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CONSTRUCTIVE APPROXIMATION

Erdös–Turán Theorems on a System of Jordan Curves and Arcs

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Dedicated to the memory of Peter Henrici

Abstract. Erdös and Turán established in [4] a qualitative result on the distribution of the zeros of a monic polynomial, the norm of which is known on $[-1, 1]$. We extend this result to a polynomial bounded on a system E of Jordan curves and arcs. If all zeros of the polynomial are real, the estimates are independent of the number of components of E for any regular compact subset E of $\mathbf{R}$. As applications, estimates for the distribution of the zeros of the polynomials of best uniform approximation and for the extremal points of the optimal error curve (generalizations of Kadec's theorem) are given.

1. Introduction

The theorems of Erdös and Turán to which we refer in the title concern the distribution of zeros of polynomials bounded on the interval $I = [-1, 1]$ or the unit disk $D = \{z: |z| \le 1\}$. More precisely, let $p \in \Pi_n$ be a monic polynomial, where Π_n denotes the set of all algebraic polynomials of degree at most n. We associate with p the zero-counting measure

$$\tau_p(A) := \frac{\text{number of zeros of } p \text{ in } A}{n}, \tag{1.1}$$

where A is any point set in $\mathbf{C}$ and the zeros are counted with their multiplicities.

If all zeros of p are in the interval $[-1, 1]$, then Erdös and Turán [4] proved, for any subinterval $[a, b]$ of $[-1, 1]$, that

$$\left| \tau_p([a, b]) - \frac{\beta - \alpha}{\pi} \right| \le \frac{8}{\log 3} \sqrt{\frac{\log P}{n}}, \tag{1.2}$$

where $\alpha = \cos \beta$, $b = \cos \alpha$, $0 \le \alpha \le \beta \le \pi$, and

$$P = 2^n \max_{x \in [-1, 1]} |p(x)|. \tag{1.3}$$

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If the polynomial p is bounded on the unit disk D such that

$$\tilde{P} = \frac{1}{\sqrt{a_0}} \max_{|z| \le 1} |p(z)|, \tag{1.4}$$

where a_0 is the constant term of $p(z), p(z) = z^n + \cdots + a_0, a_0 \neq 0$, then the number of zeros of p in the sectors

$$\tilde{S}_{\alpha,\beta} = \{re^{i\varphi}: r \geq 0, 0 \leq \alpha \leq \varphi \leq \beta \leq 2\pi\}$$

can be estimated as

$$\left| \tau_p(\tilde{S}_{\alpha,\beta}) - \frac{\beta - \alpha}{2\pi} \right| \leq 16 \sqrt{\frac{\log \tilde{P}}{n}} \tag{1.5}$$

[5].

It is interesting that (1.2) can be obtained from (1.5): Let

$$G(z) = \log|z + \sqrt{z^2 - 1}|$$

denote the Green function for $\mathbf{C}\setminus[-1, 1]$, where $\sqrt{z^2 - 1}$ is the branch which is asymptotically z near infinity. For any point $z \in \mathbf{C}$, let us define the projection $\pi(z) \in [-1, 1]$ such that z and $\pi(z)$ lie on the same orthogonal trajectory (which is a hyperbola) of the family of level lines of $G(z)$,

$$\Gamma_\sigma := \{z \in \mathbf{C}: G(z) = \log \sigma\}, \qquad \sigma \geq 1.$$

The level lines Γ_σ are ellipses with foci $+1$ and -1 and sum of semiaxes σ. Using the Joukowski transformation

$$z = \frac{1}{2}\left(\omega + \frac{1}{\omega}\right), \tag{1.6}$$

which maps the exterior of the unit disk of the ω-plane to $\mathbf{C}\setminus[-1, 1]$, we obtain

$$\pi(z) = \frac{1}{2}\left(\frac{\omega}{|\omega|} + \frac{|\omega|}{\omega}\right).$$

For $-1 \leq a \leq b \leq 1$, let us define

$$S_{a,b} = \{z \in \mathbf{C}: \pi(z) \in [a, b]\}.$$

Then $S_{a,b}$ is the point set between the two hyperbolas through a and b.

If p is a monic polynomial of degree n with P defined by (1.3), then any zero of p in $S_{a,b}$ is mapped by the inverse of the Joukowski transformation to a point in $\tilde{S}_{\alpha,\beta}$ or $\tilde{S}_{-\alpha,-\beta}$, where $a = \cos \beta$ and $b = \cos \alpha$. Furthermore, each zero of p in $[-1, 1]$ is mapped on two points of the unit circle using the continuous extensions of the inverse from above and below.

Now we consider

$$q(\omega) = p\left(\frac{1}{2}\left(\omega + \frac{1}{\omega}\right)\right) = \frac{1}{2^n \omega^n} \tilde{q}(\omega),$$

where $\tilde{q} \in \Pi_{2n}$, $\tilde{q}(\omega) = \omega^{2n} + \cdots + 1$. Then two zeros

$$\omega_v \quad \text{and} \quad \frac{1}{\omega_v}$$

of $\tilde{q}(\omega)$ correspond to each zero z_v of $p(z)$ and the number of zeros of p in $S_{a,b}$ is the same as the number of zeros in $\tilde{S}_{\alpha,\beta}$. Hence, (1.5) leads to

$$\left| \tau_p(S_{a,b}) - \frac{\beta - \alpha}{2\pi} \right| = 2 \left| \tau_{\tilde{q}}(\tilde{S}_{\alpha,\beta}) - \frac{\beta - \alpha}{2\pi} \right| \leq 32 \sqrt{\frac{\log P}{n}}, \tag{1.7}$$

where $\tau_{\tilde{q}}$ is the zero-counting measure of $\tilde{q}$. Therefore, (1.7) is a generalization of (1.2) for the case that the zeros of p are allowed to be outside of $[-1, 1]$. On the other hand, the constant 32 is worse than 8/log 3 in (1.2). Ganelius [7], [8] proved (1.5) by Fourier series methods and improved the constant 16 to 2.619.... Up to the constant, the estimate in (1.5) is sharp, but the optimal constant seems to be unknown.

The crucial key to generalizing the Erdös-Turán estimates can be found in a potential theoretical interpretation of (1.2), (1.5), and (1.7). For this, we need to introduce some terminology from potential theory.

Throughout this paper let E be a compact point set in $\mathbf{C}$ with connected complement $\Omega = \bar{\mathbf{C}} \setminus E$. We assume that E is regular, i.e., Ω has a Green function $G(z)$ with pole at infinity and boundary value 0. Hence, if $\operatorname{cap}(E)$ denotes the logarithmic capacity of E, then $\operatorname{cap}(E) > 0$. Let $\mu = \mu_E$ be the unique unit measure supported on E which minimizes the energy integral

$$I[\nu] := \iint \log \frac{1}{|z - \xi|} \, d\nu(\xi) \, d\nu(z)$$

over all unit measures ν supported on E. Then μ_E is called the *equilibrium distribution* for E and the logarithmic potential

$$U^\mu(z) := \int \log \frac{1}{|z - \xi|} \, d\mu(\xi)$$

is the *conductor potential* of E. The potential $U^\mu(z)$ is related to the Green function $G(z)$ by

$$U^\mu(z) = -G(z) - \log \operatorname{cap}(E) \qquad \text{for all} \quad z \in \Omega$$

(see [14, p. 82]).

Let p be a monic polynomial of degree n and let τ_p be its zero-counting measure. We investigate the distribution of the zeros of p if the Chebyshev norm $\|p\|_E$ on E is known. More precisely, if, for some constant M,

$$|p(z)| \leq M(\operatorname{cap}(E))^n \qquad \text{for all} \quad z \in E,$$

then $M \geq 1$ and, taking the logarithm, we obtain

$$\frac{1}{n} \log|p(z)| - \log \operatorname{cap}(E) \leq \frac{1}{n} \log M \qquad \text{for} \quad z \in E. \tag{1.8}$$

If $U^\tau(z)$ is the logarithmic potential of $\tau = \tau_p$, then

$$U^\tau(z) = \int \log \frac{1}{|z-\xi|} \, d\tau(\xi) = -\frac{1}{n} \log |p(z)|$$

and, since $U^\mu(z) \le -\log \operatorname{cap}(E)$ for all $z \in E$ (see [14, p. 60]), (1.8) can be written as

$$U^\mu(z) - U^\tau(z) = U^{\mu-\tau}(z) \le \frac{1}{n} \log M \qquad \text{for all} \quad z \in E. \tag{1.9}$$

In Ω the function $U^\mu(z) - U^\tau(z)$ is subharmonic. Hence, the maximum principle yields (1.9) for all $z \in \mathbf{C}$.

Let us now reformulate (1.5) and (1.7) in terms of

$$\varepsilon := \sup_{z \in \mathbf{C}} U^{\mu-\tau}(z).$$

Keeping in mind that

$$\mu(\tilde{S}_{\alpha,\beta}) = \frac{\beta - \alpha}{2\pi},$$

(1.5) reads

$$|(\tau - \mu)(\tilde{S}_{\alpha,\beta})| \le 16\sqrt{\varepsilon + \tfrac{1}{2} U^{\tau-\mu}(0)}. \tag{1.10}$$

In the situation of $E = [-1, 1]$ we have

$$\mu(S_{a,b}) = \frac{1}{\pi} \int_a^b \frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi} \int_\alpha^\beta dt = \frac{\beta - \alpha}{\pi}$$

and therefore (1.7) leads to

$$|(\tau - \mu)(S_{a,b})| \le 32\sqrt{\varepsilon}. \tag{1.11}$$

An equality of type (1.11) was proved by Sjögren [13] to estimate $|(\tau - \mu)(S)|$ on subarcs S of E if E is a Hölder smooth Jordan curve and if all zeros of the polynomial p lie on E.

Asymptotic results for the behavior of the zeros of a sequence (p_n) of monic polynomials are known if

$$U^\mu(z) - U^{\tau_{p_n}}(z) \le \varepsilon_n$$

and $\lim_{n \to \infty} \varepsilon_n = 0$. Blatt, Saff, and Simkani [2] have shown that in this case τ_{p_n} is weakly converging to the equilibrium measure.

The aim of this paper is to generalize estimates of type (1.10) and (1.11) to other situations, especially to the case when E is a system of Jordan curves and Jordan arcs.

2. The Main Results

In the following we use

$$\varepsilon := \sup_{z \in \mathbf{C}} U^{\mu-\tau}(z) \tag{2.1}$$

and $\tau := \tau_p$, $\mu := \mu_E$. We denote by

$$\Gamma_\sigma := \{z \in \Omega : G(z) = \log \sigma\}, \qquad \sigma > 1, \tag{2.2}$$

the level lines of the Green function G(z) and set, for $\sigma > 1$,

$$E_\sigma := \{z \in \Omega : 0 < G(z) \le \log \sigma\} \cup E. \tag{2.3}$$

Every level line Γ_σ is analytic at each point z where grad $G(z) \neq 0$. Moreover, Γ_σ consists of a finite number of Jordan curves which are mutually exterior except for a finite number of *critical points*, i.e., points where grad $G(z) = 0$ [15].

Let us now introduce the so-called *Green lines* (see [10]). These are orthogonal trajectories of the family of level lines. Hence, through every noncritical point there is a unique Green line. However, several Green lines may end in one critical point. As an example, if $E = [-2, -1] \cup [1, 2]$, the imaginary axis consists of two Green lines ending in the critical point 0, and all points $z = x + iy$, $x \neq 0$, are on a Green line ending in E.

For $z \in \mathbf{C}$, we define $L(z)$ as the union of all Green lines with z as an accumulation point. Then, for a subset J of $\mathbf{C}$, we set

$$S_\sigma^+(J) = \text{closure}\left(\bigcup_{z \in J} L(z)\right) \cap E_\sigma. \tag{2.4}$$

A function of a real variable belongs to class $C^{\alpha+}$ (α is a nonnegative integer) if its αth derivative satisfies a Hölder condition with some positive exponent. A curve (arc) belongs to class $C^{\alpha+}$ if it is rectifiable and its coordinates are $C^{\alpha+}$ functions of arc length.

Theorem 1. *Let $F \subset E$ be a Jordan arc of class C^{2+} with positive distance to $E \backslash F$, $c_0 > 0$ fixed. Then there is a constant $C > 0$, depending on E and c_0, such that, for all subarcs J of F and all $\sigma \ge 1 + c_0\sqrt{\varepsilon}$,*

$$|(\mu - \tau)(S_\sigma^+(J))| \le C\sqrt{\varepsilon}, \tag{2.5}$$

where ε is defined by (2.1).

We remark that Lemma 1 in Section 4 shows that $S_\sigma^+(J)$ defines a neighborhood of each interior point of the Jordan arc J.

Next we consider the situation that the component S of E is bounded by a Jordan curve F. Let F^* be the interior of S. Then we fix a point $z_0 \in F^*$ and proceed as above. Let $G_0(z) = G_0(z, z_0)$ be the Green function of the region F^* with pole at z_0. For $z \in F$, we define $L_0(z)$ as the union of all Green lines of $G_0(z)$, which have z as an accumulation point. Let

$$F_\sigma := \{z \in F^* : 0 < G_0(z) \le \log \sigma\} \cup F \tag{2.6}$$

for $\sigma > 1$, and define, for $J \subset F$,

$$S_\sigma^-(J) := \text{closure}\left(\bigcup_{z \in J} L_0(z)\right) \cap F_\sigma. \tag{2.7}$$

Then

$$\bigcup_{\sigma \geq 1} (S_\sigma^+(J) \cup S_\sigma^-(J))$$

is just a sector in the plane for the special case when $E = F$ is the unit disk and J is a subarc of the unit circle, a situation considered by Erdös and Turán in [5].

Theorem 2. *Let $S \subseteq E$ be bounded by a Jordan curve F of class C^{2+} with interior F^* such that S has a positive distance to $E \setminus S$ and let $z_0 \in F^*$ and $c_0 > 0$ be fixed. Then there exists a constant $C > 0$, depending on E, z_0 and c_0, such that*

$$|(\mu - \tau)(S_\sigma^+(J) \cup S_\sigma^-(J))| \leq C\sqrt{U^{\tau-\mu}(z_0) + 2\varepsilon} \tag{2.8}$$

for all subarcs J of F and all $\sigma \geq 1 + c_0\sqrt{U^{\tau-\mu}(z_0) + 2\varepsilon}$.

The proof of these theorems will show that the constant C depends on the geometry of E. Hence, it seems to be interesting that in the special case $E \subset \mathbf{R}$ the constant C can be estimated as an absolute constant independent of E if all zeros of p are real.

Theorem 3. *Let E be a compact and regular subset of $\mathbf{R}$ and let $p \in \Pi_n$ be a monic polynomial with all zeros in $\mathbf{R}$. Then, for all intervals $I \subset \mathbf{R}$,*

$$|(\mu - \tau)(I)| \leq 8\sqrt{2\varepsilon/\pi} + 8\varepsilon \tag{2.9}$$

Consequently,

$$|(\mu - \tau)(I)| \leq 8\sqrt{\varepsilon}. \tag{2.10}$$

If not all zeros of p are real, then our estimate will depend on the number of components of E. Let us therefore assume that

$$E = I_1 \cup I_2 \cup \cdots \cup I_k \subset \mathbf{R} \tag{2.11}$$

is a finite union of compact, disjoint intervals. Since E is regular, each interval I_j consists of more than a single point. Let

$$\alpha := \inf_{z \in E} z \quad \text{and} \quad \beta := \sup_{z \in E} z. \tag{2.12}$$

Theorem 4. *Let E be the union of k real compact intervals, let $p \in \Pi_n$ be a monic polynomial, and let $\delta > 0$. Then, for all intervals $I \subset [\alpha, \beta]$,*

$$|(\mu - \tau)(S_\sigma^+(I))| \leq \left(\frac{18}{\pi\delta} + 2\delta\sqrt{k}\right)\sqrt{\varepsilon} + 9\varepsilon \tag{2.13}$$

holds for arbitrary $\sigma \geq 1 + \pi\delta\sqrt{\varepsilon}$. *Especially,*

$$|(\mu - \tau)(S_\sigma^+(I))| \leq 8k^{1/4}\sqrt{\varepsilon} \tag{2.14}$$

for any $\sigma \geq 1 + 3\sqrt{\pi\varepsilon}/k^{1/4}$.

Remark 1. There is a simple way to prove

$$\tau(\Omega\setminus E_\sigma) \leq \frac{\varepsilon}{\log\sigma} \tag{2.15}$$

for all $\sigma > 1$. Fix $\sigma > 1$ and define

$$\mathrm{h}(z) := \mathrm{U}^{\mu-\tau}(z) + \frac{1}{n}\sum_{v=1}^{m} \mathrm{G}(z, z_v),$$

where $z_1, \ldots, z_m$ are the zeros of p in $\Omega\setminus E_\sigma$ and $\mathrm{G}(z, z_v)$ is the Green function for Ω with pole at z_v. Since $\mathrm{h}(z)$ is subharmonic in $\overline{\mathbb{C}}\setminus E$, the maximum principle yields

$$\mathrm{h}(\infty) = \frac{1}{n}\sum_{v=1}^{m} \mathrm{G}(\infty, z_v) = \frac{1}{n}\sum_{v=1}^{m} \mathrm{G}(z_v) \leq \varepsilon.$$

Thus,

$$\frac{m}{n} = \tau(\Omega\setminus E_\sigma) \leq \frac{\varepsilon}{\log\sigma}.$$

Remark. 2 Using the same method of proof and replacing the unbounded component of E by the Jordan region F^* of Theorem 2 as well as the point at infinity by the point $z_0 \in F^*$, we obtain, for $\sigma > 1$,

$$\tau(F^*\setminus F_\sigma) \leq \frac{\varepsilon + \mathrm{U}^{\tau-\mu}(z_0)}{\log\sigma}. \tag{2.16}$$

Remark 3. The proof of Theorem 4 shows that (2.14) can be replaced by

$$|(\mu - \tau)(S_\sigma^+(I))| \leq 8\sqrt{\varepsilon} \tag{2.17}$$

for all $\sigma \geq 1 + 3\sqrt{\pi\varepsilon}$ if all critical points of $\mathrm{G}(z)$ are outside of the interior of $E_{1+3\sqrt{\pi\varepsilon}}$.

3. Applications to Extremal Points and Zeros of Best Uniform Approximants

Let f be a real-valued or complex-valued continuous function on E, which is analytic in the interior of E, and let us denote by p_n^* the best uniform approximation of f on E with respect to Π_n. If

$$e_n = \|f - p_n^*\|_E := \max_{x\in E} |f(x) - p_n^*(x)|,$$

then Mergelyan's theorem yields $\lim_{n\to\infty} e_n = 0$.

If E is a finite union of compact intervals and f is real-valued, then there exists a point set of $n+2$ alternation points,

$$x_0^{(n)} < x_1^{(n)} < \cdots < x_{n+1}^{(n)}, \tag{3.1}$$

in E such that $e_n = |(f - p_n^*)(x_\nu^{(n)})|$ and the signs of $(f - p_n^*)(x_\nu^{(n)})$ alternate. We denote by $A_n(f, I)$ the number of points of (3.1) in an interval $I \subset \mathbf{R}$.

Corollary 1. *Let f be a real-valued continuous function on $E = \bigcup_{\nu=1}^k I_\nu$. Then there exists a constant $C > 0$ (independent of f) and infinitely many n such that*

$$|A_n(f, I) - n\mu_E(I)| \le C\sqrt{n \log n} \tag{3.2}$$

for any interval $I \subset \mathbf{R}$.

The above estimate generalizes a theorem of Kadec in [9] and sharpens a result of Fuchs [6], who proved (3.2) with the right-hand side replaced by $Cn^{2/3}$ for $I \subset E^\circ$ (resp. $Cn^{4/5}$ for general I).

The analogous result for a single interval was proved in [1].

In general, we only know that the extremal point set

$$M_n(f) := \{z \in E : |f(z) - p_n^*(z)| = \|f - p_n^*\|_E\}$$

has at least $n+2$ points. Let us denote by $\mathscr{F}_{n+2}(M_n(f))$ any $(n+2)$-point subset S of $M_n(f)$ for which the Vandermonde expression

$$V(S) := \left(\prod_{z,t \in S;\, z \neq t} |z - t| \right)^{1/2} \tag{3.3}$$

is as large as possible. The points of $\mathscr{F}_{n+2}(M_n(f))$ are called Fekete points of $M_n(f)$.

If $\tilde{A}_n(f, K)$ denotes the number of points of $\mathscr{F}_{n+2}(M_n(f))$ in a set $K \subset \mathbf{C}$, then in [3] the following result was announced for a union of finitely many intervals. But the method of proof works well in the more general situation of Theorems 1 and 2.

Corollary 2 [3]. *Suppose that any component of E has a diameter $\ge \rho$ where $\rho > 0$. Let $S \subseteq E$ bounded by F have a positive distance to $E \setminus S$. Assume that F is a Jordan curve or a Jordan arc of class C^{2+}. Then there is a constant $C > 0$ and infinitely many n such that*

$$|\tilde{A}_n(f, J) - n\mu_E(J)| \le C(n^2 \log n)^{1/3} \tag{3.4}$$

for any subarc $J \subset F$.

In [3] the verification of (3.4) was postponed, since the proof is based on Theorem 1.

For the monic Chebychev polynomial $T_n(z)$ on E of degree n, we denote by $Z_n(K)$ the number of zeros of $T_n(z)$ in K.

Since we base the proof of the next corollary on upper bounds of $\|T_n\|_E$ due to Widom [16], we assume that the boundary of E is a finite union of disjoint Jordan arcs or curves of class C^{2+}.

Corollary 3. *Let the boundary of E be a finite union of disjoint Jordan arcs or curves of class C^{2+}. Furthermore, assume that one of the components of E is a Jordan arc F. Then, for all $\sigma_0 > 1$, there exists a constant $C > 0$ such that*

$$|Z_n(S_\sigma^+(J)) - n\mu_E(S_\sigma^+(J))| \le C\sqrt{n} \tag{3.5}$$

for all integers n, all subarcs J of F, and all $\sigma \ge \sigma_0$.

We remark that, for $E \subset \mathbf{R}$, Corollary 3 yields

$$|Z_n(I) - n\mu_E(I)| \le C\sqrt{n}$$

for all compact intervals $I \subset \mathbf{R}$, since all zeros of $T_n(z)$ are real. Moreover, these zeros interlace the extremal points of $T_n(z)$ and, therefore, an analogous estimate for the extremal points is obtained.

Furthermore, a quantified version of a result of Blatt, Saff, and Simkani [2] for the zeros of the polynomials p_n^* of best uniform approximation to f can be proved. As above, let $\tau_{p_n^*}$ denote the unit measure associated with the zeros of p_n^*.

Corollary 4. *Let the boundary of E be a finite union of disjoint Jordan arcs or curves of class C^{2+}, and let F be a fixed component of this boundary. Assume that f is analytic in the interior of E, continuous on E, and not infinitely often differentiable on the boundary of E. Then, for all $\sigma_0 > 1$, there exists a constant $C > 0$ (dependent of f) and infinitely many n such that, for any subarc $J \subset F$ and $\sigma \ge \sigma_0$,*

$$|(\tau_{p_n^*} - \mu_E)(S_\sigma^+(J))| \le C\sqrt{\frac{\log n}{n}} \tag{3.6}$$

if F is a Jordan arc (resp.

$$|(\tau_{p_n^*} - \mu_E)(S_\sigma^+(J) \cup S_\sigma^-(J))| \le C\sqrt{\frac{\log n}{n}} \tag{3.7}$$

in the case of a Jordan curve F). For the special case $E \subset \mathbf{R}$ and $f \notin C^\alpha(E)$ (α is a nonnegative integer), there exist infinitely many n such that

$$|(\tau_{p_n^*} - \mu_E)(S_\sigma^+(J))| \le 8\sqrt{(3 + 2\alpha)\frac{\log n}{n}}. \tag{3.8}$$

4. Proof of the Theorems

For the proof of the results we need several lemmas.

Lemma 1. *Let $S \subseteq E$ be a continuum which has positive distance to $E \setminus S$. If M_σ denotes the component of E_σ containing S, then there is a $\sigma_0 > 1$ such that*

$$E \cap M_\sigma = S \tag{4.1}$$

for all $1 < \sigma < \sigma_0$.

Proof. Let

$$M := \bigcap_{n \in \mathbb{N}} M_{1/n}. \tag{4.2}$$

Thus $S \subseteq M \subseteq E$. We show that M is connected. Otherwise there are disjoint open sets U_1, U_2 with

$$\begin{aligned} U_1 \cap M &\neq \emptyset, \\ U_2 \cap M &\neq \emptyset, \\ M &\subset U_1 \cup U_2. \end{aligned} \tag{4.3}$$

Since each $M_{1/n}$ is connected, we can pick $z_n \in (\mathbb{C} \setminus (U_1 \cup U_2)) \cap M_{1/n}$. Every limit point z of the sequence (z_n) is in $(\mathbb{C} \setminus (U_1 \cup U_2)) \cap M$. This contradicts $M \subseteq (U_1 \cup U_2)$.

Since M is connected and S is a connected component of E, we get $S = M$. Since all $M_{1/n}$ are compact sets and S has a positive distance to $E \setminus S$, there is a number $n \in \mathbb{N}$ with $M_{1/n} \bigcap E = S$. Thus $M_\sigma \cap E = S$ for $\sigma \leq 1/n$. ■

Lemma 2. *Let $S \subseteq E$ be bounded by a Jordan curve F of class C^{2+} with positive distance to $E \setminus S$. Then* G(*z*) *and* grad G(*z*) *extend continuously to F with* $G(z) = 0$ *and*

$$|\operatorname{grad} G(z)| = \frac{\partial}{\partial n} G(z) > 0 \tag{4.4}$$

for all $z \in F$, where n denotes the outer normal on F. Moreover, the derivative

$$\frac{d}{ds} \frac{\partial}{\partial n} G(z) \tag{4.5}$$

is continuous on F, where s denotes the arc length of F.

Lemma 2 is a special case of Lemma 4.1 of Widom [16]. We include a sketch of the proof because we use a formula for $\partial G/\partial n$ in the next lemma.

Proof. Since all points of F are regular points for the Dirichlet problem, G(z) extends continuously to F with $G(z) = 0$. Concerning the continuity properties of grad G(z), the reflection principle yields (4.4) and (4.5) if F is the unit circle. Then the general case is reduced to the case that F is the unit circle. Let $t = \varphi(z)$ be a conformal mapping from the exterior of F to the exterior of the unit circle such that the point at infinity remains fixed. By a theorem of Warschawski and Kellogg (see [11]) we know that φ, φ', and φ'' can be extended continuously to F with $\varphi'(z) \neq 0$ for all $z \in F$.

Consider the Green function $G^*(z)$ of the region $\varphi(\Omega)$ with pole at infinity. Then

$$G(z) = G^*(\varphi(z))$$

is the Green function for Ω with pole at infinity and

$$\operatorname{grad} G(z) = \operatorname{grad} G^*(\varphi(z)) D\varphi(z),$$

where $D\varphi(z)$ is the Jacobian matrix:

$$D\varphi(z) = \begin{pmatrix} \frac{\partial}{\partial x} \operatorname{Re} \varphi(z) & \frac{\partial}{\partial y} \operatorname{Re} \varphi(z) \\ \frac{\partial}{\partial x} \operatorname{Im} \varphi(z) & \frac{\partial}{\partial y} \operatorname{Im} \varphi(z) \end{pmatrix}.$$

Now

$$\frac{\partial G}{\partial n}(z) = |\operatorname{grad} G(z)| = \langle \operatorname{grad} G^*(\varphi(z)), u \rangle,$$

where u is the vector

$$u = \begin{pmatrix} \frac{\partial}{\partial n} \operatorname{Re} \varphi(z) \\ \frac{\partial}{\partial n} \operatorname{Im} \varphi(z) \end{pmatrix} = \frac{\partial}{\partial n} \varphi(z).$$

Hence the vector

$$u = \frac{\partial}{\partial n} \varphi(z)$$

has the direction of the normal to the unit circle at the point $\varphi(z)$, i.e., u has the same direction as grad $G^*(\varphi(z))$. Therefore,

$$\frac{\partial G}{\partial n}(z) = |\operatorname{grad} G(z)| = |\operatorname{grad} G^*(\varphi(z))|\, |\varphi'(z)| \neq 0,$$

where we have used $|\operatorname{grad} G^*(\varphi(z))| \neq 0$ from the special case.

The continuity of (4.5) on F follows from the continuity of $\varphi''(z)$ and from the assumption that F is of class C^{2+}. ■

Lemma 3. *Let $S \subset E$ be bounded by a Jordan curve F of class C^{2+} and let $G_0(z)$ be the Green function $G(z, z_0)$ defined in the interior F^* of S with pole in z_0. Then G_0 extends continuously to F with $G_0(z) = 0$ for $z \in F$ and*

$$|\operatorname{grad} G_0(z)| = \frac{\partial}{\partial n_0} G_0(z) > 0$$

for all $z \in F^* \setminus \{z_0\}$, where n_0 denotes the inner normal on F. Moreover,

$$\frac{d}{ds} \frac{\partial}{\partial n_0} G_0(z)$$

is continuous of F, where s denotes the arc length of F.

The proof follows the same line as the proof of Lemma 2 using the conformal mapping φ which maps F^* to the unit disk with $\varphi(z_0) = 0$.

Lemma 4. *Let F be a Jordan arc of class C^{2+} in E with endpoints a and b and positive distance to $E\setminus F$. Then the functions*

$$h_{\pm}(z) = (|z-a|\,|z-b|)^{1/2}\frac{\partial \mathrm{G}}{\partial n_{\pm}}(z)$$

are continuous at the interior points of F where n_+ and n_- denote the two normals at the point z directed into Ω. Moreover, $h_{\pm}(z)$ can be continuously extended to F with $h_{\pm}(z) > 0$ for all $z \in F$. Furthermore, if we set

$$q(z) = \frac{h_+(z)}{h_-(z)},$$

then the function

$$(|z-a|\,|z-b|)^{1/2}\frac{d}{ds}q(z)$$

extends continuously to F where s denotes the arc length of F.

Proof. For simplicity we assume that ± 1 are the endpoints of F. Then we cut the z-plane along F and define on $\mathbb{C}\setminus F$ the function $\sqrt{z^2-1}$ as the branch which is asymptotically z near infinity. If we set

$$t = \varphi(z) := z + \sqrt{z^2-1},$$

then the exterior of F corresponds to the exterior of a certain closed Jordan curve $\tilde{F}$ in the t-plane. This curve contains 0 in its interior. Moreover,

$$z = \psi(t) = \frac{1}{2}\left(t + \frac{1}{t}\right)$$

extends continuously to $\tilde{F}$ as a function of t, and z traverses F twice, once in each direction, as t traverses $\tilde{F}$ once.

By Lemma 11 of Widom [16, p. 206] $\tilde{F}$ is a Jordan curve of class C^{2+} if F is of class C^{2+}. Let $\tilde{\mathrm{G}}(t)$ denote the Green function of $\varphi(\Omega)$ with pole at infinity. Then $\mathrm{G}(z) = \tilde{\mathrm{G}}(\varphi(z))$. By Lemma 2 $\tilde{\mathrm{G}}(t)$ and grad $\tilde{\mathrm{G}}(t)$ extend continuously to $\tilde{F}$ and $|\mathrm{grad}\,\tilde{\mathrm{G}}(t)| > 0$ for all $t \in \tilde{F}$.

Let us fix an interior point z of F and let us define

$$\varphi_{\pm}(z) := \lim_{\alpha\to 0_+} \varphi(z + \alpha n_{\pm}). \tag{4.6}$$

Then we obtain, as in the proof of Lemma 1,

$$\frac{\partial \mathrm{G}}{\partial n_{\pm}}(z) = |\mathrm{grad}\,\tilde{\mathrm{G}}(\varphi_{\pm}(z))|\,|\varphi'_{\pm}(z)|,$$

where

$$\varphi'_{\pm}(z) = \lim_{\alpha\to 0_+} \varphi'(z + \alpha n_{\pm}).$$

Then

$$\frac{\partial G}{\partial n_\pm}(z) = |\text{grad } \tilde{G}(\varphi_\pm(z))|\, |\varphi_\pm(z)|\, |z^2 - 1|^{-1/2}$$

or

$$h_\pm(z) = |\text{grad } \tilde{G}(\varphi_\pm(z))|\, |\varphi_\pm(z)|.$$

Hence, $h_\pm(z)$ can be extended continuously to F with $h_\pm(z) > 0$ for all $z \in F$.

Now,

$$q(z) = \frac{|\text{grad } \tilde{G}(\varphi_+(z))|\, |\varphi_+(z)|}{|\text{grad } \tilde{G}(\varphi_-(z))|\, |\varphi_-(z)|} = \frac{Z}{N},$$

where Z and N denote the numerator and the denominator. Let σ denote the arc length of $\tilde{F}$, then

$$\frac{d}{ds} q(z) = \frac{N(dZ/ds) - Z(dN/ds)}{N^2},$$

where

$$\begin{aligned}\frac{dZ}{ds} &= |\varphi_+(z)| \frac{d}{ds} |\text{grad } \tilde{G}(\varphi_+(z))| \\ &\quad + |\text{grad } \tilde{G}(\varphi_+(z))| \frac{d}{ds} |\varphi_+(z)| \\ &= |\varphi_+(z)|\, |\varphi'_+(z)| \frac{d}{d\sigma} |\text{grad } \tilde{G}(t)| \Big|_{t=\varphi_+(z)} \\ &\quad + |\text{grad } \tilde{G}(\varphi_+(z))|\, |\varphi'_+(z)|,\end{aligned}$$

and, analogously,

$$\frac{dN}{ds} = |\varphi_-(z)|\, |\varphi'_-(z)| \frac{d}{d\sigma} |\text{grad } \tilde{G}(t)| \Big|_{t=\varphi_-(z)} + |\text{grad } \tilde{G}(\varphi_-(z))|\, |\varphi'_-(z)|.$$

By Lemma 2 the derivative

$$\frac{d}{d\sigma} |\text{grad } \tilde{G}(t)|$$

is continuous on $\tilde{F}$ and because of (4.6) the same is true for the functions

$$\frac{d}{d\sigma} |\text{grad } \tilde{G}(t)| \Big|_{t=\varphi_\pm(z)}, \qquad z \in F.$$

Hence,

$$|z^2 - 1|^{1/2} \frac{d}{ds} q(z)$$

is continuous on F and the lemma is proved. ∎

Lemma 5. *For all* $0 = \tau_0 < \tau_1 < \cdots < \tau_m = 1$ *and* $\eta > 0$, *there is a function* $\psi \in C^2(\mathbf{R})$, $\psi: \mathbf{R} \to [0, 1]$, *and* $\kappa_0 > 0$ *such that, for all* $\kappa < \kappa_0$,

(i) $\psi(c) = 1$ *for* $x \le \kappa$,
(ii) $\psi(x) = 0$ *for* $x \ge 1 - \kappa$,
(iii) $\psi^{(n)}(x) = 0$ *for* $n = 1, 2$, $x \in [\tau_\nu - \kappa, \tau_\nu + \kappa]$, $\nu = 0, \ldots, m$,
(iv) $|\psi'(x)| \le (2 + \eta)m$ *for all* $x \in \mathbf{R}$,
(v) $|\psi''(x)| \le (4 + \eta)m$ *for all* $x \in \mathbf{R}$.

Proof. Let $\kappa < \frac{1}{6}$ and

$$u_\kappa(x) := \begin{cases} 0 & \text{for } x \le \kappa, \\ -1 & \text{for } x \in [2\kappa, \frac{1}{2} - \kappa], \\ +1 & \text{for } x \in [\frac{1}{2} + \kappa, 1 - 2\kappa], \\ 0 & \text{for } x \ge 1 - \kappa, \end{cases}$$

and linear in between. Define

$$\psi_1(x) := \frac{4}{(1 - 4\kappa)(1 - 2\kappa)} \int_1^x \int_1^y u_\kappa(t)\, dt\, dy. \tag{4.7}$$

Then, for κ sufficiently small, the function $\psi(x) = \psi_1(x)$ satisfies (i)-(v) for $m = 1$.

For $m > 1$, let $\delta_i = \tau_i - \tau_{i-1}$ and

$$\eta_i := \frac{\delta_i^2}{\sum_{\nu=1}^m \delta_\nu^2}$$

for $i = 1, \ldots, m$. Now, define $\psi(x) = 1$ for $x \le 0$, $\psi(x) = 0$ for $x \ge 1$, and

$$\psi(x) = 1 - \sum_{\nu=1}^{i} \eta_\nu + \eta_i \psi_1\left(\frac{1}{\delta_i}(x - \tau_{i-1})\right)$$

for $x \in (\tau_{i-1}, \tau_i)$ (κ chosen as above). Then ψ satisfies (i), (ii), and (iii) and

$$|\psi''(x)| \le (4 + \eta)\frac{\eta_i}{\delta_i^2} = \frac{4 + \eta}{\sum_{\nu=1}^m \delta_\nu^2}.$$

Since $\sum_{\nu=1}^m \delta_\nu = 1$, we get, from Hölders inequality,

$$1 = \left(\sum_{\nu=1}^m \delta_\nu\right)^2 \le m \sum_{\nu=1}^m \delta_\nu^2$$

and therefore (v) is true. Inequality (iv) can be proved analogously. ∎

First we prove Theorem 2.

Proof of Theorem 2. Let us denote by

$$\Gamma_{0,\sigma} = \{z \in F^*: G_0(z) = \log \sigma\}$$

the level curves of G_0. By Lemmas 1 and 2 there is a $\sigma_0 > 1$ such that

$$M_{\sigma_0} \cap (E \setminus S) = \varnothing \tag{4.8}$$

and

$$\frac{\partial}{\partial n} G(z) > 0 \qquad \text{for all} \quad z \in M_{\sigma_0} \backslash F^*, \tag{4.9}$$

where n denotes the outer normal on Γ_σ, $z \in \Gamma_\sigma$ (resp. on F). By Lemma 3

$$\frac{\partial}{\partial n_0} G_0(z) > 0 \qquad \text{for all} \quad z \in F_{\sigma_0},$$

where n_0 denotes the inner normal on $\Gamma_{0,\sigma}$, $z \in \Gamma_{0,\sigma}$.

First we prove, for $\sigma = 1 + c_0\sqrt{U^{\tau-\mu}(z_0) + 2\varepsilon}$, that there exists $C > 0$ depending on E, z_0, and c_0 such that

$$(\mu - \tau)(S_\sigma^+(J) \cup S_\sigma^-(J)) \leq C\sqrt{U^{\tau-\mu}(z_0) + 2\varepsilon}. \tag{4.10}$$

We may assume $\sigma \leq \sigma_0$, since otherwise, for $C = c_0/(\sigma_0 - 1)$, the right-hand side of (4.10) exceeds 1.

Let J_2 be a closed subarc of J and let J_1 and J_3 be the two subarcs forming $J \backslash J_2$. Fix

$$\delta := c_0\sqrt{U^{\tau-\mu}(z_0) + 2\varepsilon} = \sigma - 1.$$

Since we may assume $\mu(J) \geq 2\delta$, we can choose J_2 in such a way that

$$\mu(J_1) = \mu(J_3) = \delta. \tag{4.11}$$

Now we construct a test function g that is 1 on J_2 and 0 on $\mathbb{C} \backslash (S_\sigma^+(J) \cup S_\sigma^-(J))$. We start by defining g on F. Let us first consider the case $J \neq F$. Then we fix a point $t_0 \in F \backslash J$ and introduce Green's coordinates on

$$V_{\sigma_0} := M_{\sigma_0} \backslash (F^* \cup L(t_0)),$$

i.e., we choose a harmonic conjugate $\varphi(z)$ of $G(z)$ such that

$$\Phi(z) = e^{G(z) + i\varphi(z)}$$

maps V_{σ_0} onto

$$K_{\sigma_0} = \{re^{i\varphi} \colon 1 \leq r \leq \sigma_0,\ \varphi_1 < \varphi < \varphi_2\}.$$

Let $\varphi(J) = [\alpha, \beta]$. Then, by (4.11),

$$\varphi(J_2) = [\alpha + 2\pi\delta, \beta - 2\pi\delta].$$

Note that

$$\frac{1}{2\pi}\frac{d}{ds}\varphi = \frac{1}{2\pi}\frac{\partial G}{\partial n}$$

is the density function of μ_E on F.

By Lemma 5 we can define $g_1 \colon \mathbf{R} \to [0, 1]$, twice continuously differentiable, such that

$$\left|\frac{d^2}{d\varphi^2} g_1\right| \leq \frac{5}{(2\pi\delta)^2}, \qquad \left|\frac{d}{d\varphi} g_1\right| \leq \frac{3}{2\pi\delta},$$

and

$$g_1|_{\varphi(J_2)} = 1, \qquad g_1|_{\mathbb{R}\setminus\varphi(J)} = 0.$$

Analogously, we define a harmonic conjugate φ_0 of G_0 on

$$U := F^*\setminus L_0(t_0),$$

where $L_0(t_0)$ is the Green line connecting z_0 with t_0. Thus we project any point $z \in U$ to a point $\pi_0(z) \in F$ along the orthogonal trajectories to the family of level lines $\Gamma_{0,\sigma}$, i.e., $\varphi_0(z) = \varphi_0(\pi_0(z))$.

Furthermore, we set

$$g_2(z) := \begin{cases} \psi_1\left(\dfrac{e^{G(z)} - 1}{\delta}\right) & \text{for } z \in M_\sigma\setminus F^*, \\ \psi_1\left(\dfrac{e^{G_0(z)} - 1}{\delta}\right) & \text{for } z \in F^*, \\ 0 & \text{elsewhere} \end{cases}$$

where $\psi_1 = \psi$ is the function of Lemma 5 for $m = 1, \eta = 1$. Now we set

$$g(z) := \begin{cases} g_1(\varphi(z))g_2(z), & z \in V_\sigma, \\ g_1(\varphi(\pi_0(z)))g_2(z), & z \in U_\sigma, \\ 0 & \text{elsewhere} \end{cases}$$

where

$$U_\sigma := F_\sigma\setminus L_0(t_0)$$

and

$$V_\sigma := M_\sigma\setminus(F^* \cup L(t_0)).$$

In the case $F = J$ we set $g = g_2$. Then g is continuous on F and twice continuously differentiable on $\mathbb{C}\setminus F$.

We need to estimate $|\Delta g|$. Let $z \in V_\sigma\setminus F$. Since Φ is analytic, we get, by a well-known identity,

$$\Delta_z g(z) = |\Phi'(z)|^2 \Delta_w g(w),$$

where $w = \Phi(z)$. We use polar coordinates in K_{σ_0} and get

$$|\Delta_w g(w)| = \left| g_1(\varphi)\frac{\partial^2}{\partial r^2} g_2(r) + \frac{1}{r} g_1(\varphi)\frac{\partial}{\partial r} g_2(r) + \frac{1}{r^2} g_2(r)\frac{\partial^2}{\partial \varphi^2} g_1(\varphi)\right|$$
$$\le \frac{C_0}{\delta^2}$$

for some constant $C_0 > 0$ (we may assume $\delta < 1$). Thus

$$|\Delta g(z)| \le r^2\left(\frac{\partial G}{\partial n}\right)^2 \frac{C_0}{\delta^2} \tag{4.12}$$

for $z \in V_\sigma\setminus F$, where n denotes the outer normal on Γ_r, $z \in \Gamma_r$.

Next we take $z \in U_\sigma$. From Lemmas 2 and 3 we know that φ_0 is a C^2-function of φ on $F \backslash t_0$. Thus

$$\left|\frac{d^2}{d\varphi_0^2} g_1\right| = \left|\frac{d^2\varphi}{d\varphi_0^2}\frac{dg_1}{d\varphi} + \left(\frac{d\varphi}{d\varphi_0}\right)^2 \frac{d^2 g_1}{d\varphi^2}\right| \le \frac{C_2}{\delta^2}$$

on $F \backslash t_0$ with some constant $C_2 > 0$. As above, we get

$$|\Delta g(z)| \le r^2 \left(\frac{\partial G_0}{\partial n_0}\right)^2 \frac{C_1}{\delta^2} \tag{4.13}$$

for all $z \in U_\sigma$ and some constant $C_1 > 0$.

Since $\Delta g = 0$ for all $z \in \mathbf{C} \backslash (V_\sigma \cup U_\sigma)$, (4.12) and (4.13) show that $|\Delta g|$ is bounded on $\mathbf{C} \backslash F$.

Next we prove the representation

$$g(z) = \frac{1}{2\pi} \iint \Delta g(\zeta) \log |z - \zeta| \, dx \, dy \tag{4.14}$$

($\zeta = x + iy$) which is well known for $g \in C^2(\mathbf{R}^2)$ with compact support. Since $|\Delta g(z)|$ is bounded on $\mathbf{C} \backslash F$, both sides of (4.14) are continuous in z and we need only prove (4.14) for $z \notin F$. We take $\rho > 0$ so small that the circle $C_\rho(z)$ of radius ρ around z does not intersect F and apply Green's formula to the region

$$R_\rho := \mathbf{C} \backslash (B_\rho(z) \cup F),$$

where $B_\rho(z)$ is the interior of $C_\rho(z)$. Since g has compact support and is continuous on F and since

$$\frac{\partial}{\partial n} g(z) = \frac{\partial}{\partial n_0} g(z) = 0 \qquad \text{for all} \quad z \in F,$$

we get

$$\begin{aligned}
&\frac{1}{2\pi} \iint \Delta g(\zeta) \log |\zeta - z| \, dx \, dy \\
&= \lim_{\rho \to 0} \left(\frac{1}{2\pi} \int_{C_\rho(z)} \frac{\partial}{\partial n_1} g(\zeta) \log |\zeta - z| \, ds(\zeta) \right. \\
&\qquad \left. - \frac{1}{2\pi} \int_{C_\rho(z)} \frac{\partial}{\partial n_1} \log |\zeta - z| g(\zeta) \, ds(\zeta) \right) \\
&= \lim_{\rho \to 0} \left(\frac{1}{2\pi} \log \rho \iint_{B_\rho(z)} \Delta g(\zeta) \, dx \, dy + \frac{1}{2\pi\rho} \int_{C_\rho(z)} g(\zeta) \, ds(\zeta) \right),
\end{aligned}$$

where n_1 denotes the inner normal on $C_\rho(z)$. For $\rho \to 0$ the first term vanishes and the second term tends to $g(z)$.

For later use we remark

$$\iint \Delta g(\zeta)\, dx\, dy = 0. \tag{4.15}$$

This can be proved in the same way.

We are now able to prove (4.10). First,

$$\begin{aligned}(\mu - \tau)(S_\sigma^+(J) \cup S_\sigma(J)) &\le \int g(d\mu - d\tau) + \mu(J \backslash J_2) \\ &= \int g(d\mu - d\tau) + 2\delta.\end{aligned} \tag{4.16}$$

Next, using (4.12)-(4.15) and (1.9) together with Fubini's theorem

$$\begin{aligned}\left|\int g(d\mu - d\tau)\right| &= \left|\frac{1}{2\pi}\int\left(\iint \Delta g(\zeta)\log|z-\zeta|\, dx\, dy\right)(d\mu - d\tau)\right| \\ &= \left|\frac{1}{2\pi}\iint U^{\tau-\mu}(\zeta)\Delta g(\zeta)\, dx\, dy\right| \\ &= \left|\frac{1}{2\pi}\iint (U^{\tau-\mu}(\zeta) + \varepsilon)\Delta g(\zeta)\, dx\, dy\right| \\ &\le \frac{1}{2\pi}\iint_{V_\sigma} (U^{\tau-\mu} + \varepsilon)\frac{C_0}{\delta^2} r^2 \left(\frac{\partial G}{\partial n}\right)^2 dx\, dy \\ &\quad + \frac{1}{2\pi}\iint_{U_\sigma} (U^{\tau-\mu} + \varepsilon)\frac{C_1}{\delta^2} r^2 \left(\frac{\partial G_0}{\partial n_0}\right)^2 dx\, dy.\end{aligned}$$

Changing coordinates with Φ we get

$$\begin{aligned}\frac{1}{2\pi}\iint_{V_\sigma} (U^{\tau-\mu} + \varepsilon)\frac{C_0}{\delta^2} r^2 \left(\frac{\partial G}{\partial n}\right)^2 dx\, dy &= \frac{C_0}{2\pi\delta^2}\int_1^\sigma r \int_{\varphi_1}^{\varphi_2} (U^{\tau-\mu} + \varepsilon)\, d\varphi\, dr \\ &\le \frac{C_0}{2\pi\delta^2}\int_1^\sigma r \int_{\Gamma_r} (U^{\tau-\mu} + \varepsilon)\frac{\partial G}{\partial n}\, ds\, dr.\end{aligned}$$

For all functions h superharmonic in Ω, we have

$$\frac{1}{2\pi}\int_{\Gamma_r} h\frac{\partial G}{\partial n}\, ds \le h(\infty).$$

Thus, since $U^{\tau-\mu}$ is superharmonic in Ω,

$$\frac{1}{2\pi}\iint_{V_\sigma} (U^{\tau-\mu} + \varepsilon)\frac{C_0}{\delta^2} r^2 \left(\frac{\partial G}{\partial n}\right)^2 dx\, dy \le \frac{C_0\varepsilon}{2\pi\delta^2}\int_1^\sigma r\, dr \le C_3\frac{\varepsilon}{\delta} \tag{4.17}$$

for some constant $C_3 > 0$.

Since

$$\int_{\Gamma_{0,r}} h \frac{\partial G_0}{\partial n_0}\, ds \leq h(z_0)$$

for superharmonic functions h in F^*, we get analogously

$$\iint\limits_{U_\sigma} (U^{\tau-\mu} + \varepsilon) \frac{C_1}{\delta^2} r^2 \left(\frac{\partial G_0}{\partial n_0}\right)^2 dx\, dy \leq C_4 \frac{U^{\tau-\mu}(z_0) + \varepsilon}{\delta} \tag{4.18}$$

for some constant $C_4 > 0$. Combining (4.16), (4.17), and (4.18) we get (4.10).

The proof of

$$(\mu - \tau)(S_\sigma^+(J) \cup S_\sigma^-(J)) \geq C\sqrt{U^{\tau-\mu}(z_0) + 2\varepsilon}$$

follows the same line. This time we construct a test function $\tilde{g}$ which is 1 on $S_\sigma^+(J) \cup S_\sigma^-(J)$ and 0 on

$$\mathbf{C} \backslash (S_{1+2\delta}^+(J_0) \cup S_{1+2\delta}^-(J_0)),$$

where J_0 is a subarc of F such that $J_0 \backslash J$ consists of two subarcs J_1 and J_2 with $\mu(J_1) = \mu(J_2) = \delta$ if $\mu(F) - \mu(J) > 2\delta$. Otherwise $\tilde{g}$ is 1 on $S_\sigma^+(F) \cup S_\sigma^-(F)$ and 0 on the set $\mathbf{C} \backslash S_{1+2\delta}^+(F) \cup S_{1+2\delta}^-(F)$.

Then (4.16) is replaced by

$$\begin{aligned} -(\mu - \tau)(S_\sigma^+(J) \cup S_\sigma^-(J)) &\leq \int \tilde{g}(d\tau - d\mu) + 2\delta \\ &= \int -\tilde{g}(d\mu - d\tau) + 2\delta. \end{aligned}$$

The proof proceeds as above with g replaced by $-\tilde{g}$.

It remains to show (2.8) for

$$\sigma > 1 + c_0\sqrt{U^{\tau-\mu}(z_0) + 2\varepsilon}.$$

But, by Remarks 1 and 2,

$$|(\mu - \tau)((\mathbf{C} \backslash E_\sigma) \cup (F^* \backslash F_\sigma))| \leq \frac{U^{\tau-\mu}(z_0) + 2\varepsilon}{\log \sigma}.$$

Thus

$$|(\mu - \tau)((\mathbf{C} \backslash E_\sigma) \cup (F^* \backslash F_\sigma))| \leq C_5\sqrt{U^{\tau-\mu}(z_0) + 2\varepsilon},$$

where

$$C_5 = \max\left(1, \frac{1}{\log(1 + c_0)}\right).$$

The proof of Theorem 2 is now complete. ∎

Proof of Theorem 1. The proof follows the same ideas as the proof of Theorem 2. By Lemmas 1 and 4 there is a $\sigma_0 > 0$ such that (4.8) and (4.9) hold. It is sufficient to prove (2.5) for subarcs J of F which contain one of the endpoints of F. As in the proof of Theorem 2, we show that, for $\sigma = 1 + c_0\sqrt{\varepsilon}$, there is a constant $C > 0$ depending on E and c_0 such that

$$(\mu - \tau)(S_\sigma^+(J)) \le C\sqrt{\varepsilon}. \tag{4.19}$$

We may assume that $\sigma \le \sigma_0$ since otherwise (4.19) is satisfied with $C = c_0/(\sigma_0 - 1)$.

Let a and b be the endpoints of F and $a \in J$. As above we define the simply connected set

$$V_\sigma = M_\sigma \backslash (L(b) \cup F).$$

On V_σ we choose a harmonic conjugate $\varphi(z)$ of $\mathrm{G}(z)$. Then φ can be extended continuously to F from either side of F. We call these extensions φ_+ and φ_-. Again φ defines a projection π on V_σ to F along the orthogonal trajectories of the family of level lines Γ_σ.

We now decompose J into two subarcs J_1 and J_2 such that J_1 is compact, $a \in J_1$, and

$$\mu(J_2) = \delta := c_0\sqrt{\varepsilon}$$

(we may assume $\mu(J) > \delta$). Since $\mu(J_2)$ is the sum of the lengths of the intervals $\varphi_+(J_2)$ and $\varphi_-(J_2)$ we may suppose $\varphi_+(J_2) \ge \delta/2$. Using Lemma 5, we define $g_1: \mathbf{R} \to [0, 1]$ such that $g_1(\varphi_+(z)) = 1$ for all $z \in F$ in a neighborhood of J_1, $g_1(\varphi_+(z)) = 0$ for all $z \in F$ in a neighborhood of $F \backslash J$ and

$$\left|\frac{d^2}{d\varphi_+^2} g_1\right| \le \frac{5}{(\pi\delta)^2}, \qquad \left|\frac{dg_1}{d\varphi_+}\right| \le \frac{3}{\pi\delta}. \tag{4.20}$$

The function g_2 is defined by

$$g_2(z) := \begin{cases} \psi_1\left(\dfrac{e^{\mathrm{G}(z)} - 1}{\delta}\right) & \text{for } z \in M_\sigma, \\ 0 & \text{elsewhere} \end{cases}$$

($\psi = \psi_1$ from Lemma 5 for $m = 1, \eta = 1$). Finally, we set

$$g(z) := \begin{cases} g_1(\varphi_+(\pi(z)))g_2(z), & z \in M_\sigma \\ 0 & \text{elsewhere.} \end{cases}$$

We need to estimate $|\Delta g|$ and proceed as in the proof of Theorem 2. We remark that

$$\left|\frac{d^2}{d\varphi_-^2} g_1\right| = \left|\frac{d^2\varphi_+}{d\varphi_-^2}\frac{dg_1}{d\varphi_+} + \left(\frac{d\varphi_+}{d\varphi_-}\right)^2 \frac{d^2 g_1}{d\varphi_+^2}\right|.$$

By Lemma 4

$$\frac{d\varphi_+}{d\varphi_-} = \frac{\partial \mathrm{G}/\partial n_+}{\partial \mathrm{G}/\partial n_-}$$

is bounded on F, just as

$$\frac{d^2\varphi_+}{d\varphi_-^2} = \frac{ds}{d\varphi_-}\left(\frac{d}{ds}\frac{d\varphi_+}{d\varphi_-}\right) \tag{4.21}$$

$$= \frac{dq/ds}{\partial G/\partial n_-},$$

where q denotes the function in Lemma 4. Thus we deduce

$$|\Delta g(z)| \le r^2 \left(\frac{\partial G}{\partial n}\right)^2 \frac{C_1}{\delta^2}$$

for all $z \in M_\sigma \backslash F$ and some constant $C_1 > 0$.

Since $\Delta g = 0$ in a neighborhood of a and b, $|\Delta g|$ is again bounded on $\mathbf{C}\backslash F$. Therefore the representation formula (4.16) holds. Following the proof of Theorem 2 we get (4.19) and subsequently

$$(\tau - \mu)(S_\sigma^+(J)) \le C\sqrt{\varepsilon}.$$

Finally, Remarks 1 and 2 show that (2.5) is true for all $\sigma \ge 1 + c_0\sqrt{\varepsilon}$. ■

Proof of Theorem 3. We set

$$\alpha := \inf E, \qquad \beta := \sup E.$$

Since all zeros of p are real, it suffices to prove

$$(\mu - \tau)(I) \le 4\sqrt{\frac{2\varepsilon}{\pi}} + 4\varepsilon. \tag{4.22}$$

for any interval $I = (-\infty, b]$, where $b \le \beta$. We may assume that $\mu(I) > 4\sqrt{2\varepsilon/\pi}$, since otherwise (4.22) is obvious.

Let $b' \in E$ such that $\mu([b', b]) = \delta\sqrt{\varepsilon}$, where $\delta \le 4\sqrt{2/\pi}$ will be specified later. Then we consider the Green function $G^*(z)$ for $\mathbf{C}\backslash E^*$ with pole at infinity where E^* is the point set

$$E^* := E \cup [b', b].$$

E^* is again regular.

$\mu_{E^*}([b', b])$ is equal to the value at infinity of the function h^*, harmonic in $\bar{\mathbf{C}}\backslash E^*$ with boundary value 1 on $(b', b) \cap E$ and 0 on $E^*\backslash[b', b]$. Analogously, $\mu_E([b', b])$ is the value at infinity of the function h, harmonic in $\bar{\mathbf{C}}\backslash E$ which is 1 on $(b', b) \cap E$ and 0 on $E\backslash[b', b]$. Since

$$([b', b] \cap E) \subseteq [b', b],$$

it follows from the maximum principle for harmonic functions that $h^* \ge h$ in $\bar{\mathbf{C}}\backslash E^*$ and

$$\mu_{E^*}([b', b]) \ge \mu_E([b', b]) = \mu([b', b]). \tag{4.23}$$

The proof now follows the above ideas. Let $\varphi(z)$ be a harmonic conjugate of G^*

on $\mathbf{C}\setminus[\alpha, \infty)$. Then $\varphi(z)$ extends continuously to $[\alpha, \infty)$ from the upper half-plane. Moreover, (4.23) yields

$$\varphi(b') - \varphi(b) \geq \delta\pi\sqrt{\varepsilon}.$$

Fix $\eta > 0$. By Lemma 5 there is a function $g_1\colon \mathbf{R} \to [0, 1]$, twice continuously differentiable, and $\kappa > 0$ such that $g_1(\varphi) = 1$ for $\varphi \geq \varphi(b') - \kappa$, $g_1(\varphi) = 0$ for $\varphi \leq \varphi(b) + \kappa$, and

$$\left| \frac{d^2}{d\varphi^2} g_1 \right| \leq \frac{4 + \eta}{\varepsilon(\delta\pi)^2}.$$

Next, let

$$g_2(z) := \psi_1\left(\frac{e^{G^*(z)} - 1}{\delta\pi\sqrt{\varepsilon}}\right)$$

($\psi_1 = \psi$ from Lemma 5 with $m = 1, \eta > 0$ fixed). Then we set

$$g(z) := \begin{cases} g_1(\varphi(z))g_2(z) & \text{for} \quad \mathrm{Im}(z) \geq 0, \\ g_1(\varphi(\bar{z}))g_2(z) & \text{for} \quad \mathrm{Im}(z) < 0. \end{cases}$$

Because of the symmetry of G(z), namely $G(z) = G(\bar{z})$, g is continuous on $\mathbf{C}$. Since g has compact support and

$$\frac{\partial}{\partial n_+} g(z) = \frac{\partial}{\partial n_-} g(z) = 0 \qquad \text{for all} \quad z \in [\alpha, \beta],$$

where $n_+ = i$ and $n_- = -i$, the representation formula (4.14) holds.

Introducing Green coordinates (r, φ) again such that $G(z) = r$, we have

$$\begin{aligned} |\Delta g(z)| &= r^2\left(\frac{\partial G}{\partial n}\right)^2 \left| g_1 \frac{\partial^2 g_2}{\partial r^2} + \frac{1}{r} g_1 \frac{\partial g_2}{\partial r} + \frac{1}{r^2} g_2 \frac{\partial^2 g_1}{\partial \varphi^2} \right| \\ &\leq r^2\left(\frac{\partial G}{\partial n}\right)^2 \left| 1 + \frac{r-1}{r} + \frac{1}{r^2} \right| \frac{4 + \eta}{(\delta\pi)^2\varepsilon} \\ &\leq r^2\left(\frac{\partial G}{\partial n}\right)^2 \frac{8 + 2\eta}{(\delta\pi)^2\varepsilon} \end{aligned}$$

for all $z \in \mathbf{C}\setminus[\alpha, \infty)$, where we have used

$$\frac{\partial g_2}{\partial r}(r) = \int_1^r \frac{\partial^2 g_2}{\partial r^2}\, dr.$$

Again the estimates start with

$$(\mu - \tau)(I) \leq \int g(d\mu - d\tau) + \mu([b', b]).$$

Proceeding as in the proof of Theorem 2 we obtain, with $\eta \to 0$,

$$\int g(d\mu - d\tau) \leq \frac{4}{(\delta\pi)^2}(2\delta\pi\sqrt{\varepsilon} + (\delta\pi)^2\varepsilon).$$

Thus

$$(\mu - \tau)(I) \le \left(\frac{8}{\pi\delta} + \delta\right)\sqrt{\varepsilon} + 4\varepsilon.$$

Choosing $\delta = \sqrt{8/\pi}$ (4.22) follows. Furthermore, for $\sqrt{\varepsilon} < \frac{1}{8}$, inequality (2.9) yields (2.10). But, for $\sqrt{\varepsilon} \ge \frac{1}{8}$, (2.10) is obvious. ∎

Proof of Theorem 4. First we prove

$$(\mu - \tau)(S_\sigma^+(I)) \le \left(\frac{8}{\pi\delta} + \delta\sqrt{k}\right)\sqrt{\varepsilon} + 4\varepsilon \tag{4.24}$$

for any interval $I = [\alpha, b]$ where $b \le \beta$ (α and β as in the proof of Theorem 3). We may assume $\mu([\alpha, b]) > \delta\sqrt{k\varepsilon}$. Define $b' \in E$ such that

$$\mu([b', b]) = \delta\sqrt{k\varepsilon}. \tag{4.25}$$

The arguments which follow depend only on G(z), i.e., the role of G*(z) in the proof of Theorem 3 is taken over by the function G(z) itself.

For $z \in \mathbf{C}\backslash[\alpha, \infty]$, define a harmonic conjugate φ of G. We extend φ continuously to $[\alpha, \beta]$ from the upper half-plane and remark that φ is constant on each interval $S_i := [\sup I_i, \inf I_{i+1}]$. Let $m - 1$ be the number of intervals S_i in (b', b),

$$b' < S_{j+1} < S_{j+2} < \cdots < S_{j+m-1} < b.$$

Then we set

$$\tau_0 = \varphi(b'), \quad \tau_1 = \varphi(S_{j+1}), \quad \ldots, \quad \tau_{m-1} = \varphi(S_{j+m-1}), \quad \tau_m = \varphi(b),$$

and note that $\tau_m - \tau_0 = \delta\pi\sqrt{k\varepsilon}$. For this distribution $\tau_0, \ldots, \tau_m$ and $\eta > 0$, Lemma 5 yields a function $g_1 : \mathbf{R} \to [0, 1]$ such that

$$\left|\frac{d^2}{d\varphi^2} g_1\right| \le \frac{4 + \eta}{(\delta\pi)^2\varepsilon}\frac{m}{k} \le \frac{4 + \eta}{(\delta\pi)^2\varepsilon}.$$

Moreover, we set

$$g_2(z) = \psi_1\left(\frac{e^{\mathrm{G}(z)} - 1}{\delta\pi\sqrt{\varepsilon}}\right)$$

and

$$g(z) := \begin{cases} g_1(\varphi(z))g_2(z) & \text{for} \quad \mathrm{Im}(z) \ge 0, \\ g_1(\varphi(\bar{z}))g_2(z) & \text{for} \quad \mathrm{Im}(z) < 0. \end{cases}$$

The same estimates as in the proof of Theorem 3 yield

$$|\Delta g(z)| \le r^2\left(\frac{\partial \mathrm{G}}{\partial n}\right)^2 \frac{8 + 2\eta}{(\delta\pi)^2\varepsilon} \qquad \text{for} \quad \mathbf{C}\backslash[\alpha, \infty).$$

Proceeding as in that proof we get (4.24).

Of course, (4.24) holds for any interval of the form $[c, \beta]$ too. Furthermore, we deduce from Remark 1 with some calculations

$$\tau(\mathbb{C}\backslash E_{1+\pi\delta\sqrt{\varepsilon}}) \le \frac{(e-1)\sqrt{\varepsilon}}{\pi\delta} + \varepsilon.$$

Observing $(\tau - \mu)(\mathbb{C}) = 0$, we obtain (2.13). Now we insert $\delta = 3/(\sqrt{\pi}k^{1/4})$ in (2.13). This yields (2.14). ■

To prove (2.17) in Remark 3, we note that all components of $E_{1+3\sqrt{\pi\varepsilon}}$ contain exactly one interval of E. We fix $b' \in E$ such that

$$\mu([b', b]) = \delta\sqrt{\varepsilon}.$$

If $[b', b] \subset E$ we proceed as in the proof of Theorem 3 (resp. as in the proof of Theorem 4 with $k = 1$). If $[b', b] \not\subset E$ we have to modify $g(z)$ in a straightforward way. Let I_j be the interval with $b' \in I_j$ and let

$$\kappa = \max_{z \in I_j} z.$$

Then we define, for $\mathrm{Im}(z) \ge 0$,

$$g(z) := \begin{cases} g_2(z) & \text{for } \varphi(z) \le \varphi(\kappa), \\ 0 & \text{elsewhere} \end{cases}$$

and extend $g(z)$ to $\mathrm{Im}(z) < 0$ by $g(z) = g(\bar{z})$.

5. Proof of the Corollaries

Proof of Corollary 1. The zeros of the polynomial $p_{n+1}^* - p_n^* \in \Pi_{n+1}$ are all real and separate the alternation points. Let

$$p_{n+1}^* - p_n^* = \alpha_{n+1} T_{n+1} + q_n,$$

where $q_n \in \Pi_n$ and T_{n+1} is the Chebyshev polynomial of degree $n + 1$ on E. Then

$$|a_{n+1}| \, \| T_{n+1} \|_E = \| f - (p_n^* + q_n) - (f - p_{n+1}^*) \|_E.$$

A result of Widom [16] shows that there exists a constant $c > 0$ such that

$$\| T_n \|_E \le c(\mathrm{cap}\, E)^n \tag{5.1}$$

for all n. Hence,

$$|a_{n+1}| \ge \frac{e_n - e_{n+1}}{c(\mathrm{cap}\, E)^{n+1}} \tag{5.2}$$

and for the monic polynomial $p_{n+1} := (p_{n+1}^* - p_n^*)/a_{n+1}$ we obtain

$$\| p_{n+1} \|_E \le c \frac{e_n + e_{n+1}}{e_n - e_{n+1}} (\mathrm{cap}\, E)^{n+1} \tag{5.3}$$

for integers n, where $e_n \neq e_{n+1}$. Since $\lim_{n\to\infty} e_n = 0$, there exists a subsequence $(n_j)_{j=1}^{\infty}$ such that

$$e_{n+1} \leq \left(1 - \frac{1}{n^2}\right)e_n \qquad \text{for} \quad n = n_j, \quad j = 1, \ldots,$$

and therefore, for such n,

$$\frac{e_n - e_{n+1}}{e_n + e_{n+1}} \geq \frac{1}{2n^2}.$$

Then, by (5.3) there exists a constant γ such that

$$\| p_{n+1} \|_E \leq \gamma n^2 (\text{cap } E)^{n+1}$$

for $n = n_j$. Then Theorem 1 together with $\varepsilon = O((\log n)/n)$ yields

$$|(\tau_{p_{n+1}} - \mu_E)(I)| \leq C_0 \sqrt{\frac{\log n}{n}}$$

for any interval $I \subset \mathbf{R}$. Since the zeros of p_{n+1} separate the alternation points, (3.2) is true. ∎

Proof of Corollary 2. The proof is based on estimate (3.9) in Blatt, Saff, and Totik [3], namely

$$\log V(\mathscr{F}_{n+2}(M_n(f))) \geq \frac{n+2}{2} \log \| T_n \|_E - \frac{3(n+2)}{2} \log(n+2).$$

We may assume that $\text{cap}(E) = 1$. Since $\| T_n \|_E > \text{cap}(E)^n$, we get

$$\log V(\mathscr{F}_{n+2}(M_n(f))) \geq -\frac{3(n+2)}{2} \log(n+2). \tag{5.4}$$

Dropping the index n, we set

$$Z^1 := \mathscr{F}_{n+2}(M_n(f)) = \{z_0, \ldots, z_{n+1}\},$$
$$\omega_1(z) := \prod_{k=0}^{n+1} (z - z_k),$$

and define Z^j, ω^j inductively by

$$Z^j := Z^{j-1} \cup \{z_{n+j}\},$$
$$\omega_j(z) := \omega_{j-1}(z)(z - z_{n+j}),$$

where $\omega_{j-1}(z)$ attains its maximum modulus M_j on E at the point $z_{n+j} \in E$. By the construction we get, for the Vandermonde expression (3.3),

$$V(Z^{j+1}) = V(Z^j)M_j. \tag{5.5}$$

Let $n \geq 4$ and suppose, for some subarc J of F,

$$\mu_E(J)(n+2) - \tilde{A}_n(f, J) > 2m,$$

where m is a positive integer. Then $m \le n-2$ and, for $1 \le j \le m+1$,

$$\mu_E(J)(n+j+1) > m + \sum_{z \in Z^j \cap J} 1.$$

Hence Theorem 1 (resp. Theorem 2) implies that there is a constant $c > 0$ with

$$\log M_j \ge \left(\frac{m}{c}\right)^2 \frac{1}{n+j+1} \ge \frac{m^2}{2c^2 n}$$

for $1 \le j \le n-1$, and by (5.4) and (5.5)

$$\log V(Z^{m+1}) \ge \sum_{j=1}^{m} \log M_j + \log V(Z^1) \tag{5.6}$$

$$\ge \frac{m^3}{2c^2 n} - \frac{3(n+2)}{2} \log(n+2).$$

On the other hand, we use an upper bound for $V(Z^{m+1})$, namely a result of Siciak [12]. There exists a constant $M > 0$ such that, for all n,

$$\frac{2}{n(n-1)} \log V_n \le \log \operatorname{cap}(E) + \frac{6}{n-1} \log(Mn),$$

where V_n is the Vandermonde expression for an n-point Fekete point set in E. This is true because each connected component of E has a diameter greater than ρ with $\rho > 0$ fixed.

Since $\log \operatorname{cap}(E) = 0$, (5.6) yields

$$\log V(Z^{m+1}) \le \log V_{n+m+2} \le 3(n+m+2) \log[M(n+m+2)].$$

Hence, by (5.6), there exists a constant $\gamma > 0$ such that

$$m^3 \le \gamma n^2 \log n.$$

Thus

$$\mu_E(I)(n+2) - A_n(f, I) \le \operatorname{const}(n^2 \log n)^{1/3}$$

and this is equivalent to the statement of Corollary 2. ■

Proof of Corollary 3. The statement is an immediate consequence of the above-mentioned result of Widom (5.1) and Theorem 1. ■

Proof of Corollary 4. We may assume $\|f\|_E = 1$. Thus $\|p_n^*\| \le 2$ for all $n \in \mathbb{N}$. Let a_n denote the highest coefficient of p_n^* then, as in (5.2),

$$|a_n| \ge \frac{1}{c} \frac{e_{n-1} - e_n}{(\operatorname{cap} E)^n}.$$

Next, consider the monic polynomial $p_n := (1/a_n)p_n^*$, whenever $a_n \neq 0$. For such n,

$$\|p_n\|_E \leq \frac{2c}{e_{n-1} - e_n} (\text{cap } E)^n. \tag{5.7}$$

Now we show that there exists a subsequence (n_j) and a constant $s > 0$ such that

$$e_{n_j-1} - e_{n_j} \geq \frac{1}{n_j^s} \tag{5.8}$$

for all $j \in \mathrm{N}$.

Let us assume that (5.8) is false and fix a real number $s \geq 2$. Then there exists an $n_0 \in \mathrm{N}$ such that

$$e_{n-1} - e_n \leq \frac{1}{n^s}$$

for all $n \geq n_0$ and

$$\begin{aligned} \|f - p_n^*\|_E &\leq \sum_{j=1}^{\infty} \frac{1}{(n+j)^s} \\ &\leq \int_n^{\infty} \frac{1}{x^s}\, ds \\ &= \frac{1}{(s-1)n^{s-1}} \end{aligned} \tag{5.9}$$

for all $n \geq n_0$. Furthermore, the generalized Markov inequality yields a constant $C > 0$ with

$$\|q_n'\|_E \leq Cn^2 \|q_n\|_E \qquad \text{for all} \quad q_n \in \Pi_n \tag{5.10}$$

[12, Remark to Lemma 1 on p. 50].

Since s is arbitrary, it follows from (5.9) that all derivatives of p_n^* converge uniformly on E. This implies that all derivatives of f exist on E, which contradicts our assumption. This proves (5.8).

Let us first assume that F is a Jordan arc. By (5.8) and (5.7)

$$\|p_n\|_E \leq 2cn_j^s(\text{cap } E)^{n_j}.$$

Thus (1.9) is replaced by

$$U^{\mu - \tau_{p_{n_j}}}(z) \leq \varepsilon_{n_j}$$

with

$$\varepsilon_{n_j} \leq \frac{\log 2c + s \log n_j}{n_j} \leq c_0 \frac{\log n_j}{n_j} \tag{5.11}$$

for some constant $c_0 > 0$. Inequality (3.6) now follows from Theorem 1.

In the case of a Jordan curve we fix z_0 in the interior of F such that $f(z_0) \neq 0$.

Such a z_0 exists by assumption. Then

$$\begin{aligned} U^{\mu - \tau_{p_{n_j}}}(z_0) &= \frac{1}{n_j} \log |p_{n_j}(z_0)| - \log \operatorname{cap} E \\ &\le \frac{1}{n_j} \log |p^*_{n_j}(z_0)| - \frac{1}{n_j} \log |a_{n_j}| - \log \operatorname{cap} E \\ &\le c_1 \frac{\log n_j}{n_j} \end{aligned}$$

as above and (3.8) follows from Theorem 2.

In the special case $E \subset \mathbf{R}$ and $f \notin C^\alpha(E)$, the above-defined real number s satisfies

$$s \le 2 + 2\alpha.$$

Otherwise, Markov's inequality together with (5.7) imply $f \in C^\alpha(E)$. Hence, (5.11) leads to

$$\varepsilon_{n_j} \le (3 + 2\alpha) \frac{\log n_j}{n_j}$$

for a subsequence (n_j). Therefore, the critical points of $G(z)$ are outside of $E_{1+3\sqrt{\pi\varepsilon_{n_j}}}$ for $j \ge j_0$. Then Remark 3 yields the desired inequality (3.8). ■

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