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Six Generator q-Deformed Lorentz Algebra

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Abstract. The six generator deformation of the Lorentz algebra is presented. The Hopf algebra structure and the reality conditions are found. The chiral decomposition of SL(2, C) is generalized to the q-case. Casimir operators for the q-Lorentz algebra are given.

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1. Introduction

In a recent work, a q-deformed Lorentz algebra was presented [1]. Three questions were left open in this paper. First, the deformation involves seven generators, while one needs only six. Second, the whole family of algebras labelled by a parameter r was presented. This is not desirable, since (classically) the most general group acting linearly on spinors is GL(2, **C**). If, for different values of r, the generators presented in [1] were independent, together they would generate an infinite-dimensional algebra acting linearly on spinors. This contradicts geometrical intuition. Third, the reality conditions for four of the generators were not apparent. One could think that adding conjugates of these generators would produce an algebra with eleven generators.

In this Letter, we address all of these questions. We show that generators for different r can be expressed in terms of each other. This allows the value of r to be fixed. In Section 2, we prove that the deformation actually contains only six generators, and present the reality conditions for all of them. We also give formulas for the antipode in the algebra.

In the classical case, it is convenient to have chiral SL(2) groups which act only on barred or unbarred spinors. The rotation SU(2) subgroup of the Lorentz group is the diagonal in $\mathrm{SL}(2) \times \overline{\mathrm{SL}(2)}$. We discuss the generalization of this picture to the

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quantum case in Section 3. We also give expressions for the Casimir operators of the q-deformed Lorentz algebra.

Finally, in Section 4, we give explicit expressions for the generators for arbitrary r in terms of the simplest ones corresponding to the value $r = q^{-1}$.

In another work, a six generator q-deformed Lorentz algebra has also been found following an approach based on the R-matrix [2].

2. Lorentz Algebra

In a previous work, a seven generator q-deformed Lorentz algebra was discussed [1]. That algebra was arrived at by considering the representation on a complex quantum spinor space. In addition to the usual deformation parameter, that algebra had an additional parameter and the representation on spinors had two extra parameters. A further analysis shows that these extra parameters may be eliminated. In this section we recall the algebra and action for special values of these parameters. The coproduct, counit, antipode, and real structure of the generators is determined. A central element of the algebra is found which also commutes with the spinors. Thus, it is equal to unity, and the algebra actually has only six generators. In the appendix we show how the results of [1] may be recovered from this algebra.

First we outline the techniques used in [1]. Complex quantum spinors have components x and y and conjugates $\bar{x}$ and $\bar{y}$. They obey q-relations:

$$\begin{aligned} &xy = qyx, && y\bar{x} = q\bar{x}y, \\ &x\bar{x} = \bar{x}x - q\lambda\bar{y}y, && y\bar{y} = \bar{y}y, \\ &x\bar{y} = q\bar{y}x, && \bar{x}\bar{y} = q^{-1}\bar{y}\bar{x}. \end{aligned} \tag{2.1}$$

These relations come from the R-matrix for $SL_q(2)$ with deformation parameter q real. There are also q-relations between two different copies of the quantum spinors [1]. The action of generators on the spinors is then constrained by requiring consistency with all of the q-relations. The algebra is then found by considering bilinear combinations of generators acting on the spinors. Finally the action of the generators on monomials in the spinors determines the coproduct.

The generators of the $SU_q(2)$ subalgebra arise from a simple ansatz for the action on spinors. Two of the generators are uniquely determined by consistency with the q-relations. Their action on spinors is

$$\begin{aligned} &T^+x = qxT^+ + y, && T^+\bar{x} = q^{-1}\bar{x}T^+, \\ &T^+y = q^{-1}yT^+, && T^+\bar{y} = q\bar{y}T^+ - q^{-1}\bar{x}, \\ &T^-x = qxT^-, && T^-\bar{x} = q^{-1}\bar{x}T^- - q\bar{y}, \\ &T^-y = q^{-1}yT^- + x, && T^-\bar{y} = q\bar{y}T^-. \end{aligned} \tag{2.2}$$

The third generator of $SU_q(2)$ is determined by requiring the algebra to close. Its action on spinors is

$$\begin{aligned} T^3x &= q^2xT^3 - qx, & T^3\bar{x} &= q^{-2}\bar{x}T^3 + q^{-1}\bar{x}, \\ T^3y &= q^{-2}yT^3 + q^{-1}y, & T^3\bar{y} &= q^2\bar{y}T^3 - q\bar{y}. \end{aligned} \tag{2.3}$$

The algebra of these generators is then

$$\begin{aligned} & q^{-1}T^+T^- - qT^-T^+ = T^3, \\ & q^2T^3T^+ - q^{-2}T^+T^3 = (q+q^{-1})T^+, \\ & q^{-2}T^3T^- - q^2T^-T^3 = -(q+q^{-1})T^-. \end{aligned} \tag{2.4}$$

This is the form of the $SU_q(2)$ algebra introduced by Woronowicz [3].

One important property of the above generators is that they annihilate the constant monomial, i.e. $T^{\pm}1 = 0$ and $T^3 1 = 0$. In the following, it will be convenient to introduce the quantity $\tau^3 = 1 - \lambda T^3$ where $\lambda = q - q^{-1}$. Then $\tau^3 1 = 1$ and the action on spinors takes the simple form

$$\begin{aligned} \tau^3x &= q^2x\tau^3, & \tau^3\bar{x} &= q^{-2}\bar{x}\tau^3, \\ \tau^3y &= q^{-2}y\tau^3, & \tau^3\bar{y} &= q^2\bar{y}\tau^3. \end{aligned} \tag{2.5}$$

In this basis the algebra may be written

$$\tau^3T^{\pm} = q^{\mp 4}T^{\pm}\tau^3, \qquad q^{-1}T^+T^- - qT^-T^+ = \lambda^{-1}(1-\tau^3). \tag{2.6}$$

Similar quantities for the other diagonal generators will be used later.

The coproduct Δ may now be written in a simple form using τ^3. It is

$$\Delta(T^{\pm}) = T^{\pm}\otimes 1 + (\tau^3)^{1/2}\otimes T^{\pm}, \qquad \Delta(\tau^3) = \tau^3\otimes\tau^3. \tag{2.7}$$

The counit ε and antipode S are

$$\begin{aligned} \varepsilon(T^{\pm}) &= 0, & S(T^{\pm}) &= -(\tau^3)^{-1/2}T^{\pm}, \\ \varepsilon(\tau^3) &= 1, & S(\tau^3) &= (\tau^3)^{-1}. \end{aligned} \tag{2.8}$$

Finally, the real structure of the generators is found by taking the complex conjugate of their spinor action. This includes reversing the order of the elements, for example $\overline{T^3x} = \bar{x}\overline{T^3}$. Direct inspection then shows

$$\overline{T^{\pm}} = q^{\mp 2}T^{\mp} \qquad \overline{\tau^3} = \tau^3. \tag{2.9}$$

This completes the description of the $SU_q(2)$ generators and algebra.

The diagonal generator τ^3 (or T^3) was determined by requiring closure of the algebra. One can define two other diagonal generators which fulfill the requirements of consistency. Their action on spinors is

$$\begin{aligned} \tau^1_1x &= q^2x\tau^1_1, & \tau^1_1\bar{x} &= q^{-2}\bar{x}\tau^1_1, \\ \tau^1_1y &= y\tau^1_1, & \tau^1_1\bar{y} &= \bar{y}\tau^1_1, \\ \tau^2_2x &= x\tau^2_2, & \tau^2_2\bar{x} &= \bar{x}\tau^2_2, \\ \tau^2_2y &= q^{-2}y\tau^2_2, & \tau^2_2\bar{y} &= q^2\bar{y}\tau^2_2 \end{aligned} \tag{2.10}$$

and they obey $\tau^1_1 1 = 1$ and $\tau^2_2 1 = 1$. The Hopf algebra and real structure are completed by

$$\Delta(\tau^i_i) = \tau^i_i \otimes \tau^i_i, \qquad \overline{\tau^i_i} = \tau^i_i,$$
$$\varepsilon(\tau^i_i) = 1, \qquad S(\tau^i_i) = (\tau^i_i)^{-1}, \tag{2.11}$$

where $i = 1$ or 2, no sum. It is easy to see that

$$\tau^1_1 \tau^2_2 = \tau^2_2 \tau^1_1, \qquad \tau^3 = \tau^1_1 \tau^2_2. \tag{2.12}$$

The combination

$$\tau^0 = \tau^1_1 (\tau^2_2)^{-1} \tag{2.13}$$

commutes with the $SU_q(2)$ generators. Classically, $T^0 = \lambda^{-1}(\tau^0 - 1)$ reduces to the usual U(1) generator. Therefore, we say that the set of four generators T^+, T^-, T^3, and T^0 generate the quantum U(2) algebra.

As discussed in [1], quantum four vectors may be constructed as bilinears in the spinors and their conjugates. The action of the generators on spinors may be iterated to find the action on four vectors. The time coordinate is proportional to $\bar{x}x + \bar{y}y$, the central length preserved by $SU_q(2)$, so the generators introduced above do not produce Lorentz boosts. A more general ansatz for the action on spinors was used in [1] to construct generators which will produce boosts. This ansatz required four new generators to complete the algebra. Again the action on spinors must be consistent with the q-relations obeyed by the spinor components. In this case the action is not completely constrained, but there are two free parameters. Here we make a particular choice for these parameters, $a = 1$ and $d = q$, and later show how the results of [1] may be recovered from this algebra. Following the notation in [1], we have two new raising and lowering operators T^2 and S^1 and two new diagonal generators T^1 and S^2, all of which annihilate the constant monomial. Again for convenience, we define $\tau^1 = 1 + \lambda T^1$ and $\sigma^2 = 1 + \lambda S^2$. The action on spinors is

$$\begin{aligned}
T^2 x &= x T^2 + y\tau^1, & T^2 \bar{x} &= q\bar{x} T^2,\\
T^2 y &= y T^2, & T^2 \bar{y} &= q^{-1} \bar{y} T^2,\\
S^1 x &= x S^1, & S^1 \bar{x} &= q^{-1} \bar{x} S^1 + \bar{y}\sigma^2,\\
S^1 y &= y S^1, & S^1 \bar{y} &= q\bar{y} S^1,\\
\tau^1 x &= q^{-1} x \tau^1, & \tau^1 \bar{x} &= \bar{x}\tau^1 + q\lambda^2 \bar{y} T^2,\\
\tau^1 y &= q y \tau^1, & \tau^1 \bar{y} &= \bar{y}\tau^1,\\
\sigma^2 x &= q x \sigma^2 + q\lambda^2 y S^1, & \sigma^2 \bar{x} &= \bar{x}\sigma^2,\\
\sigma^2 y &= q^{-1} y \sigma^2, & \sigma^2 \bar{y} &= \bar{y}\sigma^2.
\end{aligned} \tag{2.14}$$

Using this the full algebra may be computed. Including the $SU_q(2)$ algebra from above, we find

$$\begin{array}{ll}
\tau^1 T^+ = T^+\tau^1 + \lambda T^2, & T^+T^2 = q^{-2}T^2T^+, \\
\tau^1 T^- = q^{-2}T^-\tau^1 - \lambda S^1, & T^-T^2 = T^2T^- + \lambda^{-1}(\sigma^2 - \tau^1), \\
\tau^1 T^2 = q^2 T^2 \tau^1, & T^+S^1 = q^2 S^1 T^+ + \lambda^{-1}(\tau^3\tau^1 - \sigma^2), \\
\tau^1 S^1 = S^1\tau^1, & T^-S^1 = S^1T^-, \\
 & T^+T^- = q^2T^-T^+ + q\lambda^{-1}(1-\tau^3), \\
\sigma^2 T^+ = T^+\sigma^2 - q^2\lambda\tau^3T^2, & T^2S^1 = S^1T^2, \\
\sigma^2 T^- = q^2 T^-\sigma^2 + q^2\lambda S^1, & \\
\sigma^2 T^2 = q^{-2}T^2\sigma^2, & \tau^1\sigma^2 = \sigma^2\tau^1 + q\lambda^3T^2S^1, \\
\sigma^2 S^1 = S^1\sigma^2, & \tau^3\tau^1 = \tau^1\tau^3, \\
 & \tau^3\sigma^2 = \sigma^2\tau^3. \\
\tau^3 T^+ = q^{-4}T^+\tau^3, & \\
\tau^3 T^- = q^4 T^-\tau^3, & \\
\tau^3 T^2 = q^{-4}T^2\tau^3, & \\
\tau^3 S^1 = q^4 S^1\tau^3. &
\end{array} \tag{2.15}$$

Finally, the U(1) generator τ^0 commutes with all of the other generators. The algebra may be written in a more conventional form by the substitutions $\tau^1 = 1 + \lambda T^1$, $\sigma^2 = 1 + \lambda S^2$, and $\tau^3 = 1 - \lambda T^3$.

This algebra appears to have seven generators. However, there is an extra relation in the algebra which allows elimination of one of the diagonal generators. Consider the quantity

$$Z = \tau^1\sigma^2 - q^2\lambda^2T^2S^1. \tag{2.16}$$

One finds that Z is central in the algebra and commutes with all of the spinors, e.g. $Zx = xZ$. Therefore Z is 1. Then one could eliminate τ^1 or σ^2 from the algebra, for example by the substitution $\sigma^2 = (\tau^1)^{-1}(1 + q^2\lambda^2T^2S^1)$. However, this would leave the algebra with inverse powers of the remaining diagonal generator. In the next section we show a substitution leading to a more appealing form of the algebra.

The coproduct for the new generators is found by considering their action on functions of the spinors. It has the form

$$\begin{aligned}
\Delta(\tau^1) &= \tau^1 \otimes \tau^1 + \lambda^2 S^1(\tau^3)^{-1/2} \otimes T^2, \\
\Delta(\sigma^2) &= \sigma^2 \otimes \sigma^2 + \lambda^2 T^2(\tau^3)^{1/2} \otimes S^1, \\
\Delta(T^2) &= T^2 \otimes \tau^1 + (\tau^3)^{-1/2}\sigma^2 \otimes T^2, \\
\Delta(S^1) &= S^1 \otimes \sigma^2 + (\tau^3)^{1/2}\tau^1 \otimes S^1.
\end{aligned} \tag{2.17}$$

The counit and antipode are

$$
\begin{aligned}
&\varepsilon(\tau^1) = 1, && S(\tau^1) = \sigma^2, \\
&\varepsilon(\sigma^2) = 1, && S(\sigma^2) = \tau^1, \\
&\varepsilon(T^2) = 0, && S(T^2) = -q^2(\tau^3)^{1/2}T^2, \\
&\varepsilon(S^1) = 0, && S(S^1) = -(\tau^3)^{-1/2}S^1.
\end{aligned}
\tag{2.18}
$$

In checking the antipode property one needs the fact that $Z = 1$. Finally, the reality conditions for the new generators are

$$
\begin{aligned}
&\overline{\tau^1} = (\tau^3)^{-1/2}\sigma^2, && \overline{T^2} = -(\tau^3)^{-1/2}S^1, \\
&\overline{\sigma^2} = (\tau^3)^{1/2}\tau^1, && \overline{S^1} = -q^2(\tau^3)^{1/2}T^2.
\end{aligned}
\tag{2.19}
$$

This completes the construction of the Hopf algebra $\mathrm{SL}_q(2, \mathbf{C})$.

3. Chiral Decomposition

Recall that classically one builds from rotations and boosts two copies of SL(2). Denoting rotations by J^i and boosts by K^i those copies are spanned by $M^i = J^i + iK^i$ and $N^i = J^i - iK^i$. We wish to generalize this picture to the q-case.

Analyzing the structure of the algebra in the last section we find that the proper choice for M is

$$
\begin{aligned}
&M^- = q(\tau^1)^{-1}S^1 + T^-, \\
&M^3 = \lambda^{-1}(1 - (\tau^1)^{-2}), \\
&M^+ = q(\tau^1)^{-1}T^2.
\end{aligned}
\tag{3.1}
$$

The N generators are

$$
\begin{aligned}
&N^- = q^{-1}(T^- - M^-) = -(\tau^1)^{-1}S^1, \\
&N^3 = (\tau^3)^{-1}(T^3 - M^3) = \lambda^{-1}((\tau^3)^{-1}(\tau^1)^{-2} - 1), \\
&N^+ = q^{-1}(\tau^3)^{-1}(T^+ - M^+) = (\tau^3)^{-1}(q^{-1}T^+ - (\tau^1)^{-1}T^2).
\end{aligned}
\tag{3.2}
$$

Now the generators M^i and N^i mutually q-commute. Namely, N^3 simply commutes with the M^i, M^3 simply commutes with the N^i, and for the rest we have

$$
\begin{aligned}
&N^-M^- = q^2M^-N^-, && N^-M^+ = q^{-2}M^+N^-, \\
&N^+M^+ = q^2M^+N^+, && N^+M^- = q^{-2}M^-N^+.
\end{aligned}
\tag{3.3}
$$

Among themselves they obey

$$
\begin{aligned}
&q^2M^3M^+ - q^{-2}M^+M^3 = (q + q^{-1})M^+, \\
&q^{-1}M^+M^- - qM^-M^+ = M^3, \\
&q^{-2}M^3M^- - q^2M^-M^3 = -(q + q^{-1})M^- \\
&q^{-2}N^3N^+ - q^2N^+N^3 = (q + q^{-1})N^+, \\
&qN^+N^- - q^{-1}N^-N^+ = N^3, \\
&q^2N^3N^- - q^{-2}N^-N^3 = -(q + q^{-1})N^-.
\end{aligned}
\tag{3.4}
$$

(The generators can be rescaled by powers of τ^3 so that this algebra is invariant and the M^i and N^i strictly commute, but this would result in a more complicated spinor action.) The coproduct for the redefined generators is easily found using the coproduct for $(\tau^1)^{-1}$:

$$\Delta((\tau^1)^{-1}) = (\tau^1)^{-1} \otimes (\tau^1)^{-1} \sum_{r=0}^{\infty} (q\lambda^2)^r (N^-(\tau^3)^{-1/2} \otimes M^+)^r. \tag{3.5}$$

The action of the M^i on the spinors is

$$\begin{aligned}
&M^+x = qxM^+ + y, && M^+\bar{x} = q\bar{x}M^+ - q\lambda^2\bar{y}(M^+)^2, \\
&M^+y = q^{-1}yM^+, && M^+\bar{y} = q^{-1}\bar{y}M^+, \\
&M^-x = qxM^-, && M^-\bar{x} = q^{-1}\bar{x}M^- - q\lambda\bar{y}M^3, \\
&M^-y = q^{-1}yM^- + x, && M^-\bar{y} = q\bar{y}M^-, \\
&M^3x = q^2xM^3 - qx, && M^3\bar{x} = \bar{x}M^3 + \lambda\bar{y}(1 + q^{-2})M^+(1 - \lambda M^3), \\
&M^3y = q^{-2}yM^3 + q^{-1}y, && M^3\bar{y} = \bar{y}M^3.
\end{aligned} \tag{3.6}$$

Similarly, the N^i act as follows:

$$\begin{aligned}
&N^+x = q^{-1}xN^+, && N^+\bar{x} = q\bar{x}N^+ - q\lambda\bar{x}(\tau^3)^{-1}M^+ + q^{-2}\lambda^2\bar{y}(\tau^3)^{-1}(M^+)^2, \\
&N^+y = qyN^+, && N^+\bar{y} = q^{-1}\bar{y}N^- + q^{-3}\lambda\bar{y}(\tau^3)^{-1}M^- - \bar{x}(\tau^3)^{-1}, \\
&N^-x = qxN^-, && N^-\bar{x} = q^{-1}\bar{x}N^- - \bar{y}(1 - \lambda M^3), \\
&N^-y = q^{-1}yN^-, && N^-\bar{y} = q\bar{y}N^-, \\
&N^3x = xN^3, && N^3\bar{x} = q^2\bar{x}N^3 - (q^2 + 1)\lambda\bar{y}M^+(1 + \lambda N^3) + q\bar{x}, \\
&N^3y = yN^3, && N^3\bar{y} = q^{-2}\bar{y}N^3 - q^{-1}\bar{y}.
\end{aligned} \tag{3.7}$$

Observe that the M^i (N^i) vanish when acting on functions of only $\bar{x}, \bar{y}$ (x, y). This is the analogue of the classical property of chiral SL(2) algebras. Note also that although the M^i and N^i form independent algebras, in the action they mix. Therefore they do not form independent Hopf algebras, a fact which is also seen in the term $N^-(\tau^3)^{-1/2} \otimes M^+$ in (3.5). Finally, the M^i and N^i found here are not simply related by complex conjugation as in the classical case.

The basis M^i, N^i immediately allows one to write the Casimir operators for the quantum Lorentz algebra. Since we have two independent copies of the $SL_q(2)$ algebra, the Casimir operators are given by

$$\begin{aligned}
C^m &= q^2\lambda^{-2}m^{1/2} + \lambda^{-2}m^{-1/2} + m^{-1/2}M^+M^- - \lambda^{-2}(q^2 + 1), \\
C^n &= q^{-2}\lambda^{-2}n^{1/2} + \lambda^{-2}n^{-1/2} + n^{-1/2}N^+N^- - \lambda^{-2}(q^{-2} + 1),
\end{aligned} \tag{3.8}$$

where

$$m = 1 - \lambda M^3, \qquad n = 1 + \lambda N^3. \tag{3.9}$$

The constant terms are needed for the correct classical limit.

4. Appendix

We now show how to recover the results of [1]. That work described a family of generators parameterized by two numbers a and d. More precisely, a and d appear in the spinor action. The algebra depends only on the ratio $r = a/d$. However all of those operators can be expressed in terms of the generators in section 2. (Recall that Section 2 used the results of [1] for the parameter values $a = 1$ and $d = q$.) To this end define $a = q^{2\alpha}$ and $d = q^{2\delta}$, and let

$$\tilde{\tau} = (\tau^1_1)^\alpha(\tau^2_2)^{(1/2)-\delta}. \tag{4.1}$$

Then the operators

$$\begin{aligned} \tau^{1'} &= \tilde{\tau}\tau^1, \qquad T^{2'} = d^{-1}q\tilde{\tau}T^2, \\ \sigma^{2'} &= \tilde{\tau}\sigma^2, \qquad S^{1'} = dq^{-1}\tilde{\tau}S^1, \end{aligned} \tag{4.2}$$

reproduce the algebra and spinor action in [1]. We note that the extra relation allowing the elimination of one generator can be written directly inside this algebra. It is done using the quantity

$$Z' = \tau^{1'}\sigma^{2'} - r^{-1}q\lambda^2 T^{2'}S^{1'}. \tag{4.3}$$

One finds that the commutation relations of Z' and $\tilde{\tau}^2$ with all generators and spinors coincide. Therefore we have the relation

$$\tilde{\tau}^{-2}Z' = 1 \tag{4.4}$$

which gives the reduction of the algebra to six generators.

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