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Exchangeable Processes Need Not Be Mixtures of Independent, Identically Distributed Random Variables*

Lester E. Dubins and David A. Freedman
Department of Statistics, University of California, Berkeley CA 94720, USA

Summary. According to a theorem of de Finetti's, an exchangeable stochastic process with values in a compact metric space can be represented as a mixture of sequences of independent, identically distributed random variables. This paper demonstrates the existence of a separable metric space for which the conclusion fails. In the opposite direction, an example is given of a nonstandard space for which the representation necessarily holds.

Modifications of the argument lead to examples of exchangeable stochastic processes and stationary Markov processes which take values in a separable metric space but do not satisfy the conclusions of the Kolmogorov consistency theorem.

1. Introduction

Let $(S, \mathscr{F})$ be a measurable space, and

$$(S^\infty, \mathscr{F}^\infty) = \prod_{n=1}^{\infty} (S, \mathscr{F}),$$

the usual product space. A permutation π of the positive integers is *finite* if $\pi(n) = n$ for all but finitely many n. Each π induces a measurable mapping $\tilde{\pi}$ on S^∞ as follows:

$$\tilde{\pi}(x_1, x_2, \ldots) = (x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

A probability P on $\mathscr{F}^\infty$ is *exchangeable* if P is invariant under all $\tilde{\pi}$. (Unless specified otherwise, in this paper probabilities are countably additive.)

Let S^* denote the class of all probability θ on $(S, \mathscr{F})$. Endow S^* with the "weak*" σ-field $\mathscr{F}^*$, namely, the σ-field generated by the sets $\{\theta: \theta(F) < t\}$, where

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F ranges over $\mathscr{F}$ and t over [0, 1]. For $\theta \in S^*$, let θ^∞ be the power probability on $(S^\infty, \mathscr{F}^\infty)$ which makes the coordinates independent, with common distribution θ.

If μ is a probability on $\mathscr{F}^*$, define the probability P_μ on $(S^\infty, \mathscr{F}^\infty)$ as follows:

$$(1.1)\quad P_\mu(A) = \int_{S^*} \theta^\infty(A)\, \mu(d\theta), \qquad \text{all } A \in \mathscr{F}^\infty.$$

Clearly, each θ^∞ is exchangeable, and so is P_μ. De Finetti's theorem and its generalizations give the converse: for reasonable spaces $(S, \mathscr{F})$, if P is exchangeable in S^∞, there is a μ on $\mathscr{F}^*$ with $P = P_\mu$. For instance, as proved by Hewitt and Savage [6], it is enough to assume that S is compact Hausdorff, and $\mathscr{F}$ is the Baire σ-field in S.

Hewitt and Savage raised the question of whether the conclusion holds even in the absence of topology, that is, for an abstract measurable space $(S, \mathscr{F})$. For a recent discussion, see Varadarajan [13]. The main object of this paper is to answer this question in the negative: There is a separable (nonstandard) metric space S, equipped with its Borel σ-field $\mathscr{F}$, and an exchangeable probability P on $\mathscr{F}^\infty$ which cannot be presented in the form (1.1).

In the terminology of Hewitt and Savage, if $P = P_\mu$ for some μ, then P is *presentable*. If every exchangeable P on $(S^\infty, \mathscr{F}^\infty)$ is presentable, then $\mathscr{F}$ itself is called presentable. In these terms, the main result is that there is a separable metric space whose Borel σ-field is not presentable; this theorem is proved in Sect. 2.

What happens when the mixing measure μ in (1.1) is allowed to be only finitely additive? This is discussed in Sect. 3, where it is shown that P_μ is countable additive iff μ is. So the exchangeable (and countable additive) probability P constructed in Sect. 2 cannot even be represented as a finitely additive mixture of (countably additive) power probabilities.

Section 4 contains some remarks to clarify the results of Sect. 2 and 3. Section 5 gives an example of a separable metric space which, though nonstandard is presentable. The argument is modified in Sect. 6 to make two examples: an exchangeable stochastic process and a stationary Markov process which take values in a separable metric space but which do not satisfy the conclusions of the Kolmogorov consistency theorem. Section 7 gives an example of a finitely additive exchangeable probability on the space of sequences of 0's and 1's which cannot be represented as a finitely additive mixture of countably additive power probabilities.

2. The Construction

Let I be the closed unit interval, equipped with the usual Borel σ-field $\mathscr{B}$. For $t \in I$, let t_j be the j^{th} digit in the binary expression of t, so

$$t = \sum_{j=1}^{\infty} t_j / 2^j, \qquad t_j = 0 \quad \text{or} \quad 1.$$

(2.1) For $0 \leqq p \leqq 1$, let θ_p be the probability on $(I, \mathscr{B})$ which makes the t_j's independent, with common distribution

$$\theta_p\{t_j=1\}=p, \qquad \theta_p\{t_j=0\}=1-p.$$

Let

(2.2) $Q=\int_0^1 \theta_p^\infty \lambda(dp)$, $\quad \lambda=$Lebesgue measure on $(I, \mathscr{B})$.

Thus, Q is an exchangeable probability on $(I^\infty, \mathscr{B}^\infty)$.

(2.3) Let $Z(t)=\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} t_j$, on the subset L of I where the limit exists.

Plainly, L is a Borel set and Z is a partially defined Borel function with domain L. Furthermore,

(2.4) $\theta_p(Z=p)=1$.

Let $x=(x_1, x_2, \ldots)$ be a typical point in I^∞. Let G be the set of x in I^∞ such that $x_j \in L$ and $Z(x_j)=Z(x_1)$ for all j.

(2.5) **Lemma.** (a) *$G \in \mathscr{B}^\infty$ and $Q(G)=1$.*
(b) *For $C \in \mathscr{B}$,*

$$Q\{x: Z(x_1) \in C\}=\lambda(C).$$

The proof is routine.

(2.6) **Lemma.** *Let $T \subset I$ have cardinality strictly less than c, the cardinality of the continuum. Let $\tilde{T}=\bigcup_{j=1}^{\infty} T_j$, where T_j is the set of all x in I^∞ with $x_j \in L$ and $Z(x_j) \in T$. Then $\tilde{T}$ has inner Q-measure* 0.

Proof. Let $A \in \mathscr{B}^\infty$ and $A \subset \tilde{T}$. By Lemma 2.5a, to show $Q(A)=0$, it is enough to show that $Q(A \cap G)=0$. To do this, let Z_1 map G into I as follows: $Z_1(x)=Z(x_1)$. Clearly, Z_1 is Borel, so $C=Z_1(A \cap G)$ is an analytic subset of I. But $C \subset T$, so the cardinality of C is strictly less than c. It follows that C is countable (Kuratowski, 1958, pp. 351, 387). Consequently, $\lambda(C)=0$. Plainly, $A \cap G \subset Z_1^{-1} C$, so (2.5b) implies

$$Q(A \cap G) \leqq Q(Z_1^{-1} C)=\lambda(C)=0. \quad \square$$

The main step in the construction is the next proposition. In the statement, and later, an asterisk will be used to denote outer measure.

(2.7) **Proposition.** *Define Q and Z as in* (2.2)–(2.3). *There is a subset S of the unit interval I with the following two properties:*

(2.8) $Q^*(S^\infty)=1$,

(2.9) $S \cap \{Z=p\}$ *is countable, for each* $p \in I$.

Proof. Let K be the set of ordinals of cardinality strictly less than c. Then K has cardinality c. Let $\mathscr{K}$ be the set of all $A \in \mathscr{B}^\infty$ of positive Q-measure. Then $\mathscr{K}$ too has cardinality c. Hence, there is a $1-1$ map $\alpha \to A_\alpha$ of K onto $\mathscr{K}$.

For each $\alpha \in K$, choose a point $y_\alpha \in A_\alpha$ as follows. Fix $\beta \in K$, and suppose by induction that the y_α have been chosen for all $\alpha < \beta$. Now $y_\alpha \in I^\infty$: let $y_{\alpha j}$ be its j^{th} coordinate. Let T_β be the set of relative frequencies obtained so far: $t \in T_\beta$ iff $t = Z(y_{\alpha j})$ for some $\alpha < \beta$ and some $j = 1, 2, \ldots$ with $y_{\alpha j} \in L$. The cardinality of T_β is strictly less than c, because the product of two infinite cardinals is just the larger. Define $\tilde{T}_\beta$ as in (2.6): by that result, $\tilde{T}_\beta$ has inner Q-measure 0, so $A_\beta - \tilde{T}_\beta$ is nonempty. Now choose $y_\beta \in A_\beta - \tilde{T}_\beta$.

Having chosen the y_α for all $\alpha \in K$, let

$$S = \bigcup_{\alpha \in K} \bigcup_{j=1}^{\infty} \{y_{\alpha j}\}.$$

Then $y_\alpha \in S^\infty \cap A_\alpha$, so S^∞ intersects each $A \in \mathscr{B}^\infty$ of positive Q-measure, and (2.8) follows. On the other hand, $y_\beta \in A_\beta - \tilde{T}_\beta$, so for each p, $Z(y_{\alpha j}) = p$ for at most one α, proving (2.9). □

For use in Sect. 3, the next two lemmas are stated in terms of finitely additive probabilities. Only countably additive probabilities are involved in this section. If μ is a finitely additive probability on $(I, \mathscr{B})$, the *outer measure* μ^* of $A \subset I$ is defined as usual:

$$\mu^*(A) = \inf\{\mu(B) \colon B \in \mathscr{B} \text{ and } B \supset A\}.$$

(2.10) **Lemma.** *Let (X, Σ) be an abstract measurable space. Let Y be a subset of X, not necessarily an element of Σ. Let $\Sigma_Y = Y \cap \Sigma$ be the σ-field of subsets of Y of the form $Y \cap B$, with $B \in \Sigma$.*

(a) *Let ϕ be a finitely additive probability on (Y, Σ_Y). Then ϕ induces a finitely additive probability $\eta\phi$ on (X, Σ) by the rule*

$$(\eta\phi)(B) = \phi(Y \cap B) \quad \textit{for } B \in \Sigma.$$

And $(\eta\phi)^*(Y) = 1$. If ϕ is countably additive, so is $\eta\phi$.

(b) *Let θ be a finitely additive probability on (X, Σ) with $\theta^*(Y) = 1$. Then θ has a trace finitely additive probability $\rho\theta$ on (Y, Σ_Y):*

$$(\rho\theta)(Y \cap B) = \theta(B) \quad \textit{for } B \in \Sigma.$$

If θ is countably additive, so is $\rho\theta$.

(c) *The map η, defined in* (a), *is $1-1$; its range is the set of finitely additive probabilities assigning outer measure 1 to Y, and its inverse is ρ, as defined in* (b).

(d) *Consider η as acting only on the set Y^* of countably additive probabilities on (Y, Σ_Y), and ρ as acting only on*

$$\tilde{Y} = \{\theta \colon \theta \in X^* \text{ and } 0^*(Y) = 1\}, \qquad \textit{where } \theta \in X^* \textit{ is countably additive on } (X, \Sigma).$$

Then η is (Σ_Y^*, Σ^*)-measurable, and ρ is $(\tilde{Y} \cap \Sigma^*, \Sigma_Y^*)$-measurable.

Proof. Part (a) is routine.

Part (b). First, $\rho\theta$ is well-defined: if $B_0, B_1 \in \Sigma$ and $Y \cap B_0 = Y \cap B_1$, then $B_0 \triangle B_1 \in \Sigma$ and is disjoint from Y, so $\theta(B_0 \triangle B_1) = 0$. Second, $\rho\theta$ is additive: if $Y \cap B_0$ and $Y \cap B_1$ are disjoint, then $B_0 \cap B_1 \in \Sigma$ is disjoint from Y and $\theta(B_0 \cap B_1) = 0$.

Part (c). Suppose ϕ is a finitely additive probability on (Y, Σ_Y). Then $\rho\eta\phi = \phi$, because for $B \in \Sigma$,

$$(\rho\eta\phi)(Y \cap B) = \eta\phi(B) = \phi(Y \cap B).$$

Part (d). To show η is measurable, fix $B \in \Sigma$ and $0 < t < 1$. Then

$$\eta^{-1}\{\theta\colon \theta \in X^* \text{ and } \theta(B) < t\} = \{\phi\colon \phi \in Y^* \text{ and } \phi(Y \cap B) < t\} \in \Sigma_Y^*.$$

The argument for ρ is almost the same. □

The next lemma will be helpful in the proof of the main theorem. Note that $\phi \in S^*$ is a countably additive probability on $(S, \mathscr{F})$; and $\eta\phi$, defined by (2.10), is a countably additive probability on $(I, \mathscr{B})$. However, P, Q and ν may be only finitely additive. Of course, S^∞ is a subset of I^∞; by (2.10), any finitely additive probability P on $(S^\infty, \mathscr{F}^\infty)$ induces a finitely additive probability Q on $(I^\infty, \mathscr{B}^\infty)$: write $Q = \eta^\infty P$. Here and later, $\mathscr{F} = S \cap \mathscr{B}$.

(2.11) **Lemma.** (a) $S^\infty \cap \mathscr{B}^\infty = (S \cap \mathscr{B})^\infty$

(b) $\eta^\infty \phi^\infty = (\eta\phi)^\infty$ *for* $\phi \in S^*$

(c) *If* $A \in \mathscr{B}^\infty$, *then* $\phi \to (\eta^\infty \phi^\infty)(A)$ *is* $\mathscr{F}^*$*-measurable.*

Let ν *be a finitely additive probability on* $(\mathscr{S}^*, \mathscr{F}^*)$, *and let*

$$(2.12) \quad P = \int_{S^*} \phi^\infty \, \nu(d\phi)$$

be a finitely additive, exchangeable probability on $(S^\infty, \mathscr{F}^\infty)$. *Then* P *induces a finitely additive, exchangeable probability* $\eta^\infty P$ *on* $(I^\infty, \mathscr{B}^\infty)$, *and*

$$(2.13) \quad \eta^\infty P = \int_{S^*} (\eta\phi)^\infty \, \nu(d\phi)$$

where $\eta\phi$ *is the probability induced by* ϕ *on* $(I, \mathscr{B})$.

Proof. Part (a) is routine.

Part (b). Fix n and $B_1, \ldots, B_n$ in $\mathscr{B}$. Let

$$A = \{x\colon x \in I^\infty \text{ and } x_i \in B_i \text{ for } i = 1, \ldots, n\}.$$

Then

$$\begin{aligned} \eta^\infty \phi^\infty(A) &= \phi^\infty(S^\infty \cap A) \\ &= \prod_{i=1}^{n} \phi(S \cap B_i) \\ &= \prod_{i=1}^{n} (\eta\phi)(B_i) \\ &= (\eta\phi)^\infty(A). \end{aligned}$$

The rest of the argument is routine.

Part (c) is routine.

For the rest, it is easy to verify that $\eta^\infty P$ is exchangeable. For (2.13), fix $A \in B^\infty$. Then

$$\begin{aligned}(\eta^\infty P)(A) &= P(S^\infty \cap A)\\ &= \int_{S^*} \phi^\infty(S^\infty \cap A)\, \nu(d\phi) \quad \text{by (a)}\\ &= \int_{S^*} \eta^\infty \phi^\infty(A)\, \nu(d\phi)\\ &= \int_{S^*} (\eta\phi)^\infty(A)\, \nu(d\phi) \quad \text{by (b).} \quad \square\end{aligned}$$

The next theorem shows there is a separable metric space S whose Borel σ-field is not presentable. Indeed, let S be the subset of $I=[0,1]$ constructed in (2.7). Of course, S is separable in the relative metric, and $\mathscr{F} = S \cap \mathscr{B}$ is the Borel σ-field of S. Define the exchangeable probability Q on $(I^\infty, \mathscr{B}^\infty)$ by (2.1)–(2.2). Let P be the trace of Q on $(S^\infty, \mathscr{F}^\infty)$: this is legitimate by (2.8) and Lemma 2.10b.

(2.14) **Theorem.** *The probability P on $(S^\infty, \mathscr{F}^\infty)$ is exchangeable, but cannot be presented in the form* (1.1) *as a mixture of independent, identically distributed random variables.*

Proof. Suppose by way of contradiction that P were presentable:

(2.15) $$P = \int_{S^*} \phi^\infty\, \nu(d\phi).$$

By (2.11),

(2.16) $$Q = \eta^\infty P = \int_{S^*} (\eta\phi)^\infty\, \nu(d\phi).$$

Let R be the range of the mapping $p \to \theta_p$ from I to I^*. Then $R \in \mathscr{B}^*$. To verify this, endow I^* with the weak*-topology, so I^* is compact metric and $\mathscr{B}^*$ is the Borel σ-field in I^*. As is easily verified, the map $p \to \theta_p$ is continuous, so its range is a compact set in I^*.

As is well known [6], the μ in (1.1) is unique. (This is discussed in the next section.) Comparing (2.2) and (2.16), the ν-distribution of $\phi \to \eta\phi$ coincides with the λ-distribution of $p \to \theta_p$. In particular, $\nu\eta^{-1}R = 1$. Consequently, there is at least one $\phi \in S^*$ and $p \in (0,1)$ with $\eta\phi = \theta_p$. And this is a contradiction: $(\eta\phi)^*(S) = 1$ by (2.10a) and $\theta_p^*(S) = 0$ by (2.9). $\square$

3. Finitely Additive Mixtures

The previous section constructed a separable metric space S equipped with its Borel σ-field $\mathscr{F}$, and an exchangeable probability P on $(S^\infty, \mathscr{F}^\infty)$ which could not be presented in the form (1.1), as a countably additive mixture of i.i.d. variables.

Is the representation (1.1) possible with a finitely additive μ? The answer remains, no. Note that the θ's in (1.1) are countably additive. By contrast, Hewitt and Savage [6] show that any exchangeable probability is a countably additive mixture of finitely additive power probabilities, on the field generated by the finite dimensional rectangles.

First, a lemma on finitely additive probabilities.

(3.1) **Lemma.** *Let* $(\mathscr{X}_i, \Sigma_i)$ *be abstract measurable spaces for* $i=1,2$. *Let* η *be a* (Σ_1, Σ_2)*-measurable mapping from* $\mathscr{X}_1$ *to* $\mathscr{X}_2$. *Let* v *be a finitely additive probability on* Σ_1. *Then* $\mu = v\eta^{-1}$ *is a finitely additive probability on* Σ_2. *And for any bounded, real valued,* Σ_2*-measurable function* h *on* X_2

(3.2) $$\int_{\mathscr{X}_1} h(\eta x)\, v(dx) = \int_{\mathscr{X}_2} h(y)\, \mu(dy).$$

Proof. In (3.2), the function h is a uniform limit of simple Σ_2-measurable functions. □

To state the main result of this section, let $(S, \mathscr{F})$ be an abstract measurable space, P a finitely additive exchangeable probability on $(S^\infty, \mathscr{F}^\infty)$, and v a finitely additive probability on $(S^*, \mathscr{F}^*)$. Consider the representation

(3.3) $$P(A) = \int_{S^*} \phi^\infty(A)\, v(d\phi) \qquad \text{for all } A \in \mathscr{F}^\infty.$$

This differs from (1.1) in that P and v can be finitely additive: however, $\phi \in S^*$ is still countably additive on $(S, \mathscr{F})$.

(3.4) **Proposition.** *Suppose* (3.3) *holds. Then* P *determines* v *uniquely – even if* P *and* v *are only finitely additive. Furthermore,* P *is countably additive iff* v *is.*

Proof. If v is countably additive, so is P, by a routine argument. For the other assertions, it is convenient to treat three special cases first.

Case 1. $S = I = [0,1]$, and $\mathscr{F} = \mathscr{B}$, the Borel σ-field in I. Then I^* is compact metric in the weak*-topology, and $\mathscr{B}^*$ is the Borel σ-field in I^*. For $t \in I$, let δ_t be point mass at t.

(3.5) For $x \in I^\infty$, let

$$\phi_x = \text{weak* } \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j}$$

on the set H where the limit exists.

Plainly, H is a Borel subset of S^∞, and $x \to \phi_x$ is a Borel mapping from H to S^*. For any $\phi \in S^*$, the strong law implies

(3.6) $$\phi^\infty\{\phi_x = \phi\} = 1.$$

Consequently, $\phi^\infty(H) = 1$ for all ϕ, so $P(H) = 1$ by (3.3). If $F \in \mathscr{F}^*$, then (3.6) implies

$$\nu(F)=\int_{S^*} 1_F(\phi)\,\nu(d\phi)$$
$$=\int_{S^*} \phi^\infty\{x\colon \phi_x\in F\}\,\nu(d\phi).$$

So by (3.3),

(3.7) $\nu(F)=P\{x\colon \phi_x\in F\}$.

In particular, P determines ν; and ν is countably additive when P is.

Case 2. $S\subset I$ and $\mathscr{F}=S\cap\mathscr{B}$. To begin with, P induces a finitely additive, exchangeable probability $\eta^\infty P$ on $(I^\infty,\mathscr{B}^\infty)$, by (2.10)–(2.11). The representation (3.3) extends too: by (2.13) and (3.1),

$$\eta^\infty P=\int_{S^*} (\eta\phi)^\infty\,\nu(d\phi)=\int_{I^*} \theta^\infty\,\mu(d\theta),$$

where $\mu=\nu\eta^{-1}$ is a finitely additive probability on $\mathscr{B}^*$.

By (2.10), the range of η on S^* is

$$\tilde{S}=\{\theta\colon \theta\in I^* \text{ and } \theta^*(S)=1\}.$$

Endow $\tilde{S}$ with the σ-field $\tilde{\mathscr{F}}=\tilde{S}\cap\mathscr{B}^*$. Let $\tilde{\eta}$ map S^* to $\tilde{S}$ by: $\tilde{\eta}\phi=\eta\phi$. Then $\tilde{\eta}$ is $(\mathscr{F}^*,\tilde{\mathscr{F}})$-measurable by (2.10), with a measurable inverse ρ. Thus, $\tilde{\eta}$ establishes an isomorphism between $(S^*,\mathscr{F}^*)$ and $(\tilde{S},\tilde{\mathscr{F}})$.

Clearly, $\mu^*(\tilde{S})=1$. Let $\tilde{\mu}$ be the finitely additive probability traced on $(\tilde{S},\tilde{\mathscr{F}})$ by μ. As is easily verified, $\tilde{\mu}=\nu\tilde{\eta}^{-1}$, so $\nu=\tilde{\mu}\rho^{-1}$. In particular, P determines $\eta^\infty P$, and hence μ, by Case 1. But μ determines $\tilde{\mu}$, and hence ν. Finally, if P is countably additive, so is $\eta^\infty P$ by (2.10). Then μ is countably additive by Case 1, and $\tilde{\mu}$ is countably additive by (2.10), and $\nu=\tilde{\mu}\rho^{-1}$ must be countably additive too.

Case 3. $\mathscr{F}$ is separable. As is well known, $\mathscr{F}$ must then be isomorphic to a σ-field of the type covered in Case 2.

The General Case. Let $\mathscr{F}_0$ be a separable sub σ-field of $\mathscr{F}$. Any probability θ on $\mathscr{F}$ can be restricted to $\mathscr{F}_0$: call the resulting probability $r_0\theta$. Write S_0^* for the set of probabilities on $(S,\mathscr{F}_0)$, equipped with the weak* σ-field $\mathscr{F}_0^*$. Then r_0 is $(\mathscr{F}^*,\mathscr{F}_0^*)$-measurable from S^* to S_0^*.

If P is a probability on $(S^\infty,\mathscr{F}^\infty)$, write P_0 for the restriction of P to $\mathscr{F}_0^\infty$. The whole representation (3.3) can now be restricted to $\mathscr{F}_0^\infty$:

(3.8) $$P_0(A)=\int_{S^*} (r_0\phi)^\infty(A)\,\nu(d\phi) \quad \text{for } A\in\mathscr{F}_0^\infty$$
$$=\int_{S_0^*} \theta^\infty(A)\,\nu_0(d\theta) \quad \text{by (3.1)},$$

where $\nu_0=\nu r_0^{-1}$.

To show that P determines ν in (3.3), let $F\in\mathscr{F}^*$. By a routine argument, these is a separable σ-field $\mathscr{F}_0\subset\mathscr{F}$ and a set $G\in\mathscr{F}_0^*$ such that $F=r_0^{-1}G$. Now

$$\nu(F)=\nu r_0^{-1}G=\nu_0(G)$$

is determined by P_0, and hence by P, according to Case 3.

Finally suppose P is countably additive. To show that ν in (3.3) must be countably additive, let $F_i\in\mathscr{F}$ be pairwise disjoint. Again, there is a separable σ-field $\mathscr{F}_0\subset\mathscr{F}$ and sets $G_i\in\mathscr{F}_0^*$ such that $F_i=r_0^{-1}G_i$ for $i=1,2,\ldots$. Since $F_i\cap F_j=r_0^{-1}(G_i\cap G_j)$, the sets G_i must be pairwise disjoint, modulo ν_0-null sets. Now P_0 is countably additive because P is, so ν_0 is countably additive by Case 3, and

$$\nu\left(\bigcup_{i=1}^{\infty}F_i\right)=\nu\left(r_0^{-1}\bigcup_{i=1}^{\infty}G_i\right)=\nu_0\left(\bigcup_{i=1}^{\infty}G_i\right)$$
$$=\sum_{i=1}^{\infty}\nu_0(G_i)=\sum_{i=1}^{\infty}\nu(F_i).\quad\square$$

(3.9) *Remark.* The same argument shows that if a finitely additive mixture of (countably additive) ergodic probabilities is countably additive, the mixing measure must be countably additive too. For a discussion of ergodic decompositions in standard spaces, see Oxtoby [11] and Varadarajan [13].

4. Some Remarks

Theorem (2.14) shows that in the absence of topological assumptions, the conclusions of de Finetti's theorem break down. The following discussion may clarify the reason. Continue with the notation of the previous sections, but now let $(S,\mathscr{F})$ be an abstract measurable space. Let $\mathscr{F}^f$ be the field generated by the product-measurable rectangles in I^∞, so $\mathscr{F}^f$ generates the σ-field $\mathscr{F}^\infty$. According to [1], any countably additive exchangeable probability P on $(S^\infty,\mathscr{F}^\infty)$ admits the representation

(4.1) $\quad P(A)=\int_{S^*}\theta^\infty(A)\,\mu(d\theta)\qquad$ for all $A\in\mathscr{F}^f$.

In this formula, θ runs through the countably additive probabilities on $(S,\mathscr{F})$. But μ, which is only finitely additive, need not be unique. The repesentation (4.1) may be contrasted with a result by Hewitt and Savage [6], which shows that any finitely additive exchangeable P admits a representation like (4.1), with $A\in\mathscr{F}^f$ and θ being finitely additive; but their μ is countably additive and unique.

The representation (4.1) must also be distinguished from (3.3), discussed in the previous section: in (3.3), the set A ranges over the full σ-field $\mathscr{F}^\infty$. Theorems 2.14 and 3.4 show that in general, there may be no μ for which (4.1) holds, if the field $\mathscr{F}^f$ is replaced the full σ-field $\mathscr{F}^\infty$. Even if P is countably additive, there need not be a countably additive μ which satisfies (4.1).

De Finetti's theorem is closely related to the possibility of conditioning on the σ-field of exchangeable events, as in [2, 8, 10]. The connection will now be discussed. It will be shown that there are exchangeable probabilities which do not admit regular conditional probabilities given the σ-field of exchangeable events.

Again, $(S, \mathscr{F})$ is an abstract measurable space. The event $A \in \mathscr{F}^\infty$ is exchangeable if $\pi A = A$ for all finite permutations π. Let ξ_n be the n^{th} coordinate of $x \in S^\infty$:

$$\xi_n(x_1, x_2, \ldots, x_n, \ldots) = x_n.$$

Let Q be an exchangeable probability on $(S^\infty, \mathscr{F}^\infty)$.

(4.3) **Lemma.** *Suppose $\mathscr{F}$ is separable. Then Q admits a regular conditionl distribution given the σ-field of exchangeable events only if Q is presentable.*

Proof. Let $Q(x, A)$ be a regular conditional Q-probability on $\mathscr{F}^\infty$ given the σ-field of exchangeable events, and let ϕ_x be the $Q(x, \cdot)$-distribution of ξ_1. In view of de Finetti's theorem, as formulated in [8, Th. 5, p. 151], and the separability of $\mathscr{F}$,

$$Q(x, \cdot) = \phi_x^\infty \quad \text{for } Q\text{-almost all } x.$$

But $Q = \int Q(x, \cdot)\, Q(dx)$. □

(4.4) **Corollary.** *There is a countably additive, exchangeable probability on $(S^\infty, \mathscr{F}^\infty)$ which does not admit a regular conditional probability given the σ-field of exchangeable events.*

Proof. Define $S, \mathscr{F}$ and P as in (2.14), and use (4.3). □

Since any tail set is exchangeable, and any exchangeable set differs by a null set from a tail st ([2] or [8, Th. 3, p. 150] or [10]), (4.3) and (4.4) hold with the tail σ-field in place of the exchangeable σ-field.

5. A New Presentable Space

Continue with previous notation, letting $(S, \mathscr{F})$ be an abstract measurable space. This section demonstrates the existence of a peculiar σ-field which is presentable, as defined in Section 1.

(5.1) **Theorem.** *Let S be a subset of the unit interval I such that S and its complement both meet every uncountable Borel subset of I. Let $\mathscr{F}$ be the Borel σ-field $\mathscr{B}$ relativized to S. Then $\mathscr{F}$ is presentable.*

Here are the preliminaries to (5.1). Let μ be a probability on $(S^*, \mathscr{F}^*)$, Restating (3.7),

(5.2) $P_\mu\{x\colon \phi_x \in F\} = \mu(F)$,

where ϕ_x was defined in (3.5). Now make the following definition.

(5.3) Let G be the set of $x \in H$ such that, if $\phi_x(\{t\}) > 0$ there is some $i = 1, 2, \ldots$ with $x_i = t$.

(5.4) **Lemma.** *The set G is complementary analytic in I^∞, and $P_\mu(G) = 1$ for any countably additive probability μ on $(I^*, \mathscr{B}^*)$.*

Proof. Consider the set of pairs (t,x) in $I \times H$ such that: $\phi_x(\{t\})>0$, and $x_i \neq t$ for all i. This set is Borel, and its projection onto I^∞ is $H-G$, the complement of G in H. So $H-G$ is analytic, and G is complementary analytic. As is easily verified, for every $\theta \in I^*$,

$$\theta^\infty\{x: x \in G \text{ and } \phi_x = \theta\} = 1.$$

So $\theta^\infty(G)=1$, and $P_\mu(G)=1$ too. □

As before, v^* denotes v-outer measure.

(5.5) **Lemma.** *Let A be a subset of I. Let A^d be the set of $\theta \in I^*$ with $\theta(\{t\})=0$ for all $t \notin A$. Then, for any countably additive probability μ on $(I^*, \mathscr{B}^*)$,*

$$P_\mu^*(A^\infty) \leqq \mu^*(A^d).$$

Note. A^d is the set of countably additive probabilities on $(I, \mathscr{B})$ whose discrete part (if any) lives on A.

Proof. As (5.2) implies,

$$\mu^*(A^d) \geqq P_\mu^*\{x: x \in H \text{ and } \phi_x \in A^d\}.$$

In view of (5.4), it is enough to show

$$G \cap A^\infty \subset \{x: x \in G \text{ and } \phi_x \in A^d\}$$

or, equivalently,

$$\{x: x \in G \text{ and } \phi_x \notin A^d\} \subset G - A^\infty.$$

But $\phi_x \notin A^d$ implies there is a $t \notin A$ with $\phi_x(\{t\})>0$. Then $x \in G$ implies there is an i with $x_i = t \notin A$, so $x \notin A^\infty$. □

(5.6 **Corollary.** *Let μ be a countably additive probability on $(I^*, \mathscr{B}^*)$, with $\mu\{\theta: \theta$ is discrete$\}=1$. Let A be a subset of I.*

(a) *If $\mu^*\{\theta: \theta$ is discrete and $\theta(A)=1\}=0$, then $P_\mu^*(A^\infty)=0$.*

(b) *Suppose $P_\mu^*(A^\infty)=1$. Then $\mu^*\{\theta: \theta$ is discrete and $\theta(A)=1\}=1$; furthermore, $P=\int_{A^*} \phi^\infty \, v(d\phi)$, where P is the trace of P_μ on A^∞, and $v\{\phi(A \cap B_i)>t\} = \mu\{\theta(B_i)>t\}$ for any finite collection of B_i's in $\mathscr{B}$.*

In particular, the construction for Theorem 2.14 cannot be accomplished with discrete mixands. Note: $\{\theta: \theta \text{ is discrete}\} \in \mathscr{B}^*$ by [3].

It does not seem possible to sharpen (5.6a–b) very much, for $\mu^*\{\theta: \theta(A)<1\} =1$ is compatible with $P_\mu^*(A^\infty)=1$. Indeed, a construction similar to that for (2.14) shows

(5.7) *Example.* Let $\theta_t = \frac{1}{2}\delta_t + \frac{1}{2}\delta_{1-t}$ for $0 \leqq t \leqq 1$. Let $P=\int \theta_t^\infty \, dt$. There is a subset A of $[0,1]$ with $P^*(A^\infty)=1$, but A has inner Lebesgue measure 0: in particular, $\theta_t(A) \leqq 1/2$ for a set of t's of outer Lebesgue measure 1.

Proof of Theorem 5.1. Let $\mathscr{B}$ denote the Borel σ-field of $I=[0,1]$, so $\mathscr{F}=S \cap \mathscr{B}$. Let P be an exchangeable, countably additive probability on $(S^\infty, \mathscr{F}^\infty)$. Then P

induces a countably additive, exchangeable probability Q on $(I^\infty, \mathscr{B}^\infty)$, by (2.11). By de Finetti's theorem,

$$(5.8) \quad Q = \int_{I^*} \phi^\infty \mu(d\phi)$$

where μ is a countably additive probability on $(I^*, \mathscr{B}^*)$. As (2.10a) implies, $Q^*(S^\infty) = 1$; so (5.5) implies that $\mu^*(S^d) = 1$. In other words, μ assigns outer measure 1 to those ϕ's whose discrete part (if any) lives on S. On the other hand, since every uncountable Borel set intersects S, the continuous part of any ϕ assigns full outer measure to S. Thus

$$(5.9) \quad \mu^*(\tilde{S}) = 1, \quad \text{where } \tilde{S} = \{\phi \colon \phi \in I^* \text{ and } \phi^*(S) = 1\}.$$

Endow $\tilde{S}$ with the trace σ-field $\tilde{\mathscr{F}} = \tilde{S} \cap \mathscr{B}^*$; let $\tilde{\mu}$ be the trace of μ on $(\tilde{S}, \tilde{\mathscr{F}})$, as in (2.10). As is easily verified,

$$\int_{I^*} f \, d\mu = \int_{\tilde{S}} f \, d\tilde{\mu}$$

for any bounded, real-valued, $\mathscr{B}^*$-measurable function f on I^*. In particular, (5.8) implies that

$$(5.10) \quad Q(A) = \int_{\tilde{S}} \phi^\infty(A) \, \tilde{\mu}(d\phi) \quad \text{for all } A \in \mathscr{B}^\infty.$$

As (2.10) implies, any $\phi \in \tilde{S}$ has a trace $\rho\phi$ on $(S, \mathscr{F})$; and the map ρ is $(\tilde{\mathscr{F}}, \mathscr{F}^*)$-measurable. Let ν be the $\tilde{\mu}$-distribution of ρ. So, for any bounded, real-valued, $\mathscr{F}^*$-measurable function g on S^*

$$(5.11) \quad \int_{\tilde{S}} g(\rho\phi) \, \tilde{\mu}(d\phi) = \int_{S^*} g(\theta) \, \nu(d\theta).$$

Let $A_i \in \mathscr{B}$ for $i = 1, \ldots, n$. Then

$$\begin{aligned}
& P\{x \colon x \in S^\infty \text{ and } x_i \in A_i \text{ for } i = 1, \ldots, n\} \\
&\quad = Q\{x \colon x \in I^\infty \text{ and } x_i \in A_i \text{ for } i = 1, \ldots, n\} && \text{by (2.10)} \\
&\quad = \int_{\tilde{S}} \prod_{i=1}^n \phi(A_i) \, \tilde{\mu}(d\phi) && \text{by (5.10)} \\
&\quad = \int_{\tilde{S}} \prod_{i=1}^n (\rho\phi)(S \cap A_i) \, \tilde{\mu}(d\phi) && \text{by (2.10)} \\
&\quad = \int_{S^*} \prod_{i=1}^n \theta(S \cap A_i) \, \nu(d\theta) && \text{by (5.11)}
\end{aligned}$$

That is,

$$P = \int_{S^*} \theta^\infty \, \nu(d\theta). \quad \square$$

This finishes the proof of Theorem 5.1, that the σ-field $\mathscr{F}$ in the space S is presentable. Next, it will be shown that $\mathscr{F}$ is not isomorphic to any of the previously known presentable σ-fields. Here are some preliminaries.

If $\mathscr{B}$ is a σ-field of subsets of the set X, the *universal completion* $\bar{\mathscr{B}}$ of $\mathscr{B}$ is the σ-field of subsets of X which are measurable for any probability measure on $\mathscr{B}$. The next result is easy, and taken from [1].

(5.12) **Lemma.** *If $\mathscr{B}$ is presentable, and Σ is a σ-field with $\mathscr{B} \subset \Sigma \subset \bar{\mathscr{B}}$, then Σ is presentable.*

A σ-field Σ of subsets of a set X is *regular* if X can be topologized as a compact Hausdorff space, and $\mathscr{B} \subset \Sigma \subset \bar{\mathscr{B}}$, where $\mathscr{B}$ is the Baire σ-field in X. Hewitt and Savage [6] show that $\mathscr{B}$ is presentable; so any regular σ-field is presentable by (5.12). All the presentable σ-fields known to us – except for the $\mathscr{F}$ constructed in (5.1) – are isomorphic to regular σ-fields. For instance, the wide Baire σ-field inroduced by Hewitt and Savage [6] is isomorphic to a regular σ-field.

In principle, fields can be isomorphic without the spaces being isomorphic. In the present context, however, this cannot happen. The result is probably known, but we cannot supply a reference.

(5.13) **Lemma.** *Let $\mathscr{F}$ be a field of subsets of a set S; suppose $\{x\} \in \mathscr{F}$ for all $x \in S$. Let Σ be a field of subsets of a set T; suppose $\{y\} \in \Sigma$ for all $y \in T$. Suppose $\mathscr{F}$ and Σ are isomorphic, in the sense that there is a $1-1$ function G mapping $\mathscr{F}$ onto Σ, such that $G(A \cup B) = G(A) \cup G(B)$ and $G(S-A) = G(S) - G(A)$.*

Then, there is a $1-1$ function g of S onto T, such that $G(A) = g(A)$. Furthermore, g is bimeasurable, in the sense that $g(A) \in \Sigma$ for all $A \in \mathscr{F}$, and $g^{-1}(B) \in \mathscr{F}$ for all $B \in \Sigma$.

Proof. Verify that $G(\emptyset) = \emptyset$, so $A \neq \emptyset$ implies $G(A) \neq \emptyset$, and $G\{x\}$ is a singleton too. Define the function $g(x)$ by the relation $G\{x\} = \{g(x)\}$, and verify that g has the requisite properties. □

(5.14) **Theorem.** *The σ-field $\mathscr{F}$ constructed in* (5.1) *is not isomorphic to any regular σ-field.*

Proof. In view of (5.13), only point isomorphisms need be considered. Let X be a compact Hausdorff space with the Baire σ-field $\mathscr{B}$; let $\bar{\mathscr{B}}$ be the universal completion of $\mathscr{B}$, and let Σ be a σ-field between $\mathscr{B}$ and $\bar{\mathscr{B}}$.

Suppose by way of contradiction that f were a $1-1$ bimeasurable mapping of X onto S. Let λ be Lebesgue measure retracted to $(S, \mathscr{F})$:

$$\lambda(S \cap B) = \text{Lebesgue}(B), \quad \text{all Borel } B.$$

Let μ be λ pulled back to X by f:

$$\mu(A) = \lambda\{f(A)\}, \quad \text{all } A \in \Sigma.$$

This is legitimate because f is $1-1$ and bimeasurable.

As is easily seen, there is a μ-null set $N \in \Sigma$ such that $X - N$ is a Baire subset of X, and f retracted to $X - N$ is $(\mathscr{B}, \mathscr{F})$-measurable. Since $S \subset I$, and $\mathscr{F}$ is the Borel σ-field in I relativized to S, the function f may be considered as a real-valued, Baire function on X, whose range is S. Of course, the Baire functions on X can be built up from the continuous functions by successive countable passages to the limit. So Egoroff's theorem implies there is a Baire set $B \subset X - N$, with $\mu(B)$ close to 1, such that f restricted to B is continuous. Since Baire probabilities are regular, there is a compact G_δ-set $K \subset B$, with $\mu(K)$ close to $\mu(B)$. Of course, f restricted to K is still continuous. So $f(K)$ is a compact subset of S, with positive Lebesgue measure. This is a contradiction, because S and its complement both meet every uncountable Borel subset of $[0,1]$. □

6. Two Counterexamples to the Kolmogorov Consistency Theorem

As is well known, the Kolmogorov consistency theorem can fail for sufficiently bad spaces [14]. The object of this section is to give two new examples of this phenomenon: in the first, the finite-dimensional joint distributions are exchangeable; in the second, they are stationary Markov.

Here are some preliminaries. Let X be a set, $\mathscr{A}$ a field of subsets of X and Σ the σ-field generated by $\mathscr{A}$. Let Q be a countably additive probability on (X, Σ). For $Y \subset X$, let

$$Q^*_{\mathscr{A}}(Y) = \inf\{Q(A): A \in \mathscr{A} \text{ and } A \supset Y\}.$$

As usual,

$$Q^*(Y) = \inf\{Q(A): A \in \Sigma \text{ and } A \supset Y\}.$$

Clearly, $Q^*_{\mathscr{A}}(Y) \geqq Q^*(Y)$.

Suppose $Q^*_{\mathscr{A}}(Y) = 1$. Then Q has a trace finitely additive probability P on $(Y, Y \cap \mathscr{A})$:

$$P(Y \cap A) = Q(A) \quad \text{for } A \in \mathscr{A}.$$

Let $m = Q^*(Y)$. Let $\hat{Y} \in \Sigma$ with $\hat{Y} \supset Y$ and $Q(\hat{Y}) = m$. Let $\hat{Q}(F) = Q(\hat{Y} \cap F)$, a countably additive measure whose total mass is m. Let P_0 be the trace of $\hat{Q}$ on $(Y, Y \cap \mathscr{A})$

$$P_0(Y \cap A) = \hat{Q}(A) = Q(\hat{Y} \cap A) \quad \text{for all } A \in \mathscr{A}.$$

Then P_0 is a countably additive measure on $(Y, Y \cap \mathscr{A})$ whose total mass is m.

For an example of this setup, let X be the unit interval and $\mathscr{A}$ the field generated by the intervals, so Σ is the Borel σ-field. Let Q be Lebesgue measure on Σ. Let Y be the rationals, so $Q^*_{\mathscr{A}}(Y) = 1$ while $Q^*(Y) = 0$. Then P is a finitely additive measure on $Y \cap \mathscr{A}$, the P-measure of an interval of rationals being its length. According to the next result, P is purely finitely additive.

(6.1) **Proposition.** *P_0 is the countably additive part of P: namely, $P_0(Y \cap A) \leqq P(Y \cap A)$ for all $A \in \mathscr{A}$, and P_0 is the largest such countably additive measure on $(Y, Y \cap \mathscr{A})$. In particular, P is countably additive* iff *$Q^*(Y) = 1$, and P is purely finitely additive iff $Q^*(Y) = 0$.*

Proof. First, $P_0 \leqq P$, because

$$P_0(Y \cap A) = \hat{Q}(A) = Q(\hat{Y} \cap A) \leqq Q(A) = P(Y \cap A).$$

Next, suppose P_1 is countably additive on $(Y, Y\cap\mathscr{A})$ and $P_1 \leqq P$. The σ-field generated by $Y\cap\mathscr{A}$ is just $Y\cap\Sigma$, and P_1 extends to this σ-field by Caratheodory's theorem. Now $P_1(Y\cap A)\leqq Q(A)$ for all $A\in\Sigma$, by the monotone class argument. If $A\in\mathscr{A}$, then

$$P_1(Y\cap A)=P_1(Y\cap \hat{Y}\cap A)\leqq Q(\hat{Y}\cap A)=P_0(Y\cap A). \quad \square$$

Turning now to the construction, let I be the unit interval mod 1, so $0=1$. Let $\mathscr{B}$ be the Borel σ-field in I. Let $s\sim t$ iff $s-t$ is rational. Let V be a Vitali set: that is, V selects exactly one point from each $\sim$-equivalence class. Let $S=I-V$.

Let θ be a probability on the rationals R, assigning positive mass to each $r\in R$. For $t\in I$, let θ_t be θ translated by t, that is, $\theta_t\{s\}=\theta\{s+t\}$. Let $Q=\int \theta_t^\infty\, dt$, an exchangeable probability on $(I^\infty, \mathscr{B}^\infty)$. Let $\mathscr{A}_n$ be the sub σ-field of $\mathscr{B}^\infty$ generated by the first n coordinate functions $\xi_1, \ldots, \xi_n$, where $\xi_n(x)=x_n$. Let $\mathscr{A}=\bigcup_n \mathscr{A}_n$, so $\mathscr{A}$ is a field which generates $\mathscr{B}^\infty$.

(6.2) **Proposition.** $Q_{\mathscr{A}}^*(S^\infty)=1$ *but* $Q^*(S^\infty)=0$.

Proof. Let Q_n be the Q-distribution of $\xi_1, \ldots, \xi_n$, a probability on $(I^n, \mathscr{B}^n)$. To prove $Q_{\mathscr{A}_n}^*(S^\infty)=1$, it is enough to show $Q_n^*(S^n)=1$. Now $Q_n=\int \theta_t^n\, dt$, so

$$Q_n=\sum \theta(r_1)\ldots\theta(r_n)\,\mu_{r_1,\ldots,r_n}$$

where the sum runs over all n-tuples $(r_1, \ldots, r_n)$ of rationals $r_i\in I$, and $\mu_{r_1,\ldots,r_n}$ is the uniform distribution on the linear segment

$$L_{r_1,\ldots,r_n}=\{r_1+t, \ldots, r_n+t\colon 0\leqq t\leqq 1\}.$$

Consequently,

$$P_n^*\{S^n\}=\sum \theta(r_1)\ldots\theta(r_n)\,\mu_{r_1,\ldots,r_n}^*(S^n).$$

So, it is enough to show that

$$\mu_{r_1,\ldots,r_n}^*(S^n)=1.$$

Let $M_{r_1\ldots r_n}$ map I onto $L_{r_1\ldots r_n}$ as follows:

$$t\to r_1+t, \ldots, r_n+t.$$

Then $\mu_{r_1\ldots r_n}$ is the $M_{r_1\ldots r_n}$-image of Lebesgue measure. And $L_{r_1\ldots r_n}\cap S^n$ is the $M_{r_1\ldots r_n}$ image of

(6.3) $(S-r_1)\cap\ldots\cap(S-r_n)$,

where $S-r=\{t-r\colon t\in S\}$.

The problem is therefore reduced to showing that the set (6.3) has outer Lebesgue measure 1, or equivalently, that the complement of (6.3), namely

(6.4) $\hat{V}=(V-r_1)\cup(V-r_2)\cup\ldots\cup(V-r_n)$

has inner Lebesgue measure 0. Let R_0 be the set of differences r_i-r_j as i,j run over $1, \ldots, n$. If s, s' are rationals with $s-s'\notin R_0$, then $\hat{V}+s$ and $\hat{V}+s'$ are disjoint.

Thus, $\hat{V}$ has countably many pairwise disjoint translates, so its inner Lebesgue measure is 0. This completes the proof that $Q^*_{\mathscr{A}_n}(S)=1$ for all n, and hence $Q^*_{\mathscr{A}}(S^\infty)=1$.

It will now be argued that $Q^*(S^\infty)=0$. Indeed, Q concentrates on the set $B\in\mathscr{B}^\infty$ of sequences $(x_1, x_2, \ldots)$ such that $x_i - x_1$ covers the rationals as i varies from 1 to ∞: in other words, x_i covers the $\sim$-equivalence class through x_1. And B is disjoint from S^∞. □

Let P be the trace of Q on $(S^\infty, S^\infty \cap \mathscr{A})$. As (6.1) and (6.2) show, P is purely finitely additive, although it is countably additive on $S^\infty \cap \mathscr{A}_n$ for each n. This is the desired example. To state it in more familiar terms, let $\mathscr{F}=S\cap\mathscr{B}$. Let P_n be the P-distribution of the first n coordinates $(\xi_1, \ldots, \xi_n)$. The argument just given proves

(6.5) **Theorem.** *For each n, P_n is an exchangeable, countably additive probability on $(S^n, \mathscr{F}^n)$, and the P_n are consistent However, there is no countably additive probability on $(S^\infty, \mathscr{F}^\infty)$ which projects onto the P_n's. Indeed, there is a unique finitely additive probability P on the field of measurable cylinders in S^∞ which projects onto the P_n's, and P is purely finitely additive.*

This example overlaps with Wegner's [14]. However, his process is not exchangeable.

Persi Diaconis asked whether the Kolmogorov consistency theorem held for Markov processes. The answer is negative, by a slight variation on the previous argument. Define a probability Q on $(I^\infty, \mathscr{B}^\infty)$ as follows: choose $t\in I$ from Lebesgue measure; independently, choose $r_1, r_2, \ldots$ from θ, a probability on the rationals assigning positive measure to each; then Q is the distribution of

$$(t, t+r_1, t+r_1+r_2, \ldots).$$

In other words, relative to Q, the coordinate process $\xi_1, \xi_2, \ldots$ performs a random walk: the starting position is uniform, and the steps are distributed according to θ. The transition probabilities are

(6.6) $$K(t, A)=\sum_{r\in R}\theta(r)\,1_A(t+r)$$

Recall that $\mathscr{A}$ is the field of Borel cylinders in I^∞. Exactly as before,

(6.7) $$Q^*_{\mathscr{A}}(S^\infty)=1 \quad \text{but } Q^*(S^\infty)=0.$$

Let P be the trace of Q on $(S^\infty, S^\infty\cap\mathscr{A})$, so Q is purely finitely additive, but countably additive on each $S^\infty\cap\mathscr{A}_n$. Let P_n be the P-distribution of the first n coordinates $\xi_1, \ldots, \xi_n$. Recall that $\mathscr{F}=S\cap\mathscr{B}$.

(6.7) **Theorem.** *For each n, P_n is a countably additive probability on $(S^n, \mathscr{F}^n)$. Relative to P_n, the coordinate process is stationary Markov. Furthermore, the P_n are consistent. However, there is no countably additive probability on $(S^\infty, \mathscr{F}^\infty)$ which projects onto the P_n's. Indeed, there is a unique finitely additive probability P on the field of measurable cylinders which projects onto the P_n's, and P is purely finitely additive.*

Proof. The new point is the Markov property. Let $(\zeta_1, \ldots, \zeta_n)$ be the coordinate process on S^n. It will be argued that $K(\zeta_m, \cdot)$, as defined in (6.6), is a regular conditional P_n-distribution for ζ_{m+1} given $\zeta_1, \ldots, \zeta_m$; furthermore, the ζ's are all uniformly distributed. Indeed, fix m with $1 \leqq m < n$, and Borel subsets $A_1, \ldots, A_{m+1}$ of I. By construction,

$$\begin{aligned} &P_n\{\zeta_i \in A_i \text{ for } 1 \leqq i \leqq m+1\} \\ &\quad = Q\{\xi_i \in A_i \text{ for } 1 \leqq i \leqq m+1\} \\ &\quad = \int_{\{\xi_i \in A_i \text{ for } 1 \leqq i \leqq m\}} K(\xi_m, A_{m+1})\, dQ \\ &\quad = \int_{\{\zeta_i \in A_i \text{ for } 1 \leqq i \leqq m\}} K(\zeta_m, A_{m+1})\, dP_n. \end{aligned}$$

Likewise, $P_n\{\zeta_i \in A\} = Q\{\xi_i \in A\}$ is the Lebesgue measure of A, for any Borel A, proving stationarity. □

If the ζ_i are visualized as taking values in the unit interval, then ζ_{m+1} has a regular conditional P_n-distribution given $\zeta_1, \ldots, \zeta_m$, by general theory. However, $K(t, S) < 1$ for all t. So, if the ζ_i are visualized as taking values in the nonstandard space $(S, \mathscr{F})$, then ζ_{m+1} does not have a regular conditional P_n-distribution given $\zeta_1, \ldots, \zeta_m$. Hence, Tulcea's theorem does not apply.

7. Finitely Additive, Exchangeable Probabilities in Standard Spaces

Say that the finitely additive probability P on $(I^\infty, \mathscr{B}^\infty)$ is *representable* if

(7.1) $$P(A) = \int_{I^*} \phi^\infty(A)\, \mu(d\phi) \qquad \text{for all } A \in \mathscr{B}^\infty$$

where I^* is the set of countably additive probabilites on $(I, \mathscr{B})$ and μ is a finitely additive probability on $(I^*, \mathscr{B}^*)$. To review briefly, if P is countably additive and exchangeable, and $(I, \mathscr{B})$ is a standard Borel space, the representation (7.1) necessarily holds, and μ is unique and countably additive. This is de Finetti's theorem [1] and (3.4) above. On the other hand, if $(I, \mathscr{B})$ is not standard, the representation (7.1) may fail even for countably additive, exchangeable P, by (2.14) and (3.4). One case remains: where P is finitely additive and exchangeable, but $(I, \mathscr{B})$ is standard. Then (7.1) may fail.

(7.2) **Proposition.** *Let $I = \{0, 1\}$ and $\mathscr{B}$ be the discrete σ-field in I. There is a finitely additive, exchangeable probability P on $(I^\infty, \mathscr{B}^\infty)$ which cannot be represented in the form* (7.1), *as a finitely additive mixture of countably additive power probabilities.*

Proof. Let G be the set of $x \in I^\infty$ for which $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i$ exists. By the strong law, $\phi^\infty(G) = 1$ for all $\phi \in I^*$, so $P(G) = 1$ if (7.1) holds. The next step is to construct a finitely additive, exchangeable probability P on $(I^\infty, \mathscr{B}^\infty)$ with $P(G) = 0$.

The group Π of finite permutations of the positive integers is amenable [5, 9]; let $d\pi$ be a finitely additive, invariant probability defined on all subsets of Π, so that for $\sigma \in \Pi$ and any bounded real-valued function f on Π,

(7.3) $$\int_\Pi f(\pi\sigma)\, d\pi = \int_\Pi f(\pi)\, d\pi.$$

Let P_0 be any probability on $(I^\infty, \mathscr{B}^\infty)$ with $P_0(G)=0$, and let

(7.4) $$P(A)=\int_\Pi P_0(\pi A)\,d\pi.$$

Clearly, P is a finitely additive probability on $\mathscr{B}^\infty$. And $P(G)=0$, because $\pi G = G$ for all $\pi \in \Pi$. Finally, P is exchangeable: if $\sigma \in \Pi$, then

$$\begin{aligned} P(\sigma A) &= \int_\Pi P_0(\pi\sigma A)\,d\pi && \text{by (7.4)}\\ &= \int_\Pi P_0(\pi A)\,d\pi && \text{by (7.3)}\\ &= P(A) && \text{by (7.4).} \quad \square \end{aligned}$$

Essentially the same argument can be used to make P invariant even under the larger group Γ, which changes any finite number of coordinates. Formally, $\gamma \in \Gamma$ is specified by a finite subset F of the positive integers:

$$\begin{aligned} (\gamma x)_n &= 1 - x_n && \text{if } n \in F\\ &= x_n && \text{if } n \notin F. \end{aligned}$$

Of course, if P is invariant under Γ, it agrees with coin tossing on the cylinder sets.

As noted earlier, Hewitt and Savage [6] show that any finitely additive, exchangeable P is a (unique) countably additive mixture of finitely additive power probabilities, on the field generated by measurable rectangles. This may be contrasted with (7.1)–(7.2), where ϕ^∞ is countably additive on the entire σ-field $\mathscr{B}^\infty$, but μ is only fnitely additive.

References

1. Diaconis, P., Freedman, D.: Finite exchangeable sequences. To appear in Ann. Prob.
2. Diaconis, P., Freedman, D.: de Finetti's theorem for Markov chains. To appear in Ann. Prob.
3. Dubins, L., Freedman, D.: Measurable sets of masures. Pacific J. Math. **14**, 1211–1222 (1964)
4. Dynkin, E.B.: Sufficient statistics and extreme points. Ann. Prob., **6**, 705–730 (1979)
5. Greenleaf, F.: Invariant Means on Topological Groups. New York: Van Nostrand 1969
6. Hewitt, E., Savage, L.J.: Symmetric measures on Cartesian products. Trans. Amer. Math. Soc., **80**, 470–501 (1955)
7. Kuratowski, K.: Topologie I. Warsaw. New York: Hafner 1958
8. Meyer, P.A., Probability and Potentials, Waltham: Blaisdell 1966
9. von Neumann, J.: Zur allgemeinen theorie des masses, Fund. Math., **13**, 73–116 (1929)
10. Olshen, R.: A note on exchangeable sequences. Z. Wahrscheinlichkeitstheorie verw. Gebiete **28**, 317–321 (1974)
11. Oxtoby, J.C.: Ergodic sets. Bull. Amer. Math. Soc. **58**, 116–130 (1952)
12. Sikorski, R.: Boolean Algebras. New York: Springer 1960
13. Varadarajan, V.S.: Groups of autormorphisms of Borel spaces. Trans. Amer. Math. Soc. **109**, 191–220 (1963)
14. Wegner, H., On consistency of probability measures. Z. Wahrscheinlichkeitstheorie verw. Gebiete **27**, 335–338 (1973)

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