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CONSTRUCTIVE APPROXIMATION

Super Spline Spaces of Smoothness r and Degree $d \geq 3r + 2$

Adel Kh. Ibrahim and Larry L. Schumaker

Abstract. The problem of computing the dimension of spaces of splines whose elements are piecewise polynomials of degree d with r continuous derivatives globally has attracted a great deal of attention recently. We contribute to this theory by obtaining dimension formulae for certain spaces of super splines, including the case where varying amounts of additional smoothness is enforced at each vertex. We also explicitly construct minimally supported bases for the spaces. The main tool is the Bernstein-Bézier method.

1. Introduction

Given a triangulation Δ of a set Ω in $\mathbb{R}^2$, and given integers $0 \leq r < d$, the space of *splines of degree d and smoothness r* on Δ is defined as

$$\mathcal{S}_d^r(\Delta) = \{s \in C^r(\Omega)\colon s \text{ restricted to each triangle belongs to } P_d\},$$

where P_d is the space of polynomials of total degree d. Clearly, $\mathcal{S}_d^r(\Delta)$ is a linear space. The problems of computing the dimension of this space and constructing minimally supported bases for it are difficult, in general (see [1]–[15] and the references therein).

This paper is based on several recent contributions to this problem area. Dimension formulae and local bases for the spaces $\mathcal{S}_d^r(\Delta)$ were obtained for $d \geq 4r + 1$ by Alfeld, Piper, and Schumaker [4], [6], using Bernstein-Bézier methods. These results were recently extended to $d \geq 3r + 2$ by Dong [10]. Chui and Lai [8] pointed out that certain subspaces with double smoothness at each of the interior vertices are of special importance, and they gave dimension formulae and minimally supported bases for their spaces, which they called super splines. These results were extended to a wider class of super splines for $d \geq 4r + 1$ by Schumaker [15], where the connection with finite elements was also explored. Super splines were studied in a different way by Chui and He [7], and are used by Chui and Lai [9] to prove results on the approximation power of spline spaces.

The purpose of this paper is to investigate general spaces of super splines. We require only that $d \geq 3r + 2$, and deal not only with the usual kinds of super spline

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spaces, but also with more general ones for which the amount of extra smoothness varies from vertex to vertex. The paper is organized as follows. In Section 2 we define the spaces of interest, and present our main results. In Section 3 we develop some Bernstein–Bézier tools, while in Section 4 we establish dimension formulae and construct explicit minimally supported bases for our spaces in the special case where the amount of additional vertex smoothness is the same for each vertex. In Section 5 we do the same for the general case. Finally, Section 6 is devoted to remarks.

2. The Main Results

We begin by defining the super spline spaces of interest.

Definition 2.1. Suppose that Δ is an arbitrary regular triangulation of a set Ω, and that the vertices of the triangulation are denoted by $v_1, \ldots, v_V$. Suppose d and r are nonnegative integers such that $d \geq 3r + 2$. Finally, suppose ρ_i are integers satisfying

$$r \leq \rho_i < d, \qquad i = 1, \ldots, V, \tag{2.1}$$

and let

$$\theta = (\rho_1, \ldots, \rho_V). \tag{2.2}$$

We define the space of *super splines* by

$$\mathcal{S}_d^{r,\theta}(\Delta) = \{s \in \mathcal{S}_d^r(\Delta) \colon s \in C^{\rho_i}(v_i), i = 1, \ldots, V\}, \tag{2.3}$$

where, in general, if v is a vertex of the triangulation Δ,

$$C^k(v) = \{s \colon s \text{ has derivatives up to order } k \text{ at } v\}. \tag{2.4}$$

The super splines considered in [15] correspond to the case where $\rho_i = \rho$, for all $i = 1, \ldots, V$ where $r \leq \rho < d$. For consistency with the notation used there, in this case we write

$$\mathcal{S}_d^{r,\rho}(\Delta) = \{s \in \mathcal{S}_d^r(\Delta) \colon s \in C^{\rho}(v_i), i = 1, \ldots, V\}. \tag{2.5}$$

To present formulae for the dimension of super spline spaces, we need to introduce some additional notation (see [3]–[6], [14], and [15]). Let

$$\begin{aligned}
V_I &= \text{number of interior vertices of } \Delta,\\
V_B &= \text{number of boundary vertices of } \Delta,\\
E_I &= \text{number of interior edges of } \Delta,\\
E_B &= \text{number of boundary edges of } \Delta,\\
V &= \text{total number of vertices of } \Delta,\\
E &= \text{total number of edges of } \Delta,\\
N &= \text{number of triangles of } \Delta.
\end{aligned}$$

It is well known that

$$E_B = V_B, \qquad N = V_B + 2V_I - 2, \quad \text{and} \quad E_I = V_B + 3V_I - 3. \tag{2.6}$$

We also introduce

$$S = \text{number of singular vertices of } \Delta, \tag{2.7}$$

where a *singular vertex* v is a vertex which is formed by two lines which cross at v. For each $i = 1, \ldots, V$, let E_i denote the number of interior edges attached to v_i, and let e_i denote the number of these with different slopes. Throughout this paper we assume that the vertices are numbered so that the first V_I of them are the interior vertices.

Finally, we recall some additional terminology which is by now standard (see [3]–[6], [14], and [15]). A *cell* is a subtriangulation of Δ which consists of a set of triangles sharing one common interior vertex (see [3]–[6], and [11]). We say that the spline space $\mathscr{S}_d^{r,\rho}(\Delta)$ has a *minimally supported basis* provided that it has a basis of splines, each one with support on a cell.

The first of the following theorems gives dimension formulae for the spaces in (2.5) where the order of extra smoothness is the same at each vertex; i.e., $\rho_i = \rho$ for all $i = 1, \ldots, V$. The formulae depend on the size of ρ.

Theorem 2.2. *Let Δ be an arbitrary regular triangulation. If $2r \le \rho$ and $d \ge 2\rho + 1$, then*

$$\dim(\mathscr{S}_d^{r,\rho}(\Delta)) = \binom{\rho+2}{2}V + \left[\binom{d-3r-1}{2} - 3\binom{\rho-2r}{2}\right]N + \frac{(r+1)(2d-4\rho+r-2)}{2}E. \tag{2.8}$$

If $d \ge 3r + 2$ and $r + \lfloor (r+1)/2 \rfloor \le \rho \le 2r$ with $d \ge 2\rho + 1$, then

$$\dim(\mathscr{S}_d^{r,\rho}(\Delta)) = \binom{\rho+2}{2}V + \left[\binom{d-3r-1}{2} - 3\binom{2r-\rho+1}{2}\right]N + \frac{(r+1)(2d-4\rho+r-2)}{2}E + \binom{2r-\rho+1}{2}S, \tag{2.9}$$

where S is the number of singular vertices of Δ (see (2.7)). Finally, if $d \ge 3r + 2$ and $r \le \rho \le \mu$, where

$$\mu = r + \left\lfloor \frac{r+1}{2} \right\rfloor, \tag{2.10}$$

then

$$\dim(\mathscr{S}_d^{r,\rho}(\Delta)) = \binom{\mu+2}{2}V_B + \binom{\rho+2}{2}V_I + 2\left[\binom{\mu-r+1}{2} - \binom{\rho-r+1}{2}\right]E_I + \left[\binom{d-3r-1}{2} - 3\binom{2r-\mu+1}{2}\right]N + \frac{(r+1)(2d-4\mu+r-2)}{2}E + \sigma, \tag{2.11}$$

where

$$\sigma = \sum_{i=1}^{V_I} \sum_{j=\rho-r+1}^{d} (r + j + 1 - je_i)_+. \tag{2.12}$$

In all three cases, $\mathcal{S}_d^{r,\rho}(\Delta)$ has a minimally supported basis.

Discussion. Ths first statement is contained in Theorem 2.1 of [15]. The remaining statements are extensions of Theorem 2.5 of [15] which deals with similar super spline spaces (but under the assumption that $d \geq 4r + 1$), and of the results of [10] which deal with the usual spline spaces for $d \geq 3r + 2$. We delay the proof of (2.9) and (2.11) until Section 4 (see Theorems 4.2 and 4.4).

The formulae in Theorem 2.2 can be rewritten as a single formulae which holds in all three cases, as follows:

Corollary 2.3. *In each of the three cases of Theorem 2.2,*

$$\begin{aligned}\dim(\mathcal{S}_d^{r,\rho}(\Delta)) &= \frac{(d^2 - 2rd - r^2 + d + r - 2\rho^2 + 4r\rho - 2\rho)}{2} V_B \\ &+ \frac{(2d^2 - 6rd - 3r^2 + 3r - 5\rho^2 + 12r\rho - 3\rho)}{2} V_I \\ &+ \frac{(-2d^2 + 6rd + 3r^2 - 3r + 6\rho^2 - 12r\rho + 6\rho + 2)}{2} + \sigma.\end{aligned} \tag{2.13}$$

Proof. Using the Euler relations given in (2.6), straightforward algebraic manipulation of the formulae in Theorem 2.2 lead to formula (2.13). ∎

In order to state results for the more general super spline spaces defined in (2.3), we need some additional notation. Suppose that $\{\varepsilon_i\}_{i=1}^{E}$ is the set of edges of Δ. The following theorem, which contains Theorem 2.2 as a special case, is the main dimension result of this paper.

Theorem 2.4. *Given $d \geq 3r + 2$ and $r \leq \rho_i$, $i = 1, \ldots, V$, let*

$$k_i = \max(\rho_i, \mu), \qquad i = 1, \ldots, V, \tag{2.14}$$

where μ is the integer defined in (2.10). *Suppose*

$$k_i + k_j < d \qquad \text{for all} \quad (i, j) \in \mathcal{N}, \tag{2.15}$$

where

$$\mathcal{N} = \{(i, j)\colon v_i \text{ and } v_j \text{ are neighbors}\}.$$

Then

(2.16) $\dim(\mathcal{S}_d^{r,\theta}(\Delta))$

$$= \sum_{i=1}^{V_I} \binom{\rho_i + 2}{2} + \sum_{i=V_I+1}^{V} \binom{k_i + 2}{2}$$
$$+ \sum_{i=1}^{V} \left[\binom{k_i - r + 1}{2} - \binom{\rho_i - r + 1}{2} - \binom{2r - k_i + 1}{2} - \binom{k_i - 2r}{2} \right] E_i$$
$$- \sum_{i=V_I+1}^{V} \left[\binom{2r - k_i + 1}{2} + \binom{k_i - 2r}{2} \right] + \binom{d - 3r - 1}{2} N$$
$$+ \frac{(r+1)(2d+r-2)}{2} E - (r+1) \left[2 \sum_{i=V_I+1}^{V} k_i + \sum_{i=1}^{V} k_i E_i \right]$$
$$+ \sum_{i=1}^{V_I} \sum_{j=\rho_i - r + 1}^{d} (r + 1 + j - je_i)_+ .$$

Moreover, there exists a basis of minimally supported splines.

Discussion. Formula (2.16) reduces to those in Theorem 2.2 in the special case where $\rho_i = \rho$, all $i = 1, \ldots, V$. We prove a more explicit version of this theorem in Theorem 5.2 below. ■

Example 2.5. Consider $S_8^{2,\theta}(\Delta)$, where Δ is the triangulation shown in Fig. 3, and $\theta = (3, 2, 2, 2, 2, 4, 2, 2)$.

Discussion. Using formulae (2.16), we calculate the dimension of this space to be 149. An explicit determining set of Bézier ordinates (as described in Theorem 5.2) is given in the figure.

3. Preliminaries and Tools

Our analysis of the super spline spaces introduced in Section 1 follows the Bernstein–Bézier approach pioneered in [6]. In this section we review the necessary notation, and prove several preliminary results.

Suppose the triangles of Δ are denoted by $T^{[1]}, \ldots, T^{[N]}$, and that the vertices of $T^{[l]}$ are denoted by $v_{i1(l)}^{[l]}, v_{i2(l)}^{[l]}, v_{i3(l)}^{[l]}$ in counterclockwise order. In each triangle $T^{[l]}$ we consider the set of $\binom{d+2}{2}$ points

$$\mathcal{P}_d^{[l]} := \{P_{ijk}^{[l]} = (iv_{i1(l)}^{[l]} + jv_{i2(l)}^{[l]} + kv_{i3(l)}^{[l]})/d, \qquad i + j + k = d\}.$$

Now associated with the triangulation Δ, let

$$\mathcal{P} \equiv \mathcal{P}_d := \bigcup_{l=1}^{N} \mathcal{P}_d^{[l]}. \tag{3.1}$$

Note that points on the common edge between two triangles are included in $\mathscr{P}$ just once, although they belong to two triangles. It is convenient to have the concept of *distance* of a point from a vertex or from an edge. The points $\{P^{[l]}_{d-i,j,i-j}\}_{j=0}^{i}$ are said to be at a distance i from the vertex $v^{[l]}_{i1(l)}$. Similarly, the points $\{P^{[l]}_{d-j-i,j,i}\}_{j=0}^{d-i}$ are said to be at a distance i from the edge whose endpoints are $v^{[l]}_{i1(l)}$ and $v^{[l]}_{i2(l)}$.

If v is a vertex of Δ, we introduce the *ring of order p around v*

$$R_p(v) = \{\text{points which are distance } p \text{ from } v\}. \tag{3.2}$$

We also need the *disk of order p around the vertex v* defined by

$$D_p(v) = \bigcup_{j=0}^{p} R_j(v). \tag{3.3}$$

Now given a spline $s \in \mathscr{S}^0_d(\Delta)$, we denote the restriction of s to the triangle $T^{[l]}$ by $s^{[l]}$. This is a polynomial of degree d which can be written in the Bernstein-Bézier form

$$s^{[l]}(\zeta) = \sum_{i+j+k=d} c^{[l]}_{ijk} \frac{d!}{i!\,j!\,k!} \alpha^i \beta^j \gamma^k, \tag{3.4}$$

where (α, β, γ) is the barycentric coordinate of the point ζ with respect to the triangle $T^{[l]}$.

Each $s \in \mathscr{S}^0_d(\Delta)$ is uniquely determined by the coefficients of its polynomial pieces (3.4). Each of these coefficients can be identified with a domain point in the set $\mathscr{P}$. Indeed, it will be useful to define a linear functional defined on $\mathscr{S}^0_d(\Delta)$ associated with each domain point $P \in \mathscr{P}$ as follows:

$$\lambda_P s = \text{the coefficient of } s \text{ associated with the domain point } P. \tag{3.5}$$

For splines s which are continuous, if P is a domain point on an interior edge of Δ, then the associated coefficients of the two polynomial pieces of s which join along that edge must agree. This ensures that λ_P is well defined. The set $\{(P, \lambda_P s)\}_{P \in \mathscr{P}}$ is called the *Bézier net* (see [3]–[6]). If Γ is a set of domain points, then we write

$$\Lambda_\Gamma = \{\lambda_P \colon P \in \Gamma\}.$$

As in [3]–[6], our approach to establishing the dimension of spline spaces is to use Bézier nets as a tool to obtain upper and lower bounds. Suppose $\mathscr{S} \subseteq \mathscr{S}^0_d(\Delta)$ is a linear space of splines. First, to get an upper bound on the dimension of $\mathscr{S}$, suppose $\Gamma \subset \mathscr{P}$ contains L points, and that $\Lambda_\Gamma = \{\lambda_P\}_{P \in \mathscr{P}}$ is the corresponding set of linear functionals. In addition, suppose that Λ_Γ has the property that it is a *determining set* for $\mathscr{S}$ in the sense that

$$s \in \mathscr{S} \quad \text{and} \quad \lambda s = 0 \quad \text{for all } \lambda \in \Lambda_\Gamma \qquad \text{implies} \quad s \equiv 0. \tag{3.6}$$

Then, as shown in [6], it follows that $\dim(\mathscr{S}) \le L$.

For a lower bound, we use the approach of [5]. Suppose now that Γ is such that $\{B_P\}_{P \in \Gamma}$ are splines in $\mathscr{S}$ satisfying

$$\lambda_P B_Q = \delta_{PQ}, \qquad \text{all} \quad P, Q \in \Gamma. \tag{3.7}$$

Then $\dim(\mathscr{S}_d^{r,\rho}(\Delta)) \geq L$. Clearly, if we can use the same set Γ for both the upper and lower bounds, then $\dim(\mathscr{S}) = L$, and $\{B_P\}_{P\in\Gamma}$ is a *basis* for $\mathscr{S}$ and Λ_Γ is a *dual basis*.

As in [3]–[6], [10], and [15], in order to construct an appropriate set Γ, we are going to divide the set of all domain points $\mathscr{P}$ into subsets such that the corresponding linear functionals in two disjoint subsets are linearly independent from each other. Suppose $D \subseteq \mathscr{P}$ is a typical such subset. Then the idea is to choose $\mathscr{D} \subseteq D$ with as small a cardinality as possible such that, for any spline $s \in \mathscr{S}$, the set of values $\{\lambda_P s\}_{P\in\mathscr{D}}$ determines $\{\lambda_P s\}_{P\in D}$. In this case, we say that $\mathscr{D}$ *determines* $\mathscr{S}$ *on* D. If $\mathscr{D}$ has a minimal number of points in it, we call it a *minimal determining set* for $\mathscr{S}$ on D.

The exact way in which the set of Bézier ordinates is divided up will be different for each of the cases of Theorem 2.2, as well as for Theorem 2.4. In general, the subsets D of interest will be disks, triangles, and certain polygons formed from disks and triangles. In the remainder of this section we present several lemmas which deal with the process of choosing a minimal determining set $\mathscr{D}$ associated with a given D.

Our first lemma shows how to find a minimal determining set for $\mathscr{S}_d^{r,\rho}(\Delta)$ in the case where D is a disk around a boundary vertex.

Lemma 3.1. *Let $r \leq \rho \leq p < d$, and let v be a boundary vertex of Δ with E interior edges attached. Let $D = D_p(v)$ be the disk of radius p around the vertex v. Suppose the triangles with vertices at v are numbered counterclockwise as $T^{[1]}, \ldots, T^{[E+1]}$. Finally, let $\mathscr{D}$ denote the following set of domain points:*

1. *All domain points in $T^{[1]} \cap D_p(v)$.*
2. *For each $l = 2, \ldots, E+1$, the domain points in the last $p-r$ rows of $T^{[l]} \cap D_p(v)$ adjoining $T^{[l-1]}$, and outside of $R_\rho(v)$.*

Then $\mathscr{D}$ is a minimal determining set for $\mathscr{S}_d^{r,\rho}(\Delta)$ on D with

$$\#\mathscr{D} = \binom{p+2}{2} + \left[\binom{p-r+1}{2} - \binom{\rho-r+1}{2}\right]E. \tag{3.8}$$

Proof. This lemma can be established in the same way as Lemma 2.3 in [15].■

Our next lemma deals with finding a minimal determining set for $\mathscr{S}_d^{r,\rho}(\Delta)$ in the case where D is a disk around an interior vertex.

Lemma 3.2. *Let $r \leq \rho \leq p < d$ and let v be an interior vertex of Δ with E edges attached, where e of them have different slopes. Let $D = D_p(v)$ be the disk of radius p around v. Then there exists a subset $\mathscr{D}$ of D with*

$$\#\mathscr{D} = \binom{\rho+2}{2} + \left[\binom{p-r+1}{2} - \binom{\rho-r+1}{2}\right]E + \sum_{j=\rho-r+1}^{p-r} (r+j+1-je)_+ \tag{3.9}$$

such that $\mathscr{D}$ determines $\mathscr{S}_d^{r,\rho}(\Delta)$ on D.

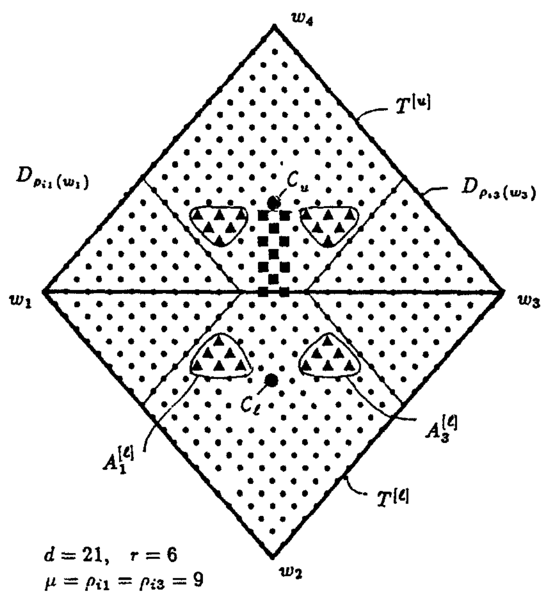

Fig. 1. Two triangles sharing a nondegenerate edge.

Proof. The statement about the cardinality can be established along the same lines as the proof of Theorem 2.2 in [14]. An explicit minimal determing set can be constructed using the methods of [14]. In general, there may be many choices which work. ■

In the remainder of this section we deal with sets of domain points which lie within a distance of at most r from an interior edge of the triangulation. In order to simplify notation, for the time being, we work with just one pair of triangles $T^{[l]}$ and $T^{[u]}$. Suppose these triangles have vertices w_1, w_2, w_3, w_4 as in Fig. 1. Then associated with the common edge $\overline{w_1 w_3}$, we define

$$\Sigma = \{P \in T^{[l]} \cup T^{[u]} \colon \text{distance}\,(P, \overline{w_1 w_3}) \le r\}. \tag{3.10}$$

Lemma 3.3. *Let $r < j \le d$, and suppose that the points in $R_j(w_1) \cap \Sigma$ are numbered as $\xi_{-r}, \dots, \xi_r$ so that ξ_{-i} and ξ_i are the points in $T^{[l]}$ and $T^{[u]}$, respectively, which are distance i from the edge. Given $s \in \mathcal{S}^r_d(\Delta)$, let $c_i = \lambda_{\xi_i} s$, $i = -r, \dots, r$. Suppose that, for some $r + 1 \ge 2q$,*

$$\lambda_P s = 0 \qquad \textit{for all} \quad P \in D_{j-1}(w_1) \cap \Sigma \tag{3.11}$$

and

$$c_i = c_{-i} = 0, \qquad i = r - q + 1, \dots, r. \tag{3.12}$$

Then either of the following two conditions

$$c_i = 0, \qquad i = 0, \dots, r - 2q, \tag{3.13}$$

or

(3.14) $$c_{-i} = 0, \qquad i = 0, \ldots, r - 2q,$$

implies

(3.15) $$c_i = 0, \qquad \text{for all} \quad i = -r, \ldots, r.$$

Proof. The C^r smoothness conditions imply that (3.13) and (3.14) are equivalent. Now suppose that (3.13)-(3.14) hold, and that the barycentric coordinates of the vertex w_4 with respect to the triangle $T^{[i]}$ are (α, β, γ); i.e., $w_4 = \alpha w_1 + \beta w_2 + \gamma w_3$. We assume that the vertices are located such that $\gamma \neq 0$. The analysis of the case where $\gamma = 0$ is similar. In addition, we restrict our attention to the case where $r + 1 < 2q$. The case where $r + 1 = 2q$ is completely analogous.

Under the assumption that $\lambda_P s = 0$ for all $P \in D_{j-1}(w_1) \cap \Sigma$, the C^l smoothness conditions across the edge $\overline{w_1 w_3}$ for $l = r, \ldots, r - 2q + 1$ can be written as

$$M\mathbf{r} = 0$$

with $\mathbf{r} = (c_{r-q}, \ldots, c_{r-2q+1}, c_{-(r-2q+1)}, \ldots, c_{-(r-q)})^{\mathrm{T}}$, and

$$M = \begin{bmatrix} O & B \\ I & C \end{bmatrix},$$

where

(3.16) $$B = \begin{bmatrix} \binom{r}{r-2q+1}\gamma^{2q-1}\beta^{r-2q+1} & \cdots & \binom{r}{r-q}\gamma^{q}\beta^{r-q} \\ \vdots & \ddots & \vdots \\ \binom{r-q+1}{r-2q+1}\gamma^{q}\beta^{r-2q+1} & \cdots & \binom{r-q+1}{r-q}\gamma\beta^{r-q} \end{bmatrix},$$

O is a $q \times q$ zero matrix, I is a $q \times q$ identity matrix, and C is a certain $q \times q$ matrix. Factoring powers of γ and β out of B and removing common factorials from the rows, we see that the determinant of B is a nonzero constant multiple of

$$G = \det \begin{vmatrix} \frac{1}{q!} & \frac{1}{(q-1)!} & \cdots & \frac{1}{1!} \\ \vdots & & \ddots & \vdots \\ \frac{1}{(2q-1)!} & \frac{1}{(2q-2)!} & \cdots & \frac{1}{q!} \end{vmatrix}.$$

Since this determinant corresponds to performing interpolation of the values $p(1), p'(1), \ldots, p^{(q-1)}(1)$ using polynomials which are linear combinations of $x^q/q!, \ldots, x^{2q-1}/(2q-1)!$, it immediately follows that the determinant of G and thus of M is nonzero, and (3.15) follows. ■

We now need some additional notation. Suppose the triangle $T^{[l]}$ has the three vertices $w_1 = v_{i1(l)}$, $w_2 = v_{i2(l)}$, $w_3 = v_{i3(l)}$ as in Fig. 1. Then associated with $T^{[l]}$ we define

$$A_1^{[l]} = \{P_{d-i,i-j,j}^{[l]}\}_{j=i-r,\ i=\rho_{i1(l)}+1}^{r,\ 2r}. \tag{3.17}$$

This is the set of domain points in $T^{[l]} \cap \Sigma$ which are within a distance r to the edge $\overline{w_1 w_2}$ but outside $D_{\rho_{i1(l)}}(w_1)$, where Σ is the strip defined in (3.10). We call $A_1^{[l]}$ the *cap* associated with the vertex $v_{i1(l)}$ of $T^{[l]}$. A typical such cap is illustrated in Fig. 1. For later use, we note that

$$\# A_1^{[l]} = \binom{2r - \rho_{i1(l)} + 1}{2}. \tag{3.18}$$

Clearly, we can define similar caps $A_2^{[l]}$ and $A_3^{[l]}$ associated with the vertices $w_2 = v_{i2(l)}$ and $w_3 = v_{i3(l)}$ of $T^{[l]}$.

Before stating our next lemma, we need to introduce another useful concept. Consider a neighboring pair of triangles as in Fig. 1 and 2. We say (see [3]) that the edge $\overline{w_1 w_3}$ is *degenerate with respect to the vertex* w_1 provided that the edges $\overline{w_1 w_2}$ and $\overline{w_1 w_4}$ are collinear (see Fig. 2). An analogous definition applies with respect to the vertex w_3.

Lemma 3.4. *Suppose that $T^{[l]}$ and $T^{[u]}$ are a pair of neighboring triangles as in Fig. 1 with vertices $w_1 = v_{i1(l)}$, $w_2 = v_{i2(l)}$, $w_3 = v_{i3(l)}$, w_4. In addition, suppose that the edge $\overline{w_1 w_3}$ is not degenerate with respect to either w_1 or w_3, and that $\mu \le \rho_{i1(l)}, \rho_{i3(l)}$ with*

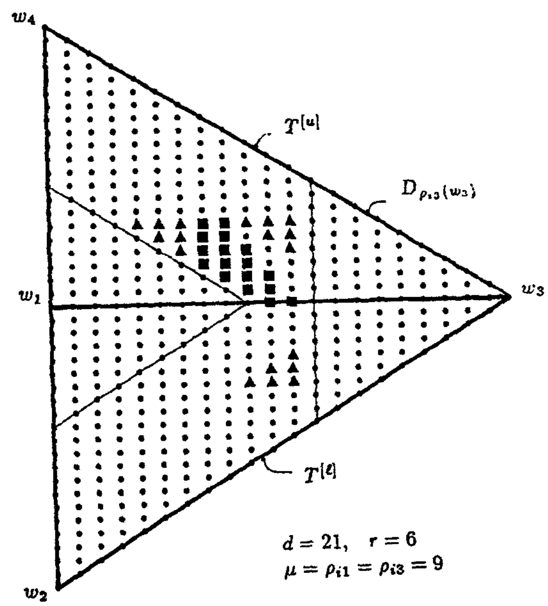

Fig. 2. Two triangles sharing a degenerate edge.

$\rho_{i1(l)} + \rho_{i3(l)} < d$. *Let*

$$A = A_1^{[l]} \cup A_1^{[u]} \cup A_3^{[l]} \cup A_3^{[u]}. \tag{3.19}$$

Then there exists a subset $\mathscr{E}$ *of* $\mathscr{F} \cap T^{[l]}$ *with*

$$\mathscr{F} = [\Sigma \backslash (D_{\rho_{i1(l)}}(w_1) \cup D_{\rho_{i3(l)}}(w_3))] \tag{3.20}$$

such that $\mathscr{E} \cup A$ *determines* $\mathscr{S}_d^r(\Delta)$ *on* $\mathscr{F}$ *and*

$$\#(\mathscr{E} \cup A) = \frac{(r+1)(2d - 2\rho_{i1(l)} - 2\rho_{i3(l)} + r - 2]}{2}. \tag{3.21}$$

Proof. We divide the analysis into three cases. To simplify notation, we drop the l when referring to $i1(l)$, $i2(l)$, and $i3(l)$.

Case 1 ($2r \le \rho_{i1} \le \rho_{i3}$). In this case the set A in (3.19) is empty, and we simply choose

$$\mathscr{E} = \mathscr{F} \cap T^{[l]}.$$

It is easy to check that (3.21) holds.

Case 2 ($\mu \le \rho_{i1} < 2r \le \rho_{i3}$). We apply Lemma 3.3 to choose a total of $r + 1$ points on each of the rings $R_{\rho_{i1}+1}(w_1), \ldots, R_{2r}(w_1)$. For each such ring we can include the points which lie in $A_1^{[u]} \cap \mathscr{F}$ and $A_1^{[l]} \cap \mathscr{F}$. To get $\mathscr{E}$ we add all of the points in $[\Sigma \backslash (D_{2r}(w_1) \cup D_{\rho_{i3}}(w_3))] \cap T^{[l]}$.

Case 3 ($\mu \le \rho_{i1} \le \rho_{i3} < 2r$). We use Lemma 3.3, starting with the ring $R_{\rho_{i1}+1}(w_1)$. We then do one ring at a time until we reach the ring $R_{d-\rho_{i3}}(w_1)$. Next we do the ring $R_{\rho_{i3}+1}(w_3)$, then ring $R_{d-\rho_{i3}+1}(w_1)$, and continue in this way alternating between rings about w_1 and w_3 until all the points in $\mathscr{F}$ are accounted for. As before, (3.21) holds. ∎

The situation is slightly different when $\overline{w_1 w_3}$ is a degenerate edge.

Lemma 3.5. *Suppose that* $T^{[l]}$ *and* $T^{[u]}$ *are a pair of triangles as in Lemma* 3.4 *such that the edge* $\overline{w_1 w_3}$ *is degenerate with respect to the vertex* w_1 *(see Fig.* 2*). Let* $\mathscr{F}$ *be the set defined in* (3.20)*, and suppose that* $\mu \le \rho_{i1(l)}, \rho_{i3(l)}$ *with* $\rho_{i1(l)} + \rho_{i3(l)} < d$. *Then there exists* $\mathscr{E} \subseteq \mathscr{F} \cap T^{[l]}$ *such that* $\mathscr{E} \cup A$ *determines* $\mathscr{S}_d^r(\Delta)$ *on* $\mathscr{F}$, *where*

$$A = A_1^{[l]} \cup A_3^{[l]} \cup A_3^{[u]} \tag{3.22}$$

and

$$\#(\mathscr{E} \cup A) = \frac{(r+1)(2d - 2\rho_{i1(l)} - 2\rho_{i2(l)} + r - 2)}{2}. \tag{3.23}$$

Proof. The proof is based on applying Lemma 3.3 to each of the rings $R_i(w_1)$ for $i = \rho_{i1(l)} - r + 1, \ldots, d - \rho_{i3(l)} - 1$. To simplify notation, we drop the l when referring to $i1(l)$, $i2(l)$, and $i3(l)$. We distinguish two cases.

Case 1 ($d > \rho_{i1} + 2r$). For $i = \rho_{i1} - r + 1, \ldots, r$, the points in $D_i(w_1) \cup A_1^{[l]}$ determine all points in $\mathscr{F}$ on the ring $R_i(w_1)$. For $i = r + 1, \ldots, \rho_{i1}$, Lemma 3.3 asserts that we can choose $r + i - \rho_{i1}$ points in $R_i(w_1) \cap T^{[l]}$. For each $i = \rho_{i1} + 1, \ldots, d - 2r - 1$, we may choose the $r + 1$ points in $R_i(w_1) \cap T^{[l]}$. Finally, for $i = d - 2r, \ldots, d - \rho_{i3} - 1$ Lemma 3.3 allows us to choose $2d - 3r - 1 - 2i$ points on each ring (in addition to the points in the two sets $A_3^{[u]}$ and $A_3^{[l]}$). The total number of points chosen is then

$$\binom{2r - \rho_{i1} + 1}{2} + 2\binom{2r - \rho_{i3} + 1}{2} + \sum_{i=r+1}^{\rho_{i1}} (r + i - \rho_{i1})$$
$$+ (r + 1)(d - 2r - \rho_{i1} - 1) + \sum_{i=d-2r}^{d-\rho_{i3}-1} (2d - 3r - 1 - 2i),$$

which is easily seen to be the number in (3.23).

Case 2 ($d \le \rho_{i1} + 2r$). For $i = \rho_{i1} - r + 1, \ldots, r$, the points in $D_{\rho_{i1}}(w_1) \cup A_1^{[l]}$ determine all points in $R_i(w_1) \cap \mathscr{F}$. For $i = r + 1, \ldots, d - 2r - 1$, Lemma 3.3 requires that we choose $r + i - \rho_{i1}$ points. For each $i = d - 2r, \ldots, \rho_{i1}$, we may choose $2d - 3r - \rho_{i1} - 2 - i$ points in $R_i(w_1) \cap T^{[l]}$. Finally, for $i = \rho_{i1} + 1, \ldots, d - \rho_{i3} - 1$ Lemma 3.3 says that we can choose $2d - 3r - 1 - 2i$ points on each ring in addition to the points in the two sets $A_3^{[u]}$ and $A_3^{[l]}$. The total number of points is then

$$\binom{2r - \rho_{i1} + 1}{2} + 2\binom{2r - \rho_{i3} + 1}{2} + \sum_{i=r+1}^{d-2r-1} (r + i - \rho_{i1})$$
$$+ \sum_{i=d-2r}^{\rho_{i1}} (2d - 3r - \rho_{i1} - 2 - i) + \sum_{i=\rho_{i1}+1}^{d-\rho_{i3}-1} (2d - 3r - 1 - 2i),$$

which is easily seen to be the number in (3.23). ■

4. Proof of Theorem 2.2

In this section we establish Theorem 2.2. Our approach is based on the idea introduced in [6] of dividing the set of domain points $\mathscr{P}$ into subsets, and then choosing minimal determining sets for each of them.

First we assume that $\mu \le \rho \le 2r$ and $d \ge 2\rho + 1$ and prove formula (2.9). As in Section 3, let $T^{[1]}, \ldots, T^{[N]}$ be the triangles of Δ. We begin by dealing with certain disks surrounding each of the vertices. For each vertex v_i in Δ, let

$$\mathscr{D}_i = D_\rho(v_i) \cap T^{[l_i]}, \tag{4.1}$$

where $T^{[l_i]}$ is some triangle with vertex at v_i. Clearly, $\mathscr{D}_i$ is a minimal determining set for $D_\rho(v_i)$, and

$$\#\mathscr{D}_i = \binom{\rho + 2}{2}.$$

We next choose certain of the caps defined in (3.17). Let v_l be a typical vertex, and let $\mathscr{G}_l$ be the union of the caps $A^{[j]}$, where $T^{[j]}$ is a triangle with vertex at v_l. If v_l is a boundary vertex, let $\mathscr{A}_l = \mathscr{G}_l$. If v_l is a nonsingular interior vertex, we take $\mathscr{A}_l$ to be the set $\mathscr{G}_l$ minus the union of those caps $A^{[j]}$ such that the first edge of $T^{[j]}$ (where the edges are ordered in counterclockwise order) is degenerate with respect to v_l. Finally, if v_l is a singular interior vertex, let $\mathscr{A}_l = A_1^{[j]}$, where $T^{[j]}$ is any triangle with vertex at v_l. Clearly, we have

$$\#\mathscr{A}_l = \binom{2r - \rho + 1}{2}[E_l - E_l^{\mathrm{D}} + \delta_l], \tag{4.2}$$

where E_l is the number of interior edges attached to v_l, E_l^{D} is the number of degenerate edges attached to v_l, and

$$\delta_l = \begin{cases} 1 & \text{if } v_l \text{ is a boundary vertex or a singular interior vertex,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Next we deal with the points in the middle of each triangle. Let

$$\mathscr{C}_l = C_l := \{(iw_1 + jw_2 + kw_3)/d \colon i > r, j > r, k > r\}, \tag{4.4}$$

where w_1, w_2, w_3 are the vertices of $T^{[l]}$. The set $\mathscr{C}_l$ is the set of Bézier points lying in the triangle $T^{[l]}$ which are at least r rows away from the boundary. Clearly, $\mathscr{C}_l$ is a minimal determining set for C_l with

$$\#\mathscr{C}_l = \binom{d - 3r - 1}{2}. \tag{4.5}$$

A typical set $\mathscr{C}_l$ is shown in Fig. 1 (where $d = 21$, $r = 6$, and $\rho = \mu = 9$).

We now consider domain points in strips near the edges $\varepsilon_1, \ldots, \varepsilon_E$ of Δ. Fix $1 \le i \le E$. If ε_i is an interior edge, we suppose that $\varepsilon_i = \overline{w_1 w_3}$, where $T^{[l]}$ and $T^{[u]}$ are the two triangles sharing the edge. We denote the vertices of these triangles as in Fig. 1. If ε_i is a boundary edge, we suppose that $\varepsilon_i = \overline{w_1 w_3}$, where w_1, w_2, w_3 are the vertices of a triangle $T^{[l]}$. Let Σ_i be the set of domain points whose distance from the edge is at most r (see. (3.10). Finally, set

$$\mathscr{F}_i = \Sigma_i \backslash [D_\rho(w_1) \cup D_\rho(w_3)]. \tag{4.6}$$

It is easy to check that

$$\#\mathscr{F}_i = \frac{(r + 1)(2d - 4\rho + r - 2)}{2}.$$

We now identify minimal determining subsets for each of the $\mathscr{F}_i$. If ε_i is a boundary edge associated with triangle $T^{[l]}$, set

$$\mathscr{U}_i = A_1^{[l]} \cup A_3^{[l]}, \tag{4.7}$$

where the sets $A_1^{[l]}$ and $A_3^{[l]}$ are defined as in (3.17). Then $\mathscr{E}_i = \mathscr{F}_i \backslash \mathscr{U}_i$ is such that $\mathscr{E}_i \cup \mathscr{U}_i$ is a determining set for $\mathscr{F}_i$.

If ε_i is an interior edge, we may use Lemmas 3.4 and 3.5. In particular, if ε_i is nondegenerate, we take $\mathscr{E}_i$ to be the set constructed in Lemma 3.4 such that $\mathscr{E}_i \cup \mathscr{U}_i$

determines s on $\mathscr{F}_i$, where

$$\mathscr{U}_i = A_1^{[l]} \cup A_3^{[l]} \cup A_1^{[u]} \cup A_3^{[u]}. \tag{4.8}$$

A typical set $\mathscr{E}_i$ is shown in Fig. 1 (which corresponds to the case where $d = 21$, $r = 6$, $\rho = \mu = 9$).

If ε_i is a degenerate edge, say with respect to w_1, we define $\mathscr{E}_i$ to be the set constructed in Lemma 3.5 such that $\mathscr{E}_i \cup \mathscr{U}_i$ determines s on $\mathscr{F}_i$, where

$$\mathscr{U}_i = A_1^{[l]} \cup A_3^{[l]} \cup A_3^{[u]}. \tag{4.9}$$

A typical set $\mathscr{E}_i$ for this case is shown in Fig. 2 (which corresponds to the case where $d = 21$, $r = 6$, $\rho = \mu = 9$).

In all cases we have

$$\#(\mathscr{E}_i \cup \mathscr{U}_i) = \#\mathscr{F}_i = \frac{(r+1)(2d - 4\rho + r - 2)}{2}.$$

We are now ready to describe the complete minimal determining set for $\mathscr{S}_d^{r,\rho}(\Delta)$.

Lemma 4.1. *Let*

$$\Gamma = \bigcup_{l=1}^{V} \mathscr{A}_l \cup \bigcup_{l=1}^{N} \mathscr{C}_l \cup \bigcup_{i=1}^{V} \mathscr{D}_i \cup \bigcup_{i=1}^{E} \mathscr{E}_i. \tag{4.10}$$

Then Γ is a determining set for $\mathscr{S}_d^{r,\rho}(\Delta)$, and its cardinality is given by the number in (2.9).

Proof. The set Γ is illustrated for $d = 8$, $r = 1$, $\rho = \mu = 2$ in Fig. 3. First we show that Γ is a determining set for $\mathscr{S}_d^{r,\rho}(\Delta)$. Suppose $s \in \mathscr{S}_d^{r,\rho}(\Delta)$. Clearly, s is completely

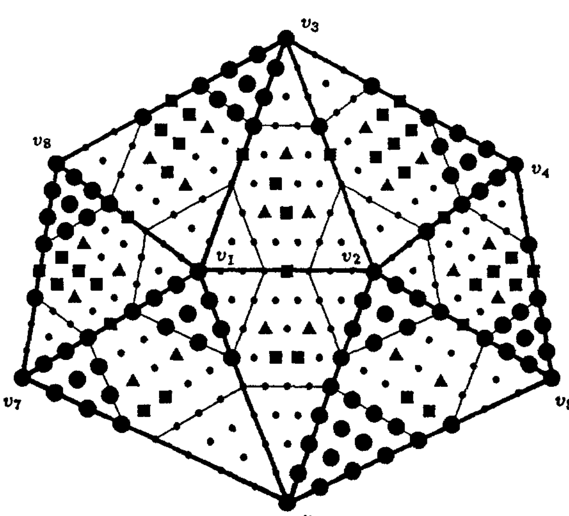

Fig. 3. The minimal determining set for Example 2.5.

determined on the disks $D_\rho(v_i)$, $i = 1, \ldots, V$, as well as in the sets $\mathscr{C}_l$, $l = 1, \ldots, N$. In addition, we claim that s is determined on all of the caps. Indeed, if v_i is a boundary vertex, then all the Bézier points in the caps near this vertex are included in Γ. Now if v_i is a singular interior vertex, then in one of the triangles with vertex at v_i, say $T^{[l_i]}$, all of the points in the cap nearest the vertex are included in the first set in (4.10). By the C^r continuity conditions, s is determined on the caps nearest v_i in the other three triangles with vertex at v_i. Finally, if v_i is a nonsingular vertex, then s is determined on those caps which are not included in Γ by using the C^r continuity conditions across degenerate edges. We have established that s is determined on all of the caps in all of the triangles. Now in view of Lemmas 3.4 and 3.5, s is also determined on the sets $\mathscr{F}_i$, $i = 1, \ldots, E$, by the points in the last term of (4.10). This completes the proof that the set Γ determines s.

We now compute the cardinality of Γ. Clearly,

$$
\begin{aligned}
(4.11)\quad \#\Gamma = \binom{\rho+2}{2}V &+ \binom{2r-\rho+1}{2}[2E_{\mathrm{I}} - E_{\mathrm{I}}^{\mathrm{D}} + V_{\mathrm{B}} + S] \\
&+ \binom{d-3r-1}{2}N + \left[\frac{(r+1)(2d-4\rho+r-2)}{2} - 2\binom{2r-\rho+1}{2}\right]E_{\mathrm{B}} \\
&+ \left[\frac{(r+1)(2d-4\rho+r-2)}{2} - 3\binom{2r-\rho+1}{2}\right]E_{\mathrm{I}}^{\mathrm{D}} \\
&+ \left[\frac{(r+1)(2d-4\rho+r-2)}{2} - 4\binom{2r-\rho+1}{2}\right]E_{\mathrm{I}}^{\mathrm{ND}},
\end{aligned}
$$

where $E_{\mathrm{I}}^{\mathrm{D}}$ is the number of degenerate interior edges and $E_{\mathrm{I}}^{\mathrm{ND}}$ is the number of nondegenerate interior edges. Now since $2E_{\mathrm{I}} + V_{\mathrm{B}} = 3N$, it follows that (4.11) reduces to (2.9). ■

We now have the ingredients to establish formula (2.9) in Theorem 2.2 and at the same time to present an explicit basis of minimally supported splines.

Theorem 4.2. *Suppose $r + \lfloor (r+1)/2 \rfloor \le \rho \le 2r$ with $d \ge 2\rho + 1$. Then, for each P in the set Γ described in Lemma 4.1, there exists a spline $B_P \in \mathscr{S}_d^{r,\rho}(\Delta)$ satisfying*

$$(4.12)\qquad \lambda_Q B_P = \delta_{PQ}, \qquad \text{all} \quad Q \in \Gamma.$$

The set of splines $\{B_P\}_{P\in\Gamma}$ is a basis for the space $\mathscr{S}_d^{r,\rho}(\Delta)$, and the set of linear functionals $\{\lambda_P\}_{P\in\Gamma}$ form a dual basis. Moreover, each of the splines B_P has local support. In particular:

1. *If P is in one of the sets $\mathscr{D}_i$ or $\mathscr{A}_i$, then B_P has support on a cell.*
2. *If P is one of the sets $\mathscr{E}_i$, then B_P has support on the union of a pair of neighboring triangles.*
3. *If P is in one of the sets $\mathscr{C}_i$, then B_P has support on the single triangle $T^{[i]}$.*

Proof. Suppose $P \in \mathscr{D}_i$ for some i. To define B_P, we need only give the value of the coefficients of each of its polynomial pieces. We set the coefficient corresponding to

P equal to one, and set all coefficients corresponding to $Q \in \Gamma$ with $Q \neq P$ equal to zero. Now it is easy to see (see the analogous arguments in [4] and [5]) that the other coefficients can be chosen to satisfy all smoothness conditions in such a way that all coefficients outside of the disk $D_\rho(v_i)$ are zero. It follows that B_P has support on the disk. Similar arguments can be used to show that the other basis splines have the stated supports.

Clearly, these splines are linearly independent because of condition (4.12). This implies dim $\mathcal{S}_d^{r,\rho}(\Delta) \geq L$, where L is the number in (2.9). On the other hand, since as shown in Lemma 4.1, Γ is a determining set for $\mathcal{S}_d^{r,\rho}(\Delta)$, if follows (see Lemma 3.3 of [11]) that dim $\mathcal{S}_d^{r,\rho}(\Delta) \leq L$, and we conclude that dim $\mathcal{S}_d^{r,\rho}(\Delta) = L$. The remaining statements are now obvious. ■

The remainder of this section is devoted to proving formula (2.11) in Theorem 2.2. In this case we work with the disks

$$D_\mu(v_i), \qquad i = 1, \ldots, V, \tag{4.13}$$

where μ is defined in (2.10). For fixed $1 \leq i \leq V$, if v_i is a boundary vertex, then by Lemma 3.1, we can choose a set $\mathcal{D}_i \subseteq D_\mu(v_i)$ with

$$\#\mathcal{D}_i = \binom{\mu+2}{2} + \left[\binom{\mu-r+1}{2} - \binom{\rho-r+1}{2}\right]E_i$$

which determines $s \in \mathcal{S}_d^{r,\rho}(\Delta)$ on $D_\mu(v_i)$, where E_i is the number of interior edges attached to v_i. Similarly, if v_i is an interior vertex, then, by Lemma 3.2, we can choose a set $\mathcal{D}_i \subseteq D_\mu(v_i)$ with

$$\#\mathcal{D}_i = \binom{\rho+2}{2} + \left[\binom{\mu-r+1}{2} - \binom{\rho-r+1}{2}\right]E_i + \sum_{j=\rho-r+1}^{\mu-r} (r+j+1-je_i)_+$$

which determines $s \in \mathcal{S}_d^{r,\rho}(\Delta)$ on $D_\mu(v_i)$, where we recall that E_i is the number of edges attached to v_i, and e_i is the number of those with different slopes.

To deal with domain points in a strip near an edge, we introduce the following analog of (4.6):

$$\mathcal{F}_i = \Sigma_i \backslash (D_\mu(w_1) \cup D_\mu(w_3)), \tag{4.14}$$

where we use the same notation as before. When ε_i is a boundary edge associated with triangle $T^{[l]}$, we take

$$\mathcal{U}_i = A_1^{[l]} \cup A_3^{[l]}, \tag{4.15}$$

where the sets $A_1^{[l]}$ and $A_3^{[l]}$ are defined as in (3.17). Note that in this case

$$\#A_1^{[l]} = \#A_3^{[l]} = \binom{2r-\mu+1}{2}. \tag{4.16}$$

Clearly, the set $\mathcal{E}_i = \mathcal{F}_i \backslash \mathcal{U}_i$ is such that $\mathcal{E}_i \cup \mathcal{U}_i$ is a determining set for $\mathcal{F}_i$.

For interior edges we may use Lemmas 3.4 and 3.5. If ε_i is nondegenerate, we take $\mathcal{E}_i$ as constructed in Lemma 3.4 such that the set $\mathcal{E}_i \cup \mathcal{U}_i$ determines $\mathcal{S}_d^{r,\rho}(\Delta)$ on $\mathcal{F}_i$ where $\mathcal{U}_i$ is defined as in (4.8). If ε_i is a degenerate edge, say with respect to

w_1, then we define $\mathscr{E}_i$ to be the set constructed in Lemma 3.5 such that $\mathscr{E}_i \cup \mathscr{U}_i$ determines $\mathscr{S}_d^{r,\rho}(\Delta)$ on $\mathscr{F}_i$ where $\mathscr{U}_i$ is defined as in (4.9). We note that in all cases

$$\#(\mathscr{E}_i \cup \mathscr{U}_i) = \#\mathscr{F}_i = \frac{(r+1)(2d-4\mu+r-2)}{2}.$$

Finally, we choose point sets $\mathscr{A}_l$ for $l = 1, \ldots, V$ and $\mathscr{C}_l$ for $l = 1, \ldots, N$ exactly as was done above in the proof of formula (2.9) of Theorem 2.2.

Lemma 4.3. *Let*

$$\Gamma = \bigcup_{l=1}^{V} \mathscr{A}_l \cup \bigcup_{l=1}^{N} \mathscr{C}_l \cup \bigcup_{i=1}^{V} \mathscr{D}_i \cup \bigcup_{i=1}^{E} \mathscr{E}_i. \tag{4.17}$$

Then Γ is a determining set for $\mathscr{S}_d^{r,\rho}(\Delta)$, and its cardinality is given by the number in (2.11).

Proof. The proof that Γ is a determining set for $\mathscr{S}_d^{r,\rho}(\Delta)$ proceeds exactly as in the proof of Lemma 4.1. Clearly,

$$\begin{aligned}\#\Gamma = {} & \binom{2r-\mu+1}{2}[2E_\mathrm{I} - E_\mathrm{I}^\mathrm{D} + V_\mathrm{B} + S] + \binom{d-3r-1}{2}N + \binom{\mu+2}{2}V_\mathrm{B} \\ & + \binom{\rho+2}{2}V_\mathrm{I} + 2\left[\binom{\mu-r+1}{2} - \binom{\rho-r+1}{2}\right]E_\mathrm{I} \\ & + \sum_{i=1}^{V_\mathrm{I}} \sum_{j=\rho-r+1}^{\mu-r} (r+j+1-je_i)_+ \\ & + \left[\frac{(r+1)(2d-4\mu+r-2)}{2} - 2\binom{2r-\mu+1}{2}\right]E_\mathrm{B} \\ & + \left[\frac{(r+1)(2d-4\mu+r-2)}{2} - 3\binom{2r-\mu+1}{2}\right]E_\mathrm{I}^\mathrm{D} \\ & + \left[\frac{(r+1)(2d-4\mu+r-2)}{2} - 4\binom{2r-\mu+1}{2}\right]E_\mathrm{I}^\mathrm{ND},\end{aligned} \tag{4.18}$$

where, as before, E_I^D is the number of degenerate interior edges and E_I^ND is the number of nondegenerate interior edges. Now using $2E_\mathrm{I} + V_\mathrm{B} = 3N$ and the fact that the number σ in (2.11) is equal to

$$\sum_{i=1}^{V_\mathrm{I}} \sum_{j=\rho-r+1}^{\mu-r} (r+j+1-je_i)_+ + \binom{2r-\mu+1}{2}S,$$

it follows that (4.18) reduces to (2.11). ∎

We can now establish the third case in Theorem 2.2 by presenting a basis of minimally supported splines for the space of super splines in (2.5).

Theorem 4.4. *Suppose $r \le \rho \le \mu$ with $d \ge 3r + 2$, where μ is the integer defined in (2.10). Then, for each P in the set Γ described in Lemma 4.3, there exists a spline $B_P \in \mathcal{S}_d^{r,\rho}(\Delta)$ satisfying (4.12). The set of splines $\{B_P\}_{P\in\Gamma}$ is a basis for the space $\mathcal{S}_d^{r,\rho}(\Delta)$, and the set of linear functionals $\{\lambda_P\}_{P\in\Gamma}$ form a dual basis. Moreover, each of the splines B_P has local support. In particular*:

1. *If P is in one of the sets $\mathcal{D}_i$ or $\mathcal{A}_i$, then B_P has support on a cell.*
2. *If P is in one of the sets $\mathcal{E}_i$, then B_P has support on the union of a pair of neighboring triangles.*
3. *If P is in one of the sets $\mathcal{C}_i$, then B_P has support on the single triangle $T^{[i]}$.*

Proof. The proof of the existence of splines satisfying (4.12) and with the stated supports proceeds exactly as in Theorem 4.2. Clearly, these splines are linearly independent because of condition (4.12). This implies $\dim \mathcal{S}_d^{r,\rho}(\Delta) \ge L$, where L is the number in (2.11). On the other hand, since, as shown in Lemma 4.3, Γ is a determining set for $\mathcal{S}_d^{r,\rho}(\Delta)$, it follows (see Lemma 3.3 of [11]) that $\dim \mathcal{S}_d^{r,\rho}(\Delta) \le L$, and we conclude that $\dim \mathcal{S}_d^{r,\rho}(\Delta) = L$. The remaining statements are now obvious. ∎

5. Proof of Theorem 2.4

In this section we establish Theorem 2.4 on the dimension of the spline space $\mathcal{S}_d^{r,\theta}(\Delta)$ defined in (2.3), and give a local basis for it. Our approach is similar to the proofs of Theorem 2.2 presented in Section 4. We need to describe a set of domain points which determine $\mathcal{S}_d^{r,\theta}(\Delta)$.

First, we consider determining $s \in \mathcal{S}_d^{r,\theta}(\Delta)$ on disks centered at the vertices of Δ. Here we use disks of radius k_i, where, as in (2.14), $k_i = \max(\rho_i, \mu)$. For fixed $1 \le i \le V$, if v_i is a boundary vertex, then, by Lemma 3.1, we can choose a set $\mathcal{D}_i \subseteq D_{k_i}(v_i)$ with

$$\#\mathcal{D}_i = \binom{k_i+2}{2} + \left[\binom{k_i - r + 1}{2} - \binom{\rho_i - r + 1}{2}\right] E_i$$

which determines $s \in \mathcal{S}_d^{r,\theta}(\Delta)$ on $D_{k_i}(v_i)$, where E_i is the number of interior edges attached to v_i.

Similarly, if v_i is an interior vertex, then, by Lemma 3.2, we can choose a set $\mathcal{D}_i \subseteq D_{k_i}(v_i)$ with

$$\#\mathcal{D}_i = \binom{\rho_i+2}{2} + \left[\binom{k_i - r + 1}{2} - \binom{\rho_i - r + 1}{2}\right] E_i + \sum_{j=\rho_i - r+1}^{k_i - r} (r + j + 1 - je_i)_+$$

which determines $s \in \mathcal{S}_d^{r,\theta}(\Delta)$ on $D_{k_i}(v_i)$, where we recall that e_i is the number of edges attached to v_i, with different slopes.

We also need to choose certain cap sets. Let v_l be a typical vertex, and let $\mathscr{G}_l$ be the union of the sets $G_j := A_1^{[j]} \setminus D_{k_l}(v_l)$, where $A_1^{[j]}$ is the cap defined in (3.17), and where $T^{[j]}$ is a triangle with vertex at v_l. If v_l is a boundary vertex, let $\mathscr{A}_l = \mathscr{G}_l$. If v_l is a nonsingular interior vertex, we take $\mathscr{A}_l$ to be the set $\mathscr{G}_l$ minus the union of those G_j corresponding to triangles $T^{[j]}$ whose first edge (where the edges are ordered in counterclockwise order) is degenerate with respect to v_l. Finally, if v_l is a singular interior vertex, let $\mathscr{A}_l = G_j$, where $T^{[j]}$ is any triangle with vertex at v_l. Clearly, we have

$$(5.1) \qquad \#\mathscr{A}_l = \binom{2r - k_l + 1}{2}[E_l - E_l^{\mathrm{D}} + \delta_l],$$

where, as before, E_l is the number of interior edges attached to v_l, E_l^{D} is the number of degenerate edges attached to v_l, and δ_l is defined in (4.3).

Next we turn to the triangles. For given $1 \le l \le N$, let $T^{[l]}$ denote the lth triangle, and let C_l be the set defined in (4.4). Suppose the vertices of the triangle are $w_1 = v_{i1(l)}$, $w_2 = v_{i2(l)}$, $w_3 = v_{i3(l)}$ as in Fig. 1. We define

$$\tilde{\mathscr{C}}_l = C_l \setminus (D_{k_{i1(l)}}(w_1) \cup D_{k_{i2(l)}}(w_2) \cup D_{k_{i3(l)}}(w_3)).$$

Note that

$$(5.2) \quad \#\tilde{\mathscr{C}}_l = \binom{d - 3r - 1}{2} - \binom{k_{i1(l)} - 2r}{2} - \binom{k_{i2(l)} - 2r}{2} - \binom{k_{i3(l)} - 2r}{2}.$$

Next, we consider domain points in strips near the edges of Δ. Fix $1 \le i \le E$. If ε_i is an interior edge, we suppose that $\varepsilon_i = \overline{w_1 w_3}$, where $T^{[l]}$ and $T^{[u]}$ are the two triangles sharing the edge. We denote the vertices of these triangles as in Fig. 1. If ε_i is a boundary edge, we suppose that $\varepsilon_i = \overline{w_1 w_3}$, where w_1, w_2, w_3 are the vertices of a triangle $T^{[l]}$. Let Σ_i be the set of domain points whose distance from the edge is at most r (see (3.10)). Finally, let

$$(5.3) \qquad \mathscr{F}_i = \Sigma_i \setminus (D_{k_{i1(l)}}(w_1) \cup D_{k_{i3(l)}}(w_3)).$$

We now identify minimal determining subsets for each of the $\mathscr{F}_i$. If ε_i is a boundary edge, let

$$(5.4) \qquad \mathscr{U}_i = A_1^{[u]} \cup A_3^{[u]}.$$

Then the set $\mathscr{E}_i = \mathscr{F}_i \setminus \mathscr{U}_i$ is such that $\mathscr{E}_i \cup \mathscr{U}_i$ is a determining set for $\mathscr{F}_i$. We have

$$\#\mathscr{E}_i = \frac{(r+1)(2d - 2k_{i1(l)} - 2k_{i3(l)} + r - 2)}{2} - \left[\binom{2r - k_{i1(l)} + 1}{2} + \binom{2r - k_{i3(l)} + 1}{2}\right].$$

If ε_i is an interior edge, we may use Lemmas 3.4 and 3.5. In particular, if ε_i is nondegenerate, we take $\mathscr{E}_i$ to be the set constructed in Lemma 3.4 such that $\mathscr{E}_i \cup \mathscr{U}_i$ determines s on $\mathscr{F}_i$, where

$$\mathscr{U}_i = A_1^{[l]} \cup A_3^{[l]} \cup A_1^{[u]} \cup A_3^{[u]}.$$

In this case

$$\#\mathscr{E}_i = \frac{(r+1)(2d - 2k_{i1(l)} - 2k_{i3(l)} + r - 2)}{2} - 2\left[\binom{2r - k_{i1(l)} + 1}{2} + \binom{2r - k_{i3(l)} + 1}{2}\right].$$

If ε_i is a degenerate edge, say with respect to w_1, we define $\mathscr{E}_i$ to be the set constructed in Lemma 3.5 such that $\mathscr{E}_i \cup \mathscr{U}_i$ determines s on $\mathscr{F}_i$, where

$$\mathscr{U}_i = A_1^{[l]} \cup A_3^{[l]} \cup A_3^{[u]}.$$

In this case

$$\#\mathscr{E}_i = \frac{(r+1)(2d - 2k_{i1(l)} - 2k_{i3(l)} + r - 2)}{2} - \left[\binom{2r - k_{i1(l)} + 1}{2} + 2\binom{2r - k_{i3(l)} + 1}{2}\right].$$

Lemma 5.1. *Let*

$$\Gamma = \bigcup_{l=1}^{V} \mathscr{A}_l \cup \bigcup_{l=1}^{N} \tilde{\mathscr{C}}_l \cup \bigcup_{i=1}^{V} \mathscr{D}_i \cup \bigcup_{i=1}^{E} \mathscr{E}_i. \tag{5.5}$$

Then Γ is a determining set for $\mathscr{S}_d^{r,\theta}(\Delta)$, and its cardinality is given by the number in (2.16).

Proof. The proof that Γ is a determining set for $\mathscr{S}_d^{r,\theta}(\Delta)$ follows along the same lines as in Lemmas 4.1 and 4.3. Clearly,

$$\begin{aligned}
\#\Gamma = {} & \sum_{l=1}^{V} \binom{2r - k_l + 1}{2}[E_l - E_l^{\mathrm{D}} + \delta_l] + \binom{d - 3r - 1}{2} N \\
& - \sum_{l=1}^{N} \left[\binom{k_{i1(l)} - 2r}{2} + \binom{k_{i2(l)} - 2r}{2} + \binom{k_{i3(l)} - 2r}{2}\right] \\
& + \sum_{l \in \mathscr{E}^{\mathrm{B}}} \left[\frac{(r+1)(2d - 2k_{i1(l)} - 2k_{i3(l)} + r - 2)}{2} \right. \\
& \qquad \left. - \binom{2r - k_{i1(l)} + 1}{2} - \binom{2r - k_{i3(l)} + 1}{2}\right] \\
& + \sum_{l \in \mathscr{E}_{\mathrm{I}}^{\mathrm{D}}} \left[\frac{(r+1)(2d - 2k_{i1(l)} - 2k_{i3(l)} + r - 2)}{2} \right. \\
& \qquad \left. - \binom{2r - k_{i1(l)} + 1}{2} - 2\binom{2r - k_{i3(l)} + 1}{2}\right]
\end{aligned} \tag{5.6}$$

$$+\sum_{l\in\mathscr{E}_{\mathrm{I}}^{\mathrm{ND}}}\left[\frac{(r+1)(2d-2k_{i1(l)}-2k_{i3(l)}+r-2)}{2}\right.$$

$$\left.-2\binom{2r-k_{i1(l)}+1}{2}-2\binom{2r-k_{i3(l)}+1}{2}\right]$$

$$+\sum_{i=1}^{V_{\mathrm{I}}}\left[\binom{\rho_i+2}{2}+\sum_{j=\rho_i-r+1}^{k_i-r}(r+1+j-je_i)_+\right]+\sum_{i=V_{\mathrm{I}}+1}^{V}\binom{k_i+2}{2}$$

$$+\sum_{i=1}^{V}\left[\binom{k_i-r+1}{2}-\binom{\rho_i-r+1}{2}\right]E_i,$$

where E_l, $E_{\mathrm{I}}^{\mathrm{D}}$, and δ_l are defined as in Section 4 (see (4.3)), and where

$\mathscr{E}^{\mathrm{B}}=\{l\colon T^{[l]}$ has $\overline{w_1w_3}$ as a boundary edge of $\Delta\}$,

$\mathscr{E}_{\mathrm{I}}^{\mathrm{D}}=\{l\colon T^{[l]}$ has an interior edge $\overline{w_1w_3}$ which is degenerate with respect to $v_{i1(l)}\}$,

$\mathscr{E}_{\mathrm{I}}^{\mathrm{ND}}=\{l\colon T^{[l]}$ has an interior edge $\overline{w_1w_3}$ which is nondegenerate with respect to $v_{i1(l)}\}$.

Now since the number of triangles attached to an interior vertex v_i is E_i, while the number attached to a boundary vertex v_i is E_i+1, it follows that

$$-\sum_{l=1}^{N}\left[\binom{k_{i1(l)}-2r}{2}+\binom{k_{i2(l)}-2r}{2}+\binom{k_{i3(l)}-2r}{2}\right]$$

$$=-\sum_{i=1}^{V}\binom{k_i-2r}{2}E_i-\sum_{i=V_{\mathrm{I}}+1}^{V}\binom{k_i-2r}{2}.$$

Moreover,

$$\left[\sum_{l\in\mathscr{E}^{\mathrm{B}}}+\sum_{l\in\mathscr{E}_{\mathrm{I}}^{\mathrm{D}}}+\sum_{l\in\mathscr{E}_{\mathrm{I}}^{\mathrm{ND}}}\right][(k_{i1(l)}+k_{i3(l)})]=2\sum_{i=V_{\mathrm{I}}+1}^{V}k_i+\sum_{i=1}^{V}k_iE_i.$$

Next, we observe that the terms involving factors of the form $\binom{2r-k_i+1}{2}$ can be combined to obtain

$$-\sum_{i=1}^{V}\binom{2r-k_i+1}{2}E_i-\sum_{i=V_{\mathrm{I}}+1}^{V}\binom{2r-k_i+1}{2}+\sum_{i=1}^{S}\binom{2r-k_{v_i}+1}{2},$$

where $v_1,\ldots,v_S$ are the indices of the singular vertices. Finally, it is easy to see that

$$\sum_{i=1}^{V_{\mathrm{I}}}\sum_{j=\rho_i-r+1}^{k_i-r}(r+1+j-je_i)_++\sum_{i=1}^{S}\binom{2r-k_{v_i}+1}{2}=\sigma,$$

where σ is the expression defined in (2.12). Now combining all these facts, we find that the cardinality of Γ is the number in (2.16). ■

We now establish Theorem 2.4 and at the same time present a basis of minimally supported splines for the space of super splines in (2.3).

Theorem 5.2. *Suppose the hypotheses of Theorem 2.4 hold. Then, for each P in the set Γ described in Lemma 5.1, there exists a spline $B_P \in \mathcal{S}_d^{r,\theta}(\Delta)$ satisfying*

$$\lambda_Q B_P = \delta_{PQ}, \qquad all \quad Q \in \Gamma. \tag{5.7}$$

The set of splines $\{B_P\}_{P\in\Gamma}$ is a basis for the space $\mathcal{S}_d^{r,\theta}(\Delta)$, and the set of linear functionals $\{\lambda_P\}_{P\in\Gamma}$ form a dual basis. Moreover, each of the splines B_P has local support. In particular.

1. *If P is in one of the sets $\mathcal{D}_i$ or $\mathcal{A}_i$, then B_P has support on a cell.*
2. *If P is in one of the sets $\mathcal{E}_i$, then B_P has support on the union of a pair of neighboring triangles.*
3. *If P is in one of the sets $\mathcal{C}_i$, then B_P has support on the single triangle $T^{[i]}$.*

Proof. The proof is completely analogous to the proof of Theorems 4.2 and 4.4. ■

6. Remarks

1. Since $\mathcal{S}_d^r(\Delta) = S_d^{r,r}(\Delta)$, Theorem 2.2 subsumes the result of Dong [10]. Our proof differs from his, however, in that the analysis in [10] is based on using disks of radius $r + \lfloor r/2 \rfloor$, while we have used disks of radius $r + \lfloor (r+1)/2 \rfloor$.

2. The super spline spaces considered in [8] correspond to $\rho = 2r$, while those considered in [9] use $\rho = r + [(d - 2r - 1)/2]$.

3. In [5], dimension results were given for triangulations with *holes*. In this paper we have restricted ourselves to the case of triangulations without holes, but it is not difficult to extend our results to such triangulations.

4. The results in [10] were established for triangulations which may include subtriangulations which are joined together only at a single vertex. The results here can also be extended to such triangulations.

5. It is easy to see that the arguments given here fail when $d < 3r + 2$. The problem of finding formulae for the dimension of spline spaces with $d < 3r + 2$ remains open (except for the one interesting and difficult special case of $S_4^1(\Delta)$ treated in [3]).

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A. K. Ibrahim
Department of Mathematics
Suez Canal University
Ismailia
Egypt

L. L. Schumaker
Department of Mathematics
Vanderbilt University
Nashville
Tennessee 37235
U.S.A