bullet Q$.

**Theorem 3.6.** *Let $\Sigma=(S,T,F,M_0)$ be an EFC system. Then $\Sigma$ is live iff $\Sigma$ satisfies the dt-property.*

*Proof.* In order to prove ($\Leftarrow$), suppose $\Sigma$ is not live. Then there is a transition $t$ and a reachable marking $M$ such that no successor marking of $M$ enables $t$. Using the EFC property it is easy to see that no transition $t'\in({}^\bullet t)^\bullet$ is enabled by a successor marking of $M$. Hence there must be a place $s\in{}^\bullet t$ and a successor marking $M'$ of $M$ such that for all successor markings $M''$ of $M'$, $M''(s) = 0$. Thus all transitions $t' \in {}^\bullet s$ are also dead at $M'$. The repetition of this argument leads to a nonempty d-set $D$ which is unmarked at some successor marking of $M_0$. $D$ cannot have contained a t-set which is marked at $M_0$.

In order to prove ($\Rightarrow$), let $\Sigma$ be live. Let $D$ be a nonempty d-set and $Q$ the maximal t-set in $D$.

By Lemma 3.5 there is a cycle-free allocation $\alpha: (D\backslash Q) \to T$ with $\alpha(D\backslash Q) \subseteq T\backslash {}^\bullet Q$. Define $\hat{T} = \alpha(D\backslash Q)$. By Lemma 3.4(a) (with $X = D\backslash Q$), there is an infinite process $\pi = (B, E, K, p)$ such that no transitions of $(D\backslash Q)^\bullet \backslash \hat{T}$ occur in $\pi$ and $|p^{-1}(\hat{T})| \notin \mathbf{N}$. For each transition $t \in \hat{T}$ with $|p^{-1}(t)| \notin \mathbf{N}$ we can find a place $s \in {}^\bullet t \cap (D\backslash Q)$ with $|p^{-1}(s)| \notin \mathbf{N}$ and a transition $t' \in {}^\bullet s$ with $|p^{-1}(t')| \notin \mathbf{N}$. We have $t' \in D^\bullet$ since $D$ is a d-set, and hence either $t' \in \hat{T}$ or $t' \in Q^\bullet$. In the first case we continue as above, eventually reaching a transition $t'' \in Q^\bullet \cap p(E)$ since $\alpha$ is cycle-free and $T$ is finite.

It follows that $Q$ is marked once during $\pi$, i.e., $p(B) \cap Q \neq \emptyset$.

With ${}^\bullet Q \subseteq {}^\bullet D \subseteq D^\bullet = (D\backslash Q)^\bullet \cup Q^\bullet$, $p^{-1}((D\backslash Q)^\bullet) \subseteq p^{-1}(\hat{T})$ and $\hat{T} \cap {}^\bullet Q = \emptyset$ we get $p^{-1}({}^\bullet Q) \subseteq p^{-1}(Q^\bullet)$. Hence for each condition $b \in p^{-1}(Q)$, either $b \in \min(\pi)$ or ${}^\bullet({}^\bullet b) \cap p^{-1}(Q) \neq \emptyset$. Thus, arguing backward from one of the conditions in $p^{-1}(Q)$, it follows that $Q$ is marked at $M_0$.

**Corollary 3.7.** *An EFC net $N$ is lively markable iff each nonempty d-set of $N$ contains a t-set.*

*Proof.* ($\Rightarrow$) is obvious. For ($\Leftarrow$) consider a marking without unmarked places.

# 4. T-Component Coverings and T-Invariants

In this section and in the next, we turn to the two main results of Hack [Hac72]; we show that live and bounded EFC systems can be covered by certain structures.

**Definition 4.1.** $N_1 = (S_1, T_1, F_1)$ is called a T-component of $N = (S, T, F)$ iff $\forall s \in S_1: |{}^\bullet s| = |s^\bullet| = 1$ and, in addition, $N_1$ is the subnet of $N$ generated by $T_1 \subseteq T$. $N$ is covered by T-components iff for each $x \in S \cup T$ there is a T-component $N_1 = (S_1, T_1, F_1)$ of $N$ such that $x \in S_1 \cup T_1$.

The following general lemma states that if a reproduction process of a bounded system uses, for any place $s$, never more than one transition $t \in s^\bullet$, then the same holds for transitions $t \in {}^\bullet s$.

**Lemma 4.2.** *Let $\Sigma = (S, T, F, M_0)$ be a bounded marked net and let $\pi = (B, E, K, p)$ be a reproduction process of $\Sigma$. If $\forall s \in S: |s^\bullet \cap p(E)| \leq 1$ then $\forall s \in S: |{}^\bullet s \cap p(E)| \leq 1$.*

*Proof.* Let $T'$ denote $p(E)$. ${}^\bullet T' = T'^\bullet$ since $\pi$ is a reproduction process. Let $S'$ denote $T'^\bullet$ and let $F'$ denote $F \cap ((S' \times T') \cup (T' \times S'))$. Assume $\forall s \in S: |s^\bullet \cap T'| \leq 1$, i.e. $\forall s \in S': |s^\bullet \cap T'| = 1$ and $\forall s \in S\backslash S': |s^\bullet \cap T'| = 0$.

Let $s$ be an arbitrary place of $S'$. Then $|s^\bullet \cap T'| = 1$ and $|{}^\bullet s \cap T'| \geq 1$. $\pi$ is a reproduction process, whence $\sum_{t \in {}^\bullet s} |p^{-1}(t)| = \sum_{t \in s^\bullet} |p^{-1}(t)|$. Hence for $t \in {}^\bullet s \cap T'$ and $t' \in s^\bullet \cap T'$ (the unique element) we get the inequality $|p^{-1}(t)| \leq |p^{-1}(t')|$. Thus $(t, t') \in F'^2$ implies $|p^{-1}(t)| \leq |p^{-1}(t')|$; by induction, $(t, t') \in$

$F'^*$ implies $|p^{-1}(t)| \leq |p^{-1}(t')|$. On the other hand, $(t, t') \in (F')^2$ implies $(t', t) \in (F')^*$ by Lemma 2.11. Thus $|p^{-1}(t)| = |p^{-1}(t')|$ and no transition of ${}^\bullet s \backslash \{t\}$ occurs in $\pi$. Hence we get $|{}^\bullet s \cap T'| = 1$.

By the definition of $S'$ we get $|{}^\bullet s \cap T'| = 0$ for all $s \in S \backslash S'$.

**Theorem 4.3.** *Let $\Sigma = (S, T, F, M_0)$ be a live and bounded EFC system and let $J$ be a minimal T-invariant of $\Sigma$. Then $T_J$ generates a strongly connected T-component. Moreover, $J$ is a realisable.*

*Proof.* ${}^\bullet T_J = T_J^\bullet$ since $J$ is a T-invariant. Let $S_J$ denote $T_J^\bullet$.

Let $\alpha : S_J \to T_J$ be an allocation of $S_J$ such that $\forall s \in S_J$: $|s^\bullet \cap \alpha(S_J)| = 1$. Such a mapping $\alpha$ exists because – by the EFC property – $s^\bullet \cap s'^\bullet \neq \emptyset$ implies $s^\bullet = s'^\bullet$ and we can choose $\alpha$ such that $\alpha(s) = \alpha(s')$.

By Lemma 3.4(b) (with $X = S_J$), there is an infinite process $\pi$ of $\Sigma$ such that only transitions of $\alpha(S_J)$ occur in $\pi$.

Since $\Sigma$ is bounded, by Lemma 2.10, there is a reproduction process $\pi'$ of $\Sigma$ in which only transitions of $\alpha(S_J)$ occur and hence, by Lemma 2.15, a T-invariant $J'$ with $T_{J'} \subseteq \alpha(S_J)$. Since $J$ is minimal we get $T_{J'} = T_J = \alpha(S_J)$ as well as $J = J'$, and $J$ is realisable.

By the definition of $\alpha$ we get $\forall s \in S_J$: $|s^\bullet \cap T_J| = 1$.

Since by the definition of $S_J$ we have $\forall s \in S \backslash S_J$: $|{}^\bullet s \cap T_J| = |s^\bullet \cap T_J| = 0$, with Lemma 4.2 we get $\forall s \in S_J$: $|{}^\bullet s \cap T_J| = |s^\bullet \cap T_J| = 1$.

Hence $T_J$ generates a T-component. It is connected because $J$ is minimal. Hence by Lemma 2.11 it is strongly connected.

**Corollary 4.4.** *A live and bounded EFC system is covered by strongly connected T-components.*

*Proof.* By Lemma 2.9, a live and bounded system has a reproduction process which uses all transitions of the system. By Lemma 2.15 there is a T-invariant $J$ such that $T_J$ is the set of all transitions, and by Lemma 2.14 there is a set of minimal T-invariants such that the union of the supports is the set of all transitions. By Theorem 4.3 each support of such a minimal T-invariant generates a strongly connected T-component.

The other part of Theorem 4.3 states that each minimal T-invariant is realisable. This result has consequences on the analysis of transition dependencies: the finiteness of the weighted synchronic distance between two sets of transitions can be decided by means of the existence of a solution of an inhomogeneous linear equation system [GoR82, Des88].

**Corollary 4.5.** *Each connected T-component of a live and bounded EFC system is strongly connected.*

*Proof.* Apply Theorem 4.3 and Lemma 2.11.

## 5. S-Component Coverings and Boundedness

We have shown in the previous sections how T-invariants and T-components are related to repetitive behaviour and reproduction processes of systems. The dual concept of T-components are S-components. They are related to safeness and boundedness.

**Definition 5.1.** $N_1$ is an S-component of $N$ iff $N_1^{-d}$ is a T-component of $N^{-d}$. $N$ is covered by S-components iff $N^{-d}$ is covered by T-components.

Since each occurrence of a transition of an S-component removes exactly one token and adds exactly one token to that S-component the sum of all tokens always remains constant. Hence the places belonging to an S-component are bounded irregardless of the initial marking. Thus a system which is covered by S-components is bounded.

In general the opposite direction does not hold true; there are bounded systems which are not covered by S-components. However, in the case of live and bounded EFC systems we will show that boundedness implies the existence of a covering by suitably marked S-components which are, moreover, strongly connected.

First we show that some properties of an EFC system can be carried over to the reverse-dual system. Since the reverse-dual net of $N^{-d}$ is $N$ again, not only the S-invariants of $N$ are the T-invariants of $N^{-d}$ but also the T-invariants of $N$ are the S-invariants of $N^{-d}$. The same holds for S- and T-components.

The nets we consider are covered by strongly connected T-components, hence their reverse-dual counterparts are covered by strongly connected S-components and are bounded for each initial marking. Furthermore, the EFC property carries over to the reverse-dual net; each violation of the EFC property for $N$ is also a violation of the EFC property for $N^{-d}$.

We will show next that given a live and bounded EFC system $\Sigma = (N, M)$, the reverse-dual net $N^{-d}$ can be provided with an appropriate marking $M'$ such that the dt-property holds for $(N^{-d}, M')$.

**Theorem 5.2.** *Let $\Sigma = (S, T, F, M_0)$ be a live and bounded EFC system. Let $U \subseteq T$ with $U^\bullet \subseteq {}^\bullet U$. Then $N$ has a T-invariant $J$ such that $T_J \subseteq U$.*

*Proof.* Let $X = U^\bullet$ and $\alpha: X \to T$ such that $Y = \alpha(X) \subseteq U$.

By Lemma 3.4(b) there is a process $\pi_b = (B_b, E_b, F_b, p_b)$ such that $p_b(E_b) = Y \subseteq U$ and $E_b$ is infinite.

By Lemma 2.10 there is a reproduction process $\bar{\pi} = (\bar{B}, \bar{E}, \bar{F}, \bar{p})$ with $\bar{p}(\bar{E}) \subseteq p_b(E) \subseteq U$.

Hence, with Lemma 2.15, $\bar{p}(\bar{E})$ is the support of a T-invariant of $N$.

From this theorem two corollaries can be deduced. Given an EFC net $N$ which is lively and boundedly markable it is shown in Corollary 5.3 that the net $N^{-d}$ can be marked lively and boundedly as well.

Hence Theorem 5.2 can be applied to $(N^{-d}, M')$ as well, where $M'$ is an appropriate live and bounded marking. Since $(N^{-d})^{-d} = N$ this yields additional properties of $N$ as shown in Corollary 5.4.

**Corollary 5.3.** *Let $N$ be an EFC net which is lively and boundedly markable. Then:*

(i) *Each nonempty d-set of $N^{-d}$ contains a set which is the support of an S-invariant of $N^{-d}$.*
(ii) *Each minimal nonempty d-set of $N^{-d}$ is a t-set and the support of an S-invariant of $N^{-d}$.*
(iii) *$N^{-d}$ is lively and boundedly markable.*

*Proof.*

(i) d-sets of $N^{-d}$ are sets of transitions $U$ of $N$ with $U^\bullet \subseteq {}^\bullet U$. T-invariants of $N$ are S-invariants of $N^{-d}$.

(ii) The support of an S-invariant is a d-set and a t-set by definition.

(iii) Apply (ii) and Corollary 3.7 for liveness. By Corollary 4.4, $N^{-d}$ is covered by S-components and thus bounded for any initial marking.

**Corollary 5.4.** *Let N be an EFC net which is lively and boundedly markable. Then:*

(i) *There is an S-invariant of N which maps all places of N to positive nonzero integers.*

(ii) *Each minimal S-invariant of N induces a strongly connected S-component.*

(iii) *N is covered by strongly connected S-components.*

(iv) *Each connected S-component of N is strongly connected.*

Finally we show that in live and bounded EFC systems, the maximum number of tokens on a place is given by the minimum marking of S-components through that place.

**Theorem 5.5.** *Let $\Sigma = (S, T, F, M_0)$ be a live and bounded EFC system. Let $s \in S$. Then:*

$$\max\{M(s) \mid M \in [M_0\rangle\} = \min\{M(S') \mid (S', T', F') \text{ is an S-component with } s \in S'\}$$

*Proof.* ($\leq$) is obvious, since $M(S') = M_0(S')$ for any $M \in [M_0\rangle$ and any S-component $(S', T', F')$ of $\Sigma$.

To prove ($\geq$), let $s \in S$ and let $m$ be the maximum number of tokens on $s$, i.e., $m = \max\{M(s) \mid M \in [M_0\rangle\}$.

By boundedness such a number $m \in \mathbf{N}$ exists and, by liveness, $m > 0$.

Let $M \in [M_0\rangle$ be any marking with $M(s) = m$.

We define a marking $\hat{M} \neq M$ by

$$\hat{M}(x) = \begin{cases} M(x) & \text{if } x \neq s \\ M(x) - m & \text{if } x = s \end{cases}$$

for all $x \in S$. Then $\hat{\Sigma} = (S, T, F, \hat{M})$ is not live since otherwise, there is a finite process $\pi$ of $\hat{\Sigma}$ with $M_{\min(\pi)} = \hat{M}$ and $M_{\max(\pi)}(s) > 0$ and hence, there is also a finite process $\pi'$ of $\Sigma$ with $M_{\min(\pi')} = M$ and $M_{\max(\pi')}(s) > m$ contradicting the maximality property of $m$ ($\pi'$ is like $\pi$ but has $m$ additional conditions which are all mapped to $s$).

From Theorem 3.6 ($\Rightarrow$) it follows that there exists a nonempty d-set $D$ of $\hat{\Sigma}$ (and hence of $\Sigma$) such that for every trap $Q$ contained in $D$, $\hat{M}(Q) = 0$.

We can assume without loss of generality that $D$ is minimal.

By Corollary 5.3(ii), $D$ is itself a t-set. Hence $\hat{M}(D) = 0$ and, since $\Sigma$ is live, $M(D) > 0$.

By the definition of $\hat{M}$, $M(D) = m$. The required result now follows from Corollary 5.3(ii) and Corollary 5.4(ii).

**Corollary 5.6.** *A live EFC system is n-bounded iff it is covered by strongly connected S-components which carry less or equal to n tokens.*

Summarising the previous results leads to the final corollary of this section:

**Corollary 5.7.** *An EFC net is lively and boundedly markable iff it is covered by strongly connected S-components, covered by strongly connected T-components, and every nonempty d-set contains a nonempty set which is both a d-set and a t-set.*

## 6. Relationship Between S-Components and T-Components

We prove a result which clarifies, to some extent, the relationship between S-components and T-components in a live and bounded EFC system. It identifies a circumstance in which every (1-token) S-component has a nonempty intersection with every T-component.

**Definition 6.1.** A marked net $\Sigma$ has a frozen token iff there exists an infinite process $\pi = (B, E, K, p)$ of $\Sigma$ and an element $b \in B$ with $b^\bullet = \emptyset$.

**Theorem 6.2.** *A live EFC system has no frozen tokens iff it is safe and every T-component has a nonempty intersection with every S-component.*

*Proof.* ($\Rightarrow$). We assume $\Sigma = (S, T, F, M_0)$ to be live and without frozen tokens. In order to prove that $\Sigma$ is safe, assume the contrary. Then there is a place $s_0 \in S$ and a marking $M \in [M_0\rangle$ with $M(s_0) \geq 2$. By Theorem 3.6 and the liveness of $\Sigma$, $\Sigma' = (S, T, F, M')$ is live, where $M'(s_0) = M(s_0) - 1$ and $M'(s) = M(s)$ for all $s \in S \backslash \{s_0\}$. Hence there is an infinite process $\pi' = (B', E, K', p')$ of $\Sigma'$. Then $\pi = (B, E, F', p)$, where $\pi$ is obtained from $\pi'$ by adding an isolated condition $b_0$ with $p(b_0) = s_0$, is an infinite process of $(S, T, F, M)$, and hence also of $\Sigma$, with $b_0^\bullet = \emptyset$, contradicting the assumption that $\Sigma$ has no frozen tokens.

In order to prove that every T-component has a nonempty intersection with every S-component, we consider an arbitrary T-component. By Theorem 4.3, this T-component corresponds to a realisable T-invariant and thus generates an infinite process. Because of the lack of frozen tokens, this process moves all tokens of the net and hence uses places of all marked S-components. Since $\Sigma$ is live all its S-components are marked.

($\Leftarrow$). We assume $\Sigma = (S, T, F, M_0)$ to be safe and such that every T-component has a nonempty intersection with every S-component.

Each reproduction process of $\Sigma$ corresponds to a T-invariant, each T-invariant is a sum of minimal T-invariants and each minimal T-invariant induces, by Theorem 4.3, a T-component. These T-components intersect with all S-components. Since $\Sigma$ is safe, it is covered by a set of S-components which carry exactly one token each (by Corollary 5.6). Hence each reproduction process $\pi$ moves all tokens of $\Sigma$, i.e., has no isolated conditions.

To prove that $\Sigma$ has no frozen tokens, let $\pi$ be an infinite process of $\Sigma$ and let $(\pi_i)$, $i = 1, 2, \ldots$ be an infinite sequence of finite processes such that $\forall i \in \mathbf{N}: M_{\max(\pi_i)} = M_{\min(\pi_{i+1})}$. Since $\Sigma$ is safe there are at most $m = 2^{|S|}$ markings reachable in $\Sigma$. Hence we can decompose $\pi$ to $\pi = \pi_0'\pi_1'\pi_2' \ldots$ such that $\pi_1', \pi_3', \pi_5', \ldots$ are reproduction processes. All of these reproduction processes use all tokens. Hence $\pi$ has no condition $b$ with $b^\bullet = \emptyset$.

## 7. Conclusion

By means of partial order semantics we have proved some properties of (bounded) extended free choice nets, most of which have already been known for the smaller class of (safe) free choice nets. In particular we have generalised the liveness criterion and the coverability properties to live and bounded marked EFC nets. New results are: minimal T-invariants of such nets are realisable and induce T-components; minimal bounds for places can be calculated by means of the token counts of the S-components and the existence of frozen tokens can be characterised by intersections of S-components and T-components. Furthermore Hack's duality result has been generalised.

The proofs of these results are considerably shorter than the ones of comparable results to be found in the literature. We believe that this is not least due to the use of partial order process semantics for nets. The advantages of this semantics include the following:

Some general results about processes (e.g. Corollary 2.8, but also Lemma 4.2) are available which are more versatile than corresponding results on sequences.

It is no longer necessary to keep arguments about switching concurrent transitions around in interleaved semantics.

Using the same formalism – nets – for systems and their behaviour leads to easy correspondences between properties (see again Corollary 2.8). This advantages becomes most apparent when arguing about the interconnection relations $F$ and $K$ (see, for instance, Lemma 2.11).

We think that in structure theory of nets as well as in other areas, partial order semantics can be used even more profitably. It is obvious, for example, that each path of a process is mapped onto a path of the system net. For live free-choice nets the opposite holds as well because for each path of the system net there exists a process containing a path which is mapped to the path mentioned first. It is much harder to formulate this property, which is related to the autonomy of conflicts in free-choice nets, in terms of execution sequences.

We are presently working on some open questions in structure theory, e.g. the characterisations of home states or the correspondence between bipolar schemata and free-choice nets without frozen tokens. Since processes did a good job in shortening other proofs they might be a suitable tool for new results as well.

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