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Contact geometry and complex surfaces

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In this paper we give a complete classification of (smooth, closed, orientable) 3-manifolds that admit a linear circle of equal volume contact forms (see Definition 1.1 for the precise meaning of this term; there we introduce the name *taut contact circle* for this type of structure). One of the most intriguing aspects of this classification is that it relies on the Enriques-Kodaira classification of compact complex surfaces and Wall's study of locally homogeneous geometric structures on these surfaces. We show that if M^3 admits a taut contact circle, then $M^3 \times S^1$ is a complex surface and the obvious free circle action is by holomorphic automorphisms. Complex surfaces of this type will be classified (up to diffeomorphism) in Section 4. The additional structure on $M^3 \times S^1$ provided by the taut contact circle allows to recover M^3 from $M^3 \times S^1$, this yields the classification Theorem 1.2. Furthermore, we relate homotopies of taut contact circles to the complex geometry, and show that any taut contact circle is homotopic to a certain distinguished type of taut contact circle that we call *Cartan structure* (Definition 1.1, Theorem 1.6).

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The motivation to study taut contact circles is twofold. First of all, they arise naturally as the Liouville-Cartan forms on the unit cotangent bundle of a Riemann surface (see Remark (2) after Definition 1.1). Secondly, we were led to these structures by our study of certain quaternionic analogues of contact structures [8, 12, 13].

In [13] the second author showed that every 3-manifold admits a triple of pointwise linearly independent contact forms. In [8] we showed that every 3-manifold admits a triple of contact forms with pointwise linearly independent Reeb vector fields, which is equivalent to saying that the differentials of the three contact forms are pointwise linearly independent.

In this context it seemed natural to ask which 3-manifolds admit a pair or triple of contact forms such that every non-trivial linear combination of these is again a contact form (pointwise linear independence of the two (or three) contact forms and their differentials, respectively, is clearly a necessary condition). We call such structure a *contact circle* or *contact sphere*, respectively.

We then observed that by imposing an additional equal volume constraint on these contact forms, one uncovers a rich complex geometric theory associated to the theory of taut contact circles. In the present paper we explore this holomorphic theory to arrive at a variety of classification results for taut contact circles and spheres.

1 Definitions and main results

Our initial object of study is a pair of contact forms ω_1, ω_2 on a 3-manifold such that any non-trivial linear combination $\lambda_1\omega_1 + \lambda_2\omega_2$ with constant coefficients $\lambda_1, \lambda_2 \in \mathbb{R}$ is also a contact form. (Recall that a contact form on a 3-manifold is a 1-form ω such that $\omega \wedge d\omega$ is nowhere zero, that is, a volume form.) Clearly it suffices to check this non-degeneracy condition for the pairs $(\lambda_1, \lambda_2) \in S^1$, where S^1 denotes the unit circle in $\mathbb{R}^2$.

Definition 1.1 *We say that a 3-manifold M^3 admits a* **contact circle** *if it admits a pair of contact forms (ω_1, ω_2) such that for any $(\lambda_1, \lambda_2) \in S^1$ the linear combination $\lambda_1\omega_1 + \lambda_2\omega_2$ is also a contact form.*

We say that this circle is a **taut contact circle** *if the contact forms $\lambda_1\omega_1 + \lambda_2\omega_2$ define the same volume form for all $(\lambda_1, \lambda_2) \in S^1$. This is equivalent to the following equations being satisfied:*

$$\begin{aligned} \omega_1 \wedge d\omega_1 &= \omega_2 \wedge d\omega_2 \ (\neq 0), \\ \omega_1 \wedge d\omega_2 &= -\omega_2 \wedge d\omega_1. \end{aligned}$$

Notice that these equations can also be written as $\omega \wedge d\omega \equiv 0$ with $\omega = \omega_1 + i\omega_2$.

We say that the pair (ω_1, ω_2) is a **Cartan structure** *on M^3 if the following equations are satisfied:*

$$\begin{aligned} \omega_1 \wedge d\omega_1 &= \omega_2 \wedge d\omega_2 \ (\neq 0), \\ \omega_1 \wedge d\omega_2 &= 0 = \omega_2 \wedge d\omega_1. \end{aligned}$$

Remarks. (1) By slight abuse of language, we shall usually refer to (ω_1, ω_2) as the (taut) contact circle. Also, we shall sometimes use the expression "equal volume contact forms" for contact forms that define the same volume form. This should cause no confusion since "volume" in its more restricted meaning of integral of a volume form over a manifold will not be used in this paper.

(2) The Liouville-Cartan forms on the unit cotangent bundle $ST^*\Sigma$ of a Riemann surface Σ can be written in local coordinates as $\omega_1 = p_1\, dq_1 + p_2\, dq_2$ and $\omega_2 = p_1\, dq_2 - p_2\, dq_1$, where $q_1 + iq_2$ is a local complex coordinate on Σ and the p_j are the dual coordinates of the q_j. It is then a straightforward check that (ω_1, ω_2) is a Cartan structure.

(3) Our terminology "Cartan structure" seems justified by the fact that the Liouville-Cartan forms play an important rôle in E. Cartan's theory of moving frames (where these forms arise in the structure equations for the natural Riemannian connection), as well as by the natural relation between Cartan structures and the Maurer-Cartan form of certain Lie groups $\mathscr{G}$; this relation will be explained below. See in particular Section 7.

(4) Clearly the forms ω_1 and ω_2 of a contact circle have to be pointwise linearly independent. One can also consider triples of contact forms $\omega_1, \omega_2, \omega_3$ such that any non-trivial linear combination with constant coefficients is a contact form. Then these forms parallelize the 3-manifold, and no such family of four or more contact forms is possible because they will be linearly dependent at every point. In analogy with Definition 1.1 we call such a triple $(\omega_1, \omega_2, \omega_3)$ a *contact 2-sphere*, and again we have the corresponding notion of a *taut contact 2-sphere*. A particular case of taut contact 2-sphere is an *S^2-Cartan structure*, defined as a triple $(\omega_1, \omega_2, \omega_3)$ where each pair (ω_i, ω_j) is a Cartan structure in the sense of Definition 1.1.

Our main classification result is the following.

Theorem 1.2 *Let M^3 be a closed 3-manifold. Then M^3 admits a taut contact circle if and only if M^3 is diffeomorphic to a quotient of the Lie group $\mathscr{G}$ under a discrete subgroup Γ acting by left multiplication, where $\mathscr{G}$ is one of the following.*

(a) $S^3 = SU(2)$, *the universal cover of* $SO(3)$.

(b) $\widetilde{SL}_2$, *the universal cover of* $PSL_2\mathbb{R}$.

(c) $\widetilde{E}_2$, *the universal cover of the Euclidean group (that is, orientation preserving isometries of* $\mathbb{R}^2$).

All these manifolds admit a Cartan structure.

It is well-known that two 3-manifolds in Theorem 1.2 of different type (a), (b), or (c) cannot be diffeomorphic. See [23], as well as Section 5.3, for a detailed description of the Lie group $\widetilde{SL}_2$, and Section 5.4 for a description of $\widetilde{E}_2$. All the manifolds in Theorem 1.2 are Seifert manifolds whose Seifert invariants can be described explicitly [21], see also [9]. We also note that the left-quotients of $\widetilde{E}_2$ are precisely the T^2-bundles over S^1 with periodic monodromy, and there are exactly five such manifolds (up to diffeomorphism).

To explain the main ideas in the proof of this theorem and in order to give a reasonable classification criterion for contact circles, we need to introduce the concept of *homothety*.

Given any smooth function v and a 1-form ω on M^3, we have

$$(v\omega) \wedge d(v\omega) = v^2\omega \wedge d\omega.$$

This implies that if (ω_1, ω_2) is a contact circle on M^3 and v a nowhere zero function, then $(v\omega_1, v\omega_2)$ is also a contact circle. Furthermore, we can rotate the forms ω_1, ω_2 by a constant angle θ. If we set

$$\omega_1' = \omega_1 \cos\theta - \omega_2 \sin\theta,$$
$$\omega_2' = \omega_1 \sin\theta + \omega_2 \cos\theta,$$

then (ω_1', ω_2') is again a contact circle, in fact, the circles spanned by ω_1, ω_2 and ω_1', ω_2', respectively, are identical. This suggests the following definition.

Definition 1.3 *The* **homothety class** *of a contact circle* (ω_1, ω_2) *is the collection of all pairs* (ω_1', ω_2') *obtained from* (ω_1, ω_2) *by multiplication by the same positive function* v *and rotation by a constant angle* θ.

The relation of a homothety class to a representative (ω_1, ω_2) is analogous to the relation of a contact structure $\mathcal{D}$ to a contact form ω defining $\mathcal{D} = \ker\omega$.

Notice that if a contact circle is taut, then so are all the contact circles homothetic to it. Hence, rather than classifying taut contact circles, one wants to classify their homothety classes.

The key step in the proof of Theorem 1.2 and the classification results that will be stated below is the following theorem (= Corollary 3.12), which points to a close relationship between the theory of taut contact circles and holomorphic geometry.

Theorem 1.4 *There is a natural bijection between the following families:*

(i) *Homothety classes of taut contact circles on* M^3, *where we identify homothety classes that are equivalent under a diffeomorphism of* M^3.

(ii) *Pairs* (S, X_c), *where* S *is a complex surface and* X_c *is a nowhere zero, holomorphic vector field on* S, *satisfying the following two conditions:*

(1) *There exists a diffeomorphism* $S \cong M^3 \times \mathbb{R}$ *taking* $X = 2\,\mathrm{Re}\,X_c$ *to* ∂_t,

(2) *There exists some holomorphic symplectic form* Ω *on* S *satisfying the identity* $L_{X_c}\Omega = \Omega$.

In particular, if M^3 *admits a taut contact circle, then* $M^3 \times S^1$ *is a compact complex surface.*

The proof of the necessity part of Theorem 1.2, that is, that no other 3-manifolds than those listed in the theorem admit a taut contact circle, is then carried out in three steps: In Section 4 we classify complex surfaces (up to diffeomorphism) of the form $M^3 \times S^1$ with holomorphic S^1-action, in Section 5 we classify (up to biholomorphism) those complex surfaces of that form that can arise from a 3-manifold M^3 admitting a taut contact circle, and finally we consider the possible holomorphic S^1-actions on these complex surfaces and determine which 3-manifolds occur as quotients under such an action.

Section 3 lays the technical groundwork for this classification scheme. There we give formulae for recovering the pair (ω_1,ω_2) from the complex surface (Proposition 3.10) and provide a universal local model for taut contact circles (Theorem 3.6).

For the sufficiency part of Theorem 1.2, that is, to prove the existence of a Cartan structure on each of the manifolds listed there, a simple Lie algebra argument is used. Let $\mathscr{G}$ be one of the (simply-connected) Lie groups in Theorem 1.2. We express the Maurer-Cartan form ω_0 of $\mathscr{G}$ in terms of a basis e_1,e_2,e_3 for the Lie algebra of $\mathscr{G}$,

$$\omega_0 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3,$$

and show that the basis e_1,e_2,e_3 can be chosen in such a way that (ω_1,ω_2) defines a Cartan structure on $\mathscr{G}$ (and such that $(\omega_1,\omega_2,\omega_3)$ defines an S^2-Cartan structure if $\mathscr{G}=SU(2)$). These structures clearly descend to any left-quotient. The details of this existence proof will be given in Section 2.

We shall also see in Section 2 that if we fix a discrete, cocompact subgroup $\Gamma \subset \mathscr{G}$ then the above construction, for all admissible choices of Lie algebra basis, yields Cartan structures on $\Gamma\setminus\mathscr{G}$ which are diffeomorphic to one another in a natural way. We thus consider these Cartan structures as one, which we call the standard Cartan structure for this choice of the subgroup Γ. In Section 6 we shall give a more concrete description of the standard Cartan structures in terms of holomorphic objects. Given a manifold M^3 from Theorem 1.2, there is a discrete, cocompact subgroup Γ of $\mathscr{G}$ and a diffeomorphism $\Gamma\setminus\mathscr{G}\to M^3$. Then we can use this diffeomorphism to push forward to M^3 a standard Cartan structure on $\Gamma\setminus\mathscr{G}$. But the diffeomorphism type of M^3 alone need not determine the choice of Γ. The conjugacy class of Γ need not be determined either. So there may be non-equivalent standard Cartan structures on M^3.

Liouville-Cartan forms are a particular case of standard Cartan structures. To see this, notice that the Lie groups $\mathscr{G}$ in Theorem 1.2 are the universal covers of the groups of orientation preserving isometries of S^2, the hyperbolic plane H^2, and the Euclidean plane E^2, respectively. If Σ is a real surface with metric of constant curvature 1, -1, or 0, then given a description of $ST^*\Sigma$ as a left-quotient of $\mathscr{G}$, the Liouville-Cartan forms on $ST^*\Sigma$ lift to a pair of linearly independent, left-invariant forms on $\mathscr{G}$ which can be used as a choice for ω_1 and ω_2 in the above construction.

We now define a further equivalence relation on taut contact circles.

Definition 1.5 *A taut contact circle* (ω_1',ω_2') *is called* **homotopic** *to a taut contact circle* (ω_1,ω_2) *if there is a smooth* 1*-parameter family of taut contact circles* (ω_1^t,ω_2^t) *with* $(\omega_1^0,\omega_2^0)=(\omega_1,\omega_2)$ *and* $(\omega_1^1,\omega_2^1)=(\omega_1',\omega_2')$.

In Section 6 we use the results of Sections 3 and 5 to classify homothety classes. In particular, we prove the following theorem.

Theorem 1.6 *Let (ω_1, ω_2) be a taut contact circle on a (compact) left-quotient M^3 of one of the Lie groups $\mathscr{G}$ listed in Theorem 1.2.*

If $\mathscr{G} = \widetilde{SL}_2$ or $\widetilde{E}_2$, then there is a particular discrete subgroup Γ of $\mathscr{G}$ and a diffeomorphism $\Gamma \setminus \mathscr{G} \to M^3$ which pulls back (ω_1, ω_2) to a taut contact circle that is homothetic to the standard Cartan structure on $\Gamma \setminus \mathscr{G}$. The same is true if M^3 is a left-quotient of S^3 under a non-abelian discrete subgroup.

If M^3 is diffeomorphic to $\Gamma \setminus S^3$ with Γ trivial or cyclic, then there are taut contact circles on M^3 that are not homothetic to any Cartan structure, and there are also Cartan structures that are not equivalent to the standard one up to homothety and diffeomorphism.

If M^3 is a left-quotient of S^3, then M^3 admits a unique taut contact circle up to homotopy and diffeomorphism.

In all cases, for any taut contact circle (ω_1, ω_2) on a left-quotient M^3 of $\mathscr{G}$ there is a discrete subgroup Γ of $\mathscr{G}$ and a diffeomorphism $\Gamma \setminus \mathscr{G} \to M^3$ which pulls back (ω_1, ω_2) to a taut contact circle that is homotopic to the standard Cartan structure.

As a by-product of the proof of Theorem 1.6, we determine the moduli space of homothety classes on the lens spaces $L(m, m-1)$ (including $L(1,0) = S^3$). To give a flavour of these results, we state the classification theorem for taut contact circles on S^3.

Proposition 1.7 *There are two disjoint families of taut contact circles on S^3, up to homothety and diffeomorphism. The first family is given by*

$$\omega_1 + i\omega_2 = j^*(az_1 dz_2 + (a-1)z_2 dz_1),$$

where j denotes the standard inclusion of S^3 as the unit sphere in $\mathbb{C}^2$, and the complex number a satisfies $0 < \mathrm{Re}(a) < 1$. Different values a and a' yield equivalent taut contact circles if and only if $a' = 1 - a$. The homothety classes containing Cartan structures correspond to the real part $(0, 1)$ of that slab.

The second family forms a discrete set $\{P_n\}$ and is given by

$$\omega_1 + i\omega_2 = j^*(nz_1 dz_2 - z_2 dz_1 + z_2^n dz_2),$$

where n ranges over the positive integers. These homothety classes do not contain any Cartan structures.

The existence of non-trivial moduli shows that the analogy described above,

contact form/contact structure $\longleftrightarrow$ taut contact circle/homothety class

fails in one important respect. Contact structures are stable, that is, if two contact forms on a closed manifold are homotopic through contact forms, then the underlying contact structures are diffeomorphic (in fact, isotopic). Thus, in spite of the existence of a universal local model for taut contact circles – a "Darboux theorem" in the language of symplectic and contact geometry – there is no global stability, i.e., homotopic homothety classes are not, in general, isotopic.

Corollary 1.8 *Homothety classes of taut contact circles do not satisfy global stability.*

It will be shown in Section 6 that Theorem 1.6 also entails the following.

Corollary 1.9 *If ω_1 is a contact form (on a closed 3-manifold) that is part of a taut contact circle (ω_1, ω_2), then the contact structure $\mathscr{D}_1 = \ker \omega_1$ is tight.*

See [5] for the definition and relevance of a contact structure being tight.

We have already pointed out that the left-quotients of $SU(2)$ admit an S^2-Cartan structure. We shall see that no other manifolds admit a taut contact 2-sphere.

Theorem 1.10 *Let M^3 be a closed 3-manifold. Then M^3 admits a taut contact 2-sphere if and only if M^3 is diffeomorphic to a quotient of $SU(2)$ under a discrete subgroup acting by left multiplication.*

This theorem will be proved in Section 8.

We close this section with a few remarks on (non-taut) contact circles. Since the first version of this paper was written, we have made considerable progress on the existence problem for contact circles, and there is evidence that such structures exist on every closed 3-manifold. For the moment, however, we only state the following theorem, which is proved in [10].

Theorem 1.11 *Let M^3 be the connected sum of any number of copies of the following manifolds.*

(a) *All the manifolds listed in Theorem* 1.2,

(b) *T^2-bundles over S^1*,

(c) $S^2 \times S^1$.

Then M^3 admits a contact circle consisting of tight contact structures.

We have an ad hoc construction of a contact circle on each of the indecomposable manifolds listed in the theorem, and we can show that at least in the neighbourhood of some point on the manifold this contact circle may be assumed to satisfy the equal volume condition. Then, based on the local model for taut contact circles and the connected sum construction in [12], we can show that one can attach 1-handles near points where the contact circle is of equal volume and extend the contact circle over this 1-handle.

Section 3 discusses the close relationship between taut contact circles and holomorphic geometry. To give a simple example that illustrates the failure of complex geometric methods in the general setting, we may consider the manifold $\mathbb{R}P^3 \# \mathbb{R}P^3$. This manifold admits a contact circle by Theorem 1.11, but $(\mathbb{R}P^3 \# \mathbb{R}P^3) \times S^1$ does not admit any complex structures (cf. [27]).

See also [6] for related results in 4-dimensional symplectic geometry.

2 Proof of Theorem 1.2 – Part I

In this section we prove that the manifolds listed in Theorem 1.2 do indeed admit a Cartan structure. In fact, on each left-quotient $\Gamma \setminus \mathcal{G}$ we construct a particular Cartan structure which depends only on the subgroup Γ, and we call this the *standard Cartan structure* for the subgroup Γ.

The reader may wish to refer to [19] for the basic facts on the relevant 3-dimensional Lie algebras, in particular, the existence of bases with the properties described below.

The Lie algebra of $\mathcal{G}$ (where $\mathcal{G} = SU(2)$, $\widetilde{SL}_2$ or $\widetilde{E}_2$) admits a basis e_1, e_2, e_3 with

$$[e_1, e_2] = \varepsilon e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2,$$

where $\varepsilon = 1$ for $SU(2)$, $\varepsilon = -1$ for $\widetilde{SL}_2$, and $\varepsilon = 0$ for $\widetilde{E}_2$, respectively.

Let $\omega_1, \omega_2, \omega_3$ be the coframe dual to e_1, e_2, e_3. In other words, $\omega_1, \omega_2, \omega_3$ are the components of the Maurer-Cartan form of $\mathcal{G}$ in terms of the basis e_1, e_2, e_3 for the Lie algebra of $\mathcal{G}$. We regard the e_i as left-invariant vector fields on $\mathcal{G}$, so the ω_i are left-invariant 1-forms on $\mathcal{G}$. Let Vol be the left-invariant volume form on $\mathcal{G}$ such that $Vol(e_1, e_2, e_3) = -1$. Then

$$\omega_1 \wedge d\omega_1(e_1, e_2, e_3) = d\omega_1(e_2, e_3) = -\omega_1([e_2, e_3]) = -1,$$

Thus $\omega_1 \wedge d\omega_1 = Vol$. Similarly, we see that

$$\omega_2 \wedge d\omega_2 = Vol,$$

$$\omega_3 \wedge d\omega_3 = \varepsilon Vol,$$

and

$$\omega_i \wedge d\omega_j = 0 \quad \text{for} \quad i \neq j.$$

Hence, (ω_1, ω_2) is a Cartan structure on $\mathcal{G}$, and this structure descends to any left-quotient.

Remark. As mentioned in Section 1, there are only five compact left-quotients of $\widetilde{E}_2$ (cf. [21]), namely, the T^2-bundles over S^1 with periodic monodromy. (Up to taking the inverse and transposition, there are exactly five periodic matrices in $SL_2\mathbb{Z}$.) This allows to give explicit descriptions of these manifolds as quotients of $\mathbb{R}^3$, and to write down explicit formulae for the Cartan structure (ω_1, ω_2).

For instance, consider the manifold M^3 corresponding to the monodromy matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of period 4. We can regard M^3 as the quotient of $\mathbb{R}^3$ under the group Γ generated by the three maps that send $(x, y, z) \in \mathbb{R}^3$ to

$$(x + \tfrac{\pi}{2}, z, -y), \; (x, y + 1, z), \; (x, y, z + 1),$$

respectively. Set

$$\omega_1 = \cos x \, dy - \sin x \, dz,$$

$$\omega_2 = \sin x \, dy + \cos x \, dz.$$

These forms are invariant under Γ and induce a Cartan structure on M^3.

We notice that in fact $(\omega_1, \omega_2, \omega_3)$ is a left-invariant S^2-Cartan structure on $SU(2)$. More generally, let V_3^* be the space of left-invariant 1-forms on $\mathcal{G}$ and define a bilinear form $B(\cdot,\cdot)$ on V_3^* as follows:

$$\alpha \wedge d\beta = B(\alpha, \beta)\, Vol; \quad \alpha, \beta \in V_3^*.$$

Then the matrix of this bilinear form with respect to the basis $\omega_1, \omega_2, \omega_3$ is diagonal, with diagonal entries $1, 1, \varepsilon$. Hence $B(\cdot,\cdot)$ is a symmetric bilinear form, and it corresponds to a quadratic form Q^* on V_3^*. The value of Q^* on the general element $\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$ of V_3^* is

(a) $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ for $\mathcal{G} = SU(2)$,

(b) $\lambda_1^2 + \lambda_2^2 - \lambda_3^2$ for $\mathcal{G} = \widetilde{SL}_2$,

(c) $\lambda_1^2 + \lambda_2^2$ for $\mathcal{G} = \widetilde{E}_2$,

that is, we have a 1-sheeted hyperboloid of left-invariant equal volume contact forms on $\widetilde{SL}_2$ (and a 2-sheeted hyperboloid of equal volume contact forms with volume form $-Vol$), and a cylinder of equal volume contact forms on $\widetilde{E}_2$.

Notice that the left-invariant Cartan structures on $\mathcal{G}$ are (constant multiples of) the orthonormal bases for the 2-planes in V_3^* on which Q^* is positive-definite.

For $\mathcal{G} = SU(2)$ or $\widetilde{SL}_2$, the coadjoint representation of $\mathcal{G}$ on V_3^* covers the full orientation-preserving isometry group of Q^*. For $\widetilde{E}_2$, it covers the group of orientation-preserving isometries of Q^* which fix every element of the null line $\mathbb{R}\omega_3$. From this it follows that given any two left-invariant Cartan structures on $\mathcal{G}$ there is an element $\gamma \in \mathcal{G}$ such that right multiplication by γ takes one Cartan structure to (a constant multiple of) the other.

Now, right multiplication by the elements of $\mathcal{G}$ descends to any left quotient $\Gamma \backslash \mathcal{G}$, and so the Cartan structure we have constructed on $\Gamma \backslash \mathcal{G}$ is unique, up to diffeomorphism and constant factor, once the subgroup $\Gamma \subset \mathcal{G}$ is given. We call this the standard Cartan structure on $\Gamma \backslash \mathcal{G}$.

3 The complex structure

In this section we establish the relationship between taut contact circles and complex surfaces.

In Section 3.1 we study the basic linear algebra of 1-jets of contact circles. A natural consequence is the construction of an almost complex structure from certain pairs of 2-forms. This construction applies to contact circles, where it yields an almost complex structure with special features. We discuss these features in Corollary 3.3 and in Proposition 3.4.

In Section 3.2 we discuss the integrability of the almost complex structure and show that it is related to tautness in a very strong sense.

In Section 3.3 we begin the investigation of the converse of this construction, where taut contact circles are derived from holomorphic objects. This construction is universal enough to allow a classification of the possible 3-manifolds admitting taut contact circles, of the homothety classes of taut contact circles (up to diffeomorphism), and of which homothety classes contain

Cartan structures. We end the section with a brief sketch of the programme for solving these classification questions.

At several places in this section we deal with open 3-manifolds. These results are not needed in the sequel, but we include them for the sake of completeness.

Notation. Throughout this section we use U to denote a not necessarily closed 3-manifold. As in the rest of the paper, we reserve the notation M^3 for closed 3-manifolds. We shall often consider the product $U \times \mathbb{R}$, and then t will denote the $\mathbb{R}$-coordinate.

Given a pair (ω_1, ω_2) of 1-forms on U, then the complex 1-form $\omega_1 + i\omega_2$ will be denoted ω. An asterisk as exponent of a real or complex 1-form on U indicates the pullback of that 1-form under the obvious projection $U \times \mathbb{R} \to U$.

3.1 Construction and properties of J

It is well-known that a 1-form ω_1 on U is a contact form if and only if the 2-form $d(e^t\omega_1^*)$ is a symplectic form on $S = U \times \mathbb{R}$. Thus, a pair (ω_1, ω_2) of 1-forms on U is a contact circle if and only if any non-trivial linear combination of $d(e^t\omega_1^*)$ and $d(e^t\omega_2^*)$ is symplectic on S.

This is a non-degeneracy condition for the pair

$$d(e^t\omega_1^*)_P \, , \; d(e^t\omega_2^*)_P$$

at each point $P \in S$. We are now going to analyze the linear algebra of this non-degeneracy condition.

To that effect, let V_4 be a 4-dimensional real vector space and set $V_6 = \Lambda^2 V_4^*$. Then on V_6 we define a symmetric bilinear form Q by

$$Q(A', A'') = A' \wedge A''; \quad A', A'' \in V_6,$$

where $\Lambda^4 V_4^*$ has been identified with $\mathbb{R}$. Note that Q has signature $(3,3)$. The problem is to understand the real 2-dimensional vector subspaces of V_6 on which Q restricts to a posititve-definite form. We call such subspaces *positive-definite planes*.

It turns out that, up to linear isomorphism of V_4, there is only one positive-definite plane in V_6. Moreover, these planes (with an orientation chosen) are in one-to-one correspondence with the complex structures on V_4.

In order to state the precise theorem, notice first that a 2-plane in V_6 is positive-definite if and only if it admits a Q-conformal basis, that is, a basis $\{A_1, A_2\}$ with

$$Q(A_1, A_1) = Q(A_2, A_2) > 0 \text{ and } Q(A_1, A_2) = 0.$$

Theorem 3.1 *Given a non-zero, complex-valued, anti-symmetric bilinear form* $\Theta = A_1 + iA_2$ *on* V_4*, the following are equivalent*:

(1) *There is a complex structure* J *on* V_4 *for which* Θ *is of type* $(2,0)$,

(2) *We have* $\Theta^2 = 0$, *and the real part of* Θ *is non-degenerate,*

(3) *The pair* A_1, A_2 *is a* Q*-conformal basis for a positive-definite* 2*-plane in* V_6.

Moreover, if A_1', A_2' *is another* Q*-conformal basis for the same plane, giving the same orientation as* A_1, A_2, *then* $\Theta' = A_1' + iA_2'$ *equals* $c\Theta$ *for some complex number* c. *Thus we have a bijection between oriented positive-definite planes in* V_6 *and complex structures on* V_4.

A 2*-plane in* V_4 *is a complex line for* J *if and only if* Θ *induces the zero form in this* 2*-plane.*

Remark. Another characterization of the Q-conformal bases A_1', A_2' defining the given orientation is

$$A_1' = A_2'(J\cdot, \cdot),$$

as A_2' ranges over the non-zero elements of the 2-plane. Therefore the opposite orientation in the plane corresponds to $-J$.

Proof. Given A_1 and A_2, consider the complex-valued 2-form $\Theta = A_1 + iA_2$. A quick calculation shows that the equations

$$Q(A_1, A_1) = Q(A_2, A_2) \text{ and } Q(A_1, A_2) = 0$$

are equivalent to $\Theta^2 = 0$. On the other hand, the algebra of anti-symmetric complex-valued forms on V_4 is identical to the exterior algebra over $\mathbb{C}$ of the complex vector space $V_4^* + iV_4^*$. Hence, from $\Theta^2 = 0$ we deduce the existence of a pair of complex-valued linear forms ℓ and ℓ' on V_4 such that $\Theta = \ell \wedge \ell'$. Separating these linear forms into their real and imaginary part,

$$\ell = \ell_1 + i\ell_2,$$
$$\ell' = \ell_1' + i\ell_2',$$

we get the identities

$$A_1 = \ell_1 \wedge \ell_1' - \ell_2 \wedge \ell_2',$$
$$A_2 = \ell_1 \wedge \ell_2' + \ell_2 \wedge \ell_1',$$

and the extra requirement that A_1 have rank 4 implies that $\{\ell_1, \ell_2, \ell_1', \ell_2'\}$ is a basis for V_4^*. These formulae provide a universal model for the Q-conformal bases of positive definite planes; in particular, this implies that such planes are unique up to linear isomorphisms of V_4.

The map

$$(\ell, \ell') : V_4 \longrightarrow \mathbb{C}^2$$

is a real isomorphism, and it is clear that the only complex structure on V_4 for which Θ is of type $(2,0)$ is the one pulled back from $\mathbb{C}^2$ by this isomorphism. Then ℓ and ℓ' become complex linear coordinates for V_4. If one wants to see this complex structure as an endomorphism of V_4 whose square is -1, one only has to take the unique automorphism J of V_4 which satisties the identity

$$A_1(\cdot, \cdot) = A_2(J\cdot, \cdot).$$

It is a straightforward check that

$$A_i(J\cdot,J\cdot) = -A_i(\cdot,\cdot),\ i = 1,2.$$

If B_1,B_2 is a different positive Q-conformal basis for the 2-plane spanned by A_1,A_2, we have

$$\begin{aligned} B_1 &= rA_1 - sA_2, \\ B_2 &= sA_1 + rA_2, \end{aligned}$$

or

$$B_1 + iB_2 = (r + is)(A_1 + iA_2).$$

This implies that B_1 and B_2 are related by the same complex structure J. Conversely, we see that the automorphism relating a basis B_1,B_2 that is not Q-conformal is not a complex structure, because it equals $c' \cdot (\text{identity}) + c''J$ with $c' + c''i \neq \pm i$.

To prove the last statement of Theorem 3.1, we first observe that the $(2,0)$-form Θ induces the zero form on any complex line, since $\Theta(e,Je) = i\Theta(e,e) = 0$ for any $e \in V$. On the other hand, if $\{e_1,e_2\}$ is the basis of a 2-plane in V_4 which is not a complex line, then the vectors e_1,e_2 are also linearly independent over $\mathbb{C}$ (which acts on V_4 via J), and in the factorization $\Theta = \ell \wedge \ell'$ we can choose ℓ and ℓ' so that

$$\ell(e_1) = 1\ ,\ \ell(e_2) = 0\ ,\ \ell'(e_2) \neq 0,$$

and $\Theta(e_1,e_2) = \ell'(e_2) \neq 0$.

This completes the proof of the theorem.

The following is immediate.

Corollary 3.2 *Let Ω_1,Ω_2 be 2-forms on a 4-manifold W such that any non-trivial linear combination is of rank 4. Then there is a unique almost complex structure J on W such that the (non-vanishing) forms of type $(2,0)$ for J are precisely those of the form $A_1 + iA_2$, where at each point A_1,A_2 is a Q-conformal basis for the plane spanned by Ω_1,Ω_2 and defining the same orientation as Ω_1,Ω_2.*

A real surface C in W is a J-holomorphic curve if and only if it is almost Lagrangian for both Ω_1 and Ω_2, that is, $\Omega_i|TC \equiv 0$, $i = 1,2$.

Remark. The characterization of holomorphic curves in this corollary is similar to the class of special Lagrangian surfaces in $\mathbb{R}^4$, as studied in [14].

The unique 2-form $\Omega^{(1)}$ of type $(2,0)$ and with real part Ω_1 is

$$\Omega^{(1)} = \Omega_1 + i\Omega_1(-J\cdot,\cdot).$$

Stated a different way, we have $\Omega^{(1)} = \Omega_1 + i\Omega_1'$, where Ω_1' is determined by the following three conditions:

- $\{\Omega_1,\Omega_1'\}$ and $\{\Omega_1,\Omega_2\}$ define the same oriented plane at each point,
- $\Omega_1 \wedge \Omega_1 = \Omega_1' \wedge \Omega_1'$,
- $\Omega_1 \wedge \Omega_1' \equiv 0$.

Thus $\Omega^{(1)}$ equals $\Omega_1 + i\Omega_2$ if and only if $(\Omega_1 + i\Omega_2)^2 \equiv 0$.

Corollary 3.3 *Let (ω_1, ω_2) be a contact circle on a 3-manifold U. Then the forms $\Omega_i = d(e^t\omega_i^*)$, $i = 1, 2$, on $S = U \times \mathbb{R}$ satisfy the conditions of Corollary 3.2, and so S inherits an almost complex structure J. This structure is invariant under translation in the direction of the factor $\mathbb{R}$ of $U \times \mathbb{R}$, i.e., the flow of ∂_t is by automorphisms of J.*

Moreover, if we define $\Omega = \Omega_1 + i\Omega_2$, we have the identities

$$L_{\partial_t}\Omega = \Omega \text{ and } \omega = j_0^*(\partial_t \rfloor \Omega),$$

where j_0 maps U into $U \times \mathbb{R}$ by $p \mapsto (p, 0)$. The contact circle is taut if and only if Ω is of type $(2,0)$ with respect to J, that is, if and only if $\Omega^2 \equiv 0$.

Proof. The identities are straightforward. Also, we compute

$$\Omega^2 = 2e^{2t} dt \wedge \omega^* \wedge d\omega^*,$$

so the identity $\Omega^2 \equiv 0$ is equivalent to $\omega \wedge d\omega \equiv 0$, which is the tautness condition.

Finally, we have to check the statement on translational invariance of J. The flow in time t_0 pulls back Ω_1, Ω_2 to $e^{t_0}\Omega_1, e^{t_0}\Omega_2$, respectively. So the pulled-back 2-forms still span the same oriented plane at each point, and therefore determine the same J. Because of the naturality of the construction in Corollary 3.2, the pulled-back J equals the almost complex structure induced by the pulled-back forms. This proves the statement.

The tangent bundle of an almost complex manifold S diffeomorphic to $U \times \mathbb{R}$ (or to $U \times S^1$) splits into the complex tangencies to the level sets $U_{t_0} = \{t = t_0\}$ and the complex lines spanned by ∂_t. In the case of the construction in Corollary 3.3, this specializes further, as explained in the next proposition.

Proposition 3.4 *Let J be the almost complex structure constructed in Corollary 3.3. The complex tangencies to the level sets $U_{t_0} = \{t = t_0\}$ are spanned, as real planes, by the Reeb vector fields ξ_1 and ξ_2 of ω_1 and ω_2, respectively. The complex line spanned by ∂_t equals the real 2-plane spanned by ∂_t and the line*

$$\ker \omega_1 \cap \ker \omega_2 \text{ in } TU_{t_0}.$$

Thus TS is topologically trivial as a complex vector bundle.

The contact circle (ω_1, ω_2) is taut if and only if $J\xi_1 = \xi_2$.

Proof. Lift ξ_1 and ξ_2 to $U \times \mathbb{R}$ as vector fields tangent to the level sets of t. Then we have

$$\Omega_1(\xi_1, \xi_2) = e^t(dt \wedge \omega_1^* + d\omega_1^*)(\xi_1, \xi_2) = e^t(\xi_1 \rfloor d\omega_1^*)(\xi_2) \equiv 0.$$

Likewise $\Omega_2(\xi_1, \xi_2) \equiv 0$. By Theorem 3.1, the plane of ξ_1, ξ_2 is a complex line.

Let Y be a non-zero vector field on U which spans $\ker \omega_1 \cap \ker \omega_2$, and lift it to $U \times \mathbb{R}$ as tangent to the level sets of t. Then, for $k = 1, 2$, we have

$$\Omega_k(\partial_t, Y) = e^t(\partial_t \rfloor (dt \wedge \omega_k^* + d\omega_k^*))(Y) = e^t\omega_k^*(Y) \equiv 0.$$

Thus the span of ∂_t and Y is a complex line.

Finally, we prove the equivalence between tautness and the equality $J\xi_1 = \xi_2$. We have seen in Corollary 3.3 that if $\Omega^{(1)} = \Omega_1 + i\Omega_1'$ is the (2,0)-form with real part Ω_1 (that is, $\Omega_1'(J\cdot,\cdot) = \Omega_1$), then the contact circle is taut if and only if $\Omega_1' = \Omega_2$. We have

$$\xi_2 \rfloor \Omega_2 = -e^t dt = \xi_1 \rfloor \Omega_1 = (J\xi_1) \rfloor \Omega_1'.$$

Hence, if $\Omega_1' = \Omega_2$, then the non-degeneracy of this form implies $J\xi_1 = \xi_2$.

Conversely, suppose $J\xi_1 = \xi_2$. Then $\xi_2 \rfloor \Omega_1' = -e^t dt$. Now Ω_1' is a (pointwise) linear combination of Ω_2 and Ω_1. But $\xi_2 \rfloor \Omega_1$ induces a non-zero form on the level sets of t, since $\xi_2 \rfloor d\omega_1 \neq 0$ (because of the linear independence of $d\omega_1$ and $d\omega_2$). Thus we must have $\Omega_1' = \Omega_2$.

We now address the issue of integrability of J, first for the construction in Corollary 3.2, and then, in Section 3.2, for that of Corollary 3.3.

Proposition 3.5 *Let Ω_1, Ω_2 be as in Corollary 3.2 and suppose Ω_1 is closed. Then J is integrable if and only if the (2,0)-form $\Omega^{(1)} = \Omega_1 + i\Omega_1'$ is closed.*

Proof. The assumption that the (2,0)-form $\Omega^{(1)}$ has closed real part Ω_1 can be written as

$$d\Omega^{(1)} + d\overline{\Omega^{(1)}} = 0.$$

But for integrable J the form $d\Omega^{(1)}$ is of type (2,1), while the form $d\overline{\Omega^{(1)}}$ is of type (1,2). Thus they must be zero separately, so $\Omega^{(1)}$ is closed.

Conversely, if $\Omega^{(1)}$ is closed, then we deduce integrability of J by using the Newlander-Nirenberg Theorem as follows. If X_1 and X_2 are complex vector fields of type (0,1), then $\Omega^{(1)}(X_1,\cdot) = \Omega^{(1)}(X_2,\cdot) \equiv 0$ and one computes

$$0 \equiv d\Omega^{(1)}(X_1,X_2,Z) = -\Omega^{(1)}([X_1,X_2],Z),$$

for any complex vector field Z. Hence $[X_1,X_2] \rfloor \Omega^{(1)} \equiv 0$, thus $[X_1,X_2]$ is of type (0,1), which implies that J is integrable.

Remark. Theorem 3.1 and Proposition 3.5 together imply that a complex, closed 2-form Ω on a 4-manifold W is holomorphic symplectic with respect to a (necessarily unique) complex structure J on W if and only if $\Omega^2 \equiv 0$ and $\mathrm{Re}(\Omega)$ is symplectic. Proposition 3.5 says that if Ω_1 is closed and if Ω_1, Ω_2 induce an integrable almost complex structure, then we can induce the same structure from a holomorphic symplectic form. See [6] for more information on these symplectic aspects of the theory.

3.2 Tautness and integrability

The almost complex structure induced from a taut contact circle is always integrable. We can deduce this from Proposition 3.5 and Corollary 3.3, in which case we would be quoting the Newlander-Nirenberg Theorem. But actually we do not need this theorem, because the local model for taut contact circles that we develop next provides a special holomorphic atlas. This approach was suggested to us by the referee.

If w, z are complex-valued functions on some domain in U, we are going to use w^*, z^* to denote their respective pullbacks to $U \times \mathbb{R}$ as functions independent of t.

Theorem 3.6 *A taut contact circle* (ω_1, ω_2) *on U always admits the following local expression,*

$$\omega_1 + i\omega_2 = w\,dz,$$

where w and z are suitable local complex-valued functions.

Conversely, given w and z, the real and imaginary part of $w\,dz$ *form a taut contact circle if and only if w is nowhere zero and the map* (w, z) *is an immersion transverse to the radial directions of the first factor of* $(\mathbb{C} - \{0\}) \times \mathbb{C}$*. That is, the map* $(w/|w|, z)$ *must be an immersion into* $S^1 \times \mathbb{C}$*.*

If we take all such pairs (w, z) *for a given taut circle on U, with domains restricted to make them embeddings, then the maps* $(e^t w^*, z^*)$ *on* $U \times \mathbb{R}$ *form a holomorphic atlas. The corresponding complex structure is that from Corollary* 3.3.

Proof. Let Y be a vector field on U which spans $\ker\omega_1 \cap \ker\omega_2$. Since ω_1 and ω_2 are everywhere linearly independent, the complex 1-form $\omega = \omega_1 + i\omega_2$ defines a real isomorphism from the quotients $T_pU/\langle Y\rangle$ onto $\mathbb{C}$. Let J_0 be the unique complex structure on $TU/\langle Y\rangle$ making ω a complex isomorphism on each fibre. Since the flow of Y preserves Y, it induces isomorphisms between fibres of $TU/\langle Y\rangle$. These isomorphisms preserve J_0 if and only if $L_Y\omega$ is a complex multiple of ω.

Now, taking interior product with Y in the tautness condition $\omega \wedge d\omega \equiv 0$, we get $-\omega \wedge L_Y\omega \equiv 0$, which implies that $L_Y\omega$ is a complex multiple of ω since ω is nowhere zero. Thus J_0 is indeed invariant under the flow of Y.

Let now (U_0, x_1, x_2, x_3) be a flow box for Y as follows. The image of U_0 under the coordinate maps (x_1, x_2, x_3) is $D \times$ (interval) for some domain $D \subseteq \mathbb{R}^2$, and in these coordinates Y is ∂_{x_3}. The invariance of J_0 under the flow of Y now means that there is a unique almost complex structure on D of which J_0 is the natural lift. As D has complex dimension one, we have by the classical result of Gauss (extended to non-analytic metrics, as for example in the work of Ahlfors-Bers) that there are local holomorphic coordinates $z = x + iy$ for this structure on D. We can shrink U_0 so as to make z defined on all of D. Lift z to $D \times$ (interval) as constant in the interval direction, and then pull it back to U_0. Now dz is a complex 1-form on U_0 with the same null line $\langle Y\rangle$ as ω, and defining the same transverse complex structure J_0. Therefore $\omega = w\,dz$ for some complex-valued function w on U_0.

For the converse, start with complex functions $z = x + iy$ and w, defined on a domain $U_0 \subseteq U$. Clearly w must be nowhere zero if $\mathrm{Re}(w\,dz)$ and $\mathrm{Im}(w\,dz)$ are to be contact forms. Then we can locally write $w = re^{i\theta}$, for suitable real functions r, θ, with $r > 0$, and $(\mathrm{Re}(re^{i\theta}dz), \mathrm{Im}(re^{i\theta}dz))$ is a taut contact circle if and only if $(\mathrm{Re}(e^{i\theta}dz), \mathrm{Im}(e^{i\theta}dz))$ is a taut contact circle. The latter is

$$(\cos\theta\, dx - \sin\theta\, dy, \sin\theta\, dx + \cos\theta\, dy),$$

which is the pullback under the map (θ, x, y) of the Liouville-Cartan forms on the unit tangent bundle of the Euclidean plane. Therefore we have a taut contact circle if and only if the map (θ, x, y) has rank 3 everywhere, and this is the same as $(w/|w|, z)$ being an immersion.

As for the last statement, it is obvious that $e^t w^* dz^* = e^t \omega^*$, so that if $z_1 = e^t w^*$ and if $z_2 = z^*$ then $dz_1 \wedge dz_2 = d(e^t \omega^*)$, hence (z_1, z_2) are complex coordinates for the structure J which makes $d(e^t\omega^*)$ a $(2,0)$-form.

This concludes the proof of the theorem.

Remark. It is very easy to check directly that the coordinate changes between the maps $(e^t w^*, z^*)$ are holomorphic. This gives a construction of J, in the case of taut contact circles, which avoids the linear algebra we have developed since Theorem 3.1. We have taken the longer route because it allows to treat more general situations such as that of Corollary 3.2 and the case of non-taut contact circles. Furthermore, studying taut contact circles in this more general context, the integrability of J is seen to be equivalent to the tautness condition, in a sense made precise in the following propositions. This equivalence holds in a particularly strong sense for closed 3-manifolds (Corollary 3.9), but first we formulate two slightly more technical propositions. With these results we also begin the analysis of how to recover taut contact circles from the induced complex structure.

Proposition 3.7 *If the almost complex structure J induced on $U \times \mathbb{R}$ by a contact circle (ω_1, ω_2) is integrable, then it is also induced by a taut contact circle of the form*

$$(\omega_1, \eta_2) = (\omega_1, (h_1/h_2)\omega_1 - (1/h_2)\omega_2),$$

where $h_2 < 0$ and $h = h_1 + ih_2$ is a complex function on U which extends over $U \times \mathbb{R}$ as a ∂_t-invariant holomorphic function.

Proposition 3.8 *If $\omega = \omega_1 + i\omega_2$ represents a taut contact circle on U, inducing J on $U \times \mathbb{R}$, then the contact circles (π_1, π_2) inducing J are precisely those given by*

$$\pi_1 + i\pi_2 = h'\omega + \overline{h''\omega},$$

for any $h', h'' : U \to \mathbb{C}$ that extend as ∂_t-invariant holomorphic functions on $U \times \mathbb{R}$ and satisfy $|h''| < |h'|$ everywhere. The circle (π_1, π_2) is taut if and only if $h'' \equiv 0$.

Remark. If $\eta = h'\omega$, then $(\mathrm{Re}(\eta), \mathrm{Im}(\eta))$ is a taut contact circle inducing J. We have $\pi_1 + i\pi_2 = \eta + \overline{g\eta}$, with $g = h''/h'$, and g being constant is equivalent to $(\pi_1, c_1\pi_1 + c_2\pi_2)$ being taut for a suitable choice of constants c_1, c_2 with $c_2 > 0$. In general (for instance, for a small domain U), there is much more freedom in the choice of g (∂_t-invariance of g essentially means that g is a holomorphic function in one variable), so there are many contact circles (π_1, π_2) that yield an integrable J but which are not taut, even after passing to $(\pi_1, c_1\pi_1 + c_2\pi_2)$ for any constants c_1, c_2. On the other hand, there will be cases where U is not closed but g is still forced to be constant, since it takes values in the unit disk.

Corollary 3.9 *For a closed manifold M^3, the only contact circles inducing integrable almost complex structures on $M^3 \times \mathbb{R}$ are those of the form $(\omega_1, c_1\omega_1 + c_2\omega_2)$, where (ω_1, ω_2) is a taut contact circle and c_1, c_2 are constants, with $c_2 > 0$. Such circles are the (positively oriented) ellipses centered at the origin in the plane of ω_1 and ω_2.*

Proof of Proposition 3.7. We have $\Omega_k = d(e^t\omega_k^*)$, and $\Omega^{(k)} = \Omega_k + i\Omega_k'$ of type (2,0), k=1,2. By Proposition 3.5 we know that $\Omega^{(1)}$ and $\Omega^{(2)}$ are holomorphic symplectic. Thus $\Omega^{(1)} = h\Omega^{(2)}$, where $h = h_1 + ih_2$ is a nowhere zero, holomorphic function. It follows that $\Omega_2 = h_1\Omega_1 - h_2\Omega_1'$, and the orientation requirements imply $h_2 < 0$. We can then solve for Ω_1' :

$$\Omega_1' = \frac{h_1}{h_2}\Omega_1 - \frac{1}{h_2}\Omega_2.$$

Since $L_{\partial_t}\Omega_k = \Omega_k$, $k = 1,2$, and since Ω_1, Ω_1' is a Q-conformal basis for the plane of Ω_1, Ω_2, we conclude that $L_{\partial_t}\Omega_1' = \Omega_1'$, and so $L_{\partial_t}\Omega^{(1)} = \Omega^{(1)}$. Likewise $L_{\partial_t}\Omega^{(2)} = \Omega^{(2)}$. This has the following consequences.

- $\partial_t h \equiv 0$.
- $\eta_2^* = e^{-t}\partial_t \rfloor \Omega_1'$ is a 1-form on $U \times \mathbb{R}$ satisfying $L_{\partial_t}\eta_2^* \equiv 0$ and $\eta_2^*(\partial_t) \equiv 0$. That is, η_2^* is pulled back from a 1-form on U, which we denote η_2.
- The identities $L_{\partial_t}\Omega_1' = \Omega_1'$ and $d\Omega_1' \equiv 0$ imply $\Omega_1' = d(e^t\eta_2^*)$. So the holomorphic symplectic form $\Omega^{(1)}$ can be written as $d(e^t(\omega_1^* + i\eta_2^*))$, which means that (ω_1, η_2) is a taut contact circle inducing J.

It only remains to express η_2 in terms of ω_1 and ω_2. We have

$$\begin{aligned}\eta_2^* &= \partial_t \rfloor (e^{-t}\Omega_1') = \partial_t \rfloor \left(\frac{h_1}{h_2}e^{-t}\Omega_1 - \frac{1}{h_2}e^{-t}\Omega_2\right) \\ &= \frac{h_1}{h_2}\omega_1^* - \frac{1}{h_2}\omega_2^*,\end{aligned}$$

and restricting this identity to $\{t = 0\}$ gives the desired expression for η_2.

Proof of Proposition 3.8. Let π_k^*, $k = 1,2$, denote the pullback of π_k under the obvious projection $U \times \mathbb{R} \to U$, define $\Pi_k = d(e^t\pi_k^*)$, and define Π_k' by the requirement that $\Pi^{(k)} = \Pi_k + i\Pi_k'$ be of type (2,0) with respect to J, for $k = 1,2$. We know that the $\Pi^{(k)}$ are holomorphic.

Since $\Omega = d(e^t(\omega_1^* + i\omega_2^*))$ is holomorphic symplectic for J, there are ∂_t-invariant holomorphic functions $h^{(1)}, h^{(2)}$ such that $\Pi^{(k)} = h^{(k)}\Omega$, $k = 1,2$.

We consider $\Pi = \Pi_1 + i\Pi_2 = d(e^t(\pi_1^* + i\pi_2^*))$. This form is important for recovering π_1 and π_2, because $e^t(\pi_1^* + i\pi_2^*) = \partial_t \rfloor \Pi$. We compute

$$\begin{aligned}\Pi &= \operatorname{Re}\Pi^{(1)} + i\operatorname{Re}\Pi^{(2)} \\ &= \frac{1}{2}\left(h^{(1)}\Omega + \overline{h^{(1)}\Omega}\right) + \frac{i}{2}\left(h^{(2)}\Omega + \overline{h^{(2)}\Omega}\right) \\ &= \frac{h^{(1)} + ih^{(2)}}{2}\Omega + \frac{\overline{h^{(1)}} + i\,\overline{h^{(2)}}}{2}\overline{\Omega},\end{aligned}$$

which displays Π as a form without $(1,1)$-part. Let

$$h' = \frac{h^{(1)} + ih^{(2)}}{2} \qquad \text{and} \qquad h'' = \frac{h^{(1)} - ih^{(2)}}{2},$$

then

$$\begin{aligned}e^t(\pi_1^* + i\pi_2^*) &= \partial_t \rfloor \Pi \\ &= h'\partial_t \rfloor \Omega + \overline{h''\partial_t \rfloor \Omega} \\ &= e^t(h'\omega^* + \overline{h''\omega^*}),\end{aligned}$$

hence $\pi_1 + i\pi_2 = h'\omega + \overline{h''\omega}$.

We now have to prove that if h', h'' are ∂_t-invariant holomorphic functions, then $\pi_1 + i\pi_2 = h'\omega + \overline{h''\omega}$ represents a contact circle inducing J if and only if $|h''| < |h'|$.

The holomorphic 1-forms $dh', dh'', e^t\omega^*$ all annihilate ∂_t, therefore

$$dh' \wedge \omega^* = dh'' \wedge \omega^* \equiv 0,$$

and so

$$d(e^t(\pi_1^* + i\pi_2^*)) = h'd(e^t\omega^*) + \overline{h''d(e^t\omega^*)} = h'\Omega + \overline{h''\Omega}.$$

At a point where $h' = 0$, the form $d(e^t(\pi_1^* + i\pi_2^*))$ is of type $(2,0)$ with respect to the conjugate structure $-J$, and so (π_1, π_2) would induce $-J$ there. We conclude that h' is nowhere zero.

Then $\Psi = h'\Omega$ is a holomorphic symplectic form for J, and

$$d(e^t(\pi_1^* + i\pi_2^*)) = \Psi + \overline{g\Psi},$$

with $g = h''/h'$. Consider the real and imaginary parts,

$$\Psi = \Psi_1 + i\Psi_2 \qquad \text{and} \qquad g = g_1 + ig_2,$$

then (π_1, π_2) is a contact circle inducing J if and only if $d(e^t\pi_1^*)$ and $d(e^t\pi_2^*)$ are linearly independent and define the same orientation as Ψ_1, Ψ_2. Direct calculation yields

$$\Psi + \overline{g\Psi} = (1 + g_1)\Psi_1 - g_2\Psi_2 + i(-g_2\Psi_1 + (1 - g_1)\Psi_2),$$

and the condition becomes

$$0 < \det \begin{pmatrix} 1+g_1 & -g_2 \\ -g_2 & 1-g_1 \end{pmatrix} = 1 - g_1^2 - g_2^2,$$

that is, $|g| < 1$.

Notice that if $|g| > 1$, we still get a contact circle, but the complex structure it induces is the conjugate one $-J$.

Finally, the circle (π_1, π_2) is taut if and only if $d(e^t(\pi_1^* + i\pi_2^*))$ is of type $(2,0)$ with respect to J, and it is clear that this is equivalent to $h'' \equiv 0$.

This finishes the proof of Proposition 3.8.

3.3 Construction and study via holomorphic objects

For the remainder of this paper, we consider taut contact circles only.

Proposition 3.8 provides a construction of a family of (taut) contact circles starting from a taut contact circle that serves as reference. The formula $\omega = j_0^*(\partial_t \rfloor \Omega)$ of Corollary 3.3 gives a clue as to how we can construct a family of taut contact circles without starting with a reference one. Instead, we need a holomorphic 2-form Ω and a vector field X (playing the role of ∂_t) and, to keep the situation of Corollary 3.3, we impose the identity $L_X\Omega = \Omega$ and that the flow of X be made of holomorphic maps.

It is convenient to consider the complex vector field $X_c \stackrel{\text{def}}{=} (1/2)(X - iJX)$, from which X is easily recovered as $2\,\mathrm{Re}\,X_c$. Note that X_c is holomorphic if and only if the flow of X is made of holomorphic maps. Also, as Ω is a $(2,0)$-form, we have $X \rfloor \Omega = X_c \rfloor \Omega$ and $L_X\Omega = L_{X_c}\Omega$. Now all the relevant conditions can be formulated in terms of the holomorphic objects Ω, X_c.

Proposition 3.10 *Let S be a complex surface on which we have*
(1) *a nowhere zero, holomorphic 2-form Ω,*
(2) *a nowhere zero, holomorphic vector field X_c,*
(3) *a real hypersurface $j : U \to S$,*
and suppose that
(i) $L_{X_c}\Omega = \Omega$,
(ii) *j is transverse to $X = 2\,\mathrm{Re}\,X_c$.*
Then $\omega_1 + i\omega_2 = j^*(X_c \rfloor \Omega)$ defines a taut contact circle on U which induces the complex structure of S if we identify a neighbourhood of $U \times \{0\}$ in $U \times \mathbb{R}$ with a neighbourhood of $j(U)$ in S, taking ∂_t to X.

A complex surface S comes from a taut contact circle as in Corollary 3.3 *if and only if it has a pair X_c, Ω, satisfying* (1), (2), *and* (i), *and where $2\,\mathrm{Re}\,X_c$ is complete, with open orbits, and admits a global transversal which pierces each orbit exactly once.*

Proof. Let $\tilde{\omega} = X_c \rfloor \Omega$. This is a nowhere zero, holomorphic 1-form which satisfies

$$d\tilde{\omega} = \Omega, \quad L_{X_c}\tilde{\omega} = L_X\tilde{\omega} = \tilde{\omega}, \quad \text{and} \quad \tilde{\omega}(X_c) = \tilde{\omega}(X) \equiv 0.$$

Since we are in complex dimension 2, we have local expressions $\widetilde{\omega} = z_1 dz_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are holomorphic functions, and z_1 has no zeros. Now $d\widetilde{\omega} = \Omega$ translates to $dz_1 \wedge dz_2 = \Omega$, hence z_1, z_2 define local complex coordinates and we are back to the situation in Theorem 3.6, so let us compute the radial vector field in the z_1-direction.

The equation $X_c \rfloor (dz_1 \wedge dz_2) = z_1 dz_2$ implies $X_c = z_1 \partial_{z_1}$, therefore

$$X = 2\,\mathrm{Re}\,(z_1 \partial_{z_1}) = x_1 \partial_{x_1} + y_1 \partial_{y_1}$$

is the radial vector field in the z_1-direction. By Theorem 3.6, the real and imaginary part of $\omega = j^*\widetilde{\omega}$ define a taut contact circle if and only if j is transverse to X.

Consider now the immersion Φ, from a neighbourhood of $U \times \{0\}$ in $U \times \mathbb{R}$ to a neighbourhood of $j(U)$ in S, which maps $(p, 0) \in U \times \{0\}$ to $j(p)$ and whose differential takes ∂_t to X. Let J^* be the pullback under Φ of the complex structure of S. This is the unique almost complex structure for which $\Phi^*\Omega$ is a $(2,0)$-form. We have to show that J^* agrees with the complex structure induced on $U \times \mathbb{R}$ by the taut contact circle $(\mathrm{Re}\,(\omega), \mathrm{Im}\,(\omega))$.

The identities $\widetilde{\omega}(X) \equiv 0$ and $L_X\widetilde{\omega} = \widetilde{\omega}$ pull back to the identities

$$(\Phi^*\widetilde{\omega})(\partial_t) \equiv 0 \ \text{ and } \ L_{\partial_t}(\Phi^*\widetilde{\omega}) = \Phi^*\widetilde{\omega},$$

hence $\Phi^*\widetilde{\omega} = e^t\omega^*$, where ω^* is the pullback under the projection $U \times \mathbb{R} \to U$ of $j_0^*\Phi^*\widetilde{\omega} = j^*\widetilde{\omega} = \omega$, here $j_0(p) = (p, 0)$. Thus J^* is the almost complex structure for which $\Phi^*\Omega = d(e^t\omega^*)$ is a $(2,0)$-form, i.e., the complex structure induced by the taut contact circle $(\mathrm{Re}\,(\omega), \mathrm{Im}\,(\omega))$.

A real vector field X on S is ∂_t, for some product structure $S \cong U \times \mathbb{R}$, if and only if it is complete, with open orbits, and it has a global transversal piercing each orbit exactly once.

The proof of Proposition 3.10 is now complete.

For $\omega_1 + i\omega_2 = j^*(X_c \rfloor \Omega)$ to define a taut contact circle, $X = 2\,\mathrm{Re}\,X_c$ need not be ∂_t for any product structure $S \cong U \times \mathbb{R}$ (indeed, X need not be complete), and the transverse immersion j need not be injective. But obviously we already get all taut contact circles if we restrict ourselves to quadruples (S, X_c, Ω, j) satisfying (1), (2), and (i) of Propostion 3.10, and where $X = 2\,\mathrm{Re}\,X_c$ is complete, with open orbits, and j is an embedding transverse to X and piercing each orbit exactly once. The pair (X, j) represents a product structure $S \cong U \times \mathbb{R}$.

The choice of one such product structure is implicit in Proposition 3.8, because it gives all (taut) contact circles (π_1, π_2) such that J is the only almost complex structure for which $d(e^t(\pi_1^* + i\pi_2^*))$ is of type $(2,0)$. It is then clear that the projection $S \to U$ and the function $t : S \to \mathbb{R}$ have been fixed in Proposition 3.8, and this is the same as fixing a product structure $S \cong U \times \mathbb{R}$.

Of the two data (X, j) which determine the product structure, we now allow changes in the second part j. This gives a much larger family of taut contact circles, which we again restrict by fixing a choice (up to a multiplicative constant) of the holomorphic 2-form.

Theorem 3.11 *There is a natural bijection between the following families:*

(i) *Homothety classes of taut contact circles on U, where we identify homothety classes that are equivalent under a diffeomorphism of U.*

(ii) *Triples (S, X_c, Ω), where S is a complex surface, X_c and Ω satisfy* (1), (2), *and* (i) *of Proposition* 3.10, *and $X = 2\,\mathrm{Re}\,X_c$ is complete, with open orbits, and has a global transversal diffeomorphic to U piercing each orbit exactly once. We identify two such triples (S, X_c, Ω) and (S', X_c', Ω') if there is a biholomorphism between S and S' taking X_c to X_c' and pulling Ω' back to a constant multiple of Ω.*

Proof. The embeddings of U into $U \times \mathbb{R}$, transverse to ∂_t and piercing each orbit exactly once are, up to reparametrization, the graph embeddings

$$\begin{array}{rcl} j_u : U & \longrightarrow & U \times \mathbb{R} \\ p & \longmapsto & (p, u(p)) \end{array}$$

for u any smooth function on U.

We conclude that, for a quadruple (S, X_c, Ω, j) as above, the set of embeddings transverse to $2\,\mathrm{Re}\,X_c$ and piercing each orbit exactly once is the set of reparametrizations of the embeddings

$$j_u'(p) = \varphi_{u(p)}(j(p)),$$

where φ_t is the flow of $2\,\mathrm{Re}\,X_c$, and u ranges over the smooth functions on U. Then $j = j_0'$, and the identity $L_X(X_c \rfloor \Omega) = (X_c \rfloor \Omega)$ implies

$$(j_u')^*(X_c \rfloor \Omega) = e^u j^*(X_c \rfloor \Omega).$$

Therefore, if we let $\omega_1 + i\omega_2 = j^*(X_c \rfloor \Omega)$, then as j' ranges over all such embeddings, the taut contact circle made of the real and imaginary part of $j'^*(X_c \rfloor \Omega)$ ranges over all diffeomorphic copies of the taut contact circles $(e^u\omega_1, e^u\omega_2)$, for any smooth function u on U.

If we further let z_0 range over all complex constants, then the taut contact circle made of the real and imaginary part of $j'^*(X_c \rfloor (z_0\Omega))$ ranges over all diffeomorphic copies of all taut contact circles homothetic to (ω_1, ω_2).

If we drop the choice of j and consider the triple (S, X_c, Ω), up to isomorphism and multiplication of Ω by complex constants, then we determine, up to diffeomorphism, a homothety class of taut contact circles on U, in the explicit way we describe next.

We first recover U as the orbit space of $X = 2\,\mathrm{Re}\,X_c$. A more concrete model for U is provided by any transversal to X which pierces each orbit exactly once. Then the homothety class is recovered by inducing the 1-form $X_c \rfloor \Omega$ in that transversal, and taking real and imaginary part of the induced form.

Conversely, fix a taut contact circle (ω_1, ω_2) on U, and let (η_1, η_2) be a homothetic taut contact circle. If we set $\omega = \omega_1 + i\omega_2$ and $\eta = \eta_1 + i\eta_2$, then there are a function u and a complex constant z_0 such that $\eta = z_0 e^u \omega$. It follows from the proof of Proposition 3.10 that if Φ is the diffeomorphism of $U \times \mathbb{R}$ which preserves ∂_t and satisfies $\Phi \circ j_0 = j_u$, then Φ pulls $e^t\omega^*$ back to $e^t(e^u\omega)^*$, and therefore it pulls back the quadruple (S, X_c, Ω, j_0) to the quadruple $(S', X_c', (1/z_0)\Omega', j_u)$. Here S is $U \times \mathbb{R}$ with the structure J making $\Omega = d(e^t\omega^*)$ a (2,0)-form, and $X_c = (1/2)(\partial_t - iJ\partial_t)$; likewise S' is $U \times \mathbb{R}$ with the structure J' making $\Omega' = d(e^t\eta^*)$ a (2,0)-form, and $X_c' = (1/2)(\partial_t - iJ'\partial_t)$.

In particular, we see that homothetic taut contact circles induce complex structures on $U \times \mathbb{R}$ isomorphic through a diffeomorphism which preserves ∂_t. Moreover, this diffeomorphism makes the corresponding holomorphic symplectic forms isomorphic up to a multiplicative constant.

Theorem 3.11 is now proved.

Corollary 3.12 *Classifying homothety classes of taut contact circles on closed 3-manifolds, up to diffeomorphism, is the same as classifying pairs (S, X_c) where S is a complex surface and X_c is a nowhere zero, holomorphic vector field on S, satisfying the following two conditions:*

(1) *There exists a diffeomorphism $S \cong M^3 \times \mathbb{R}$ taking $X = 2\,\mathrm{Re}\, X_c$ to ∂_t.*

(2) *There exists some holomorphic symplectic form Ω on S satisfying the identity $L_{X_c}\Omega = \Omega$.*

Proof. The only fact that needs to be checked is that Ω is unique up to a multiplicative constant. The holomorphic 2-forms Ω' satisfying $L_{X_c}\Omega' = \Omega'$ are given by $\Omega' = h\Omega$, with h a ∂_t-invariant, holomorphic function. But we have $S \cong M^3 \times \mathbb{R}$, with $X = \partial_t$ and M^3 compact, hence h must be constant.

This proves the corollary.

From here to the end of the paper, we consider only the case of a closed 3-manifold M^3.

We shall next give an outline of how the classification of homothety classes and of the manifolds M^3 is carried out in this paper. The main tools will be Theorem 3.11, the classification of compact complex surfaces, and the description of geometric complex surfaces (in the sense of [27]) as quotients under discrete group actions.

For the classification of the triples (S, X_c, Ω), let $\widetilde{S}$ be the universal cover of S, and let $(\widetilde{X}_c, \widetilde{\Omega})$ be the lift to $\widetilde{S}$ of (X_c, Ω). The real vector field $\widetilde{X} = 2\,\mathrm{Re}\,\widetilde{X}_c$ is the lift of $X = 2\,\mathrm{Re}\, X_c$.

Let Γ be the group $\pi_1(M^3) = \pi_1(S)$, considered as a transformation group of $\widetilde{S}$ by the monodromy representation. Then Γ is a group of holomorphic automorphisms of $\widetilde{S}$ which preserve $\widetilde{X}_c$ and $\widetilde{\Omega}$. From the quadruple $(\widetilde{S}, \Gamma, \widetilde{X}_c, \widetilde{\Omega})$ we determine the triple (S, X_c, Ω) as follows. The surface S is obtained as the quotient $S = \Gamma \setminus \widetilde{S}$. Now $\widetilde{X}_c$ and $\widetilde{\Omega}$ descend to this quotient, thus defining X_c and Ω and allowing to recover the homothety class as explained in the proof of Theorem 3.11.

Notice that the possible 3-manifolds M^3 are also obtained here. There is a diffeomorphism $\widetilde{S} \cong \widetilde{M}^3 \times \mathbb{R}$ taking $\widetilde{X}$ to ∂_t, where $\widetilde{M}^3$ is the universal cover of M^3. The hypersurface $M^3 \times \{0\} \subset S = M^3 \times \mathbb{R}$ lifts to a hypersurface H_0 in $\widetilde{S}$ which is Γ-invariant and pierces each orbit of $\widetilde{X}$ exactly once and transversely. Clearly H_0 is diffeomorphic to the orbit space of $\widetilde{X}$. As $\widetilde{X}$ is Γ-invariant, the transformations $\gamma \in \Gamma$ map orbits of $\widetilde{X}$ to orbits of $\widetilde{X}$ and therefore induce diffeomorphisms of the orbit space of $\widetilde{X}$. Then we have

$$M^3 = \Gamma \setminus (\text{orbit space of}\, \widetilde{X}) = \Gamma \setminus H_0 \cong \Gamma \setminus H_1,$$

where H_1 is any Γ-invariant hypersurface piercing each orbit of $\widetilde{X}$ once and transversely. The choice of H_1 does not matter because they are all isotopic to one another, due to the uniqueness up to isotopy of the global transversals of X in S.

Consequently, it is sufficient to classify the quadruples $(\widetilde{S}, \Gamma, \widetilde{X}_c, \widetilde{\Omega})$. We relate them to compact surfaces in the following obvious way.

Use φ_t to denote both the flow of $\widetilde{X}$ and the flow of X. Then, for each positive number t_0, we define

$$S(t_0) = \langle \varphi_{t_0} \rangle \setminus S,$$

which is a *compact* complex surface diffeomorphic to $M^3 \times S^1$.

Notice that $\widetilde{S}$ is also the universal cover of $S(t_0)$, for all $t_0 > 0$.

The vector field X_c descends to each $S(t_0)$, defining there a vector field which we also denote by X_c. Similarly for X, and the relation $X = 2\,\mathrm{Re}\,X_c$ holds also on $S(t_0)$. The pair $(S(t_0), X_c)$ determines t_0 as the common period of all orbits of X.

The diffeomorphism Φ, constructed in the proof of Theorem 3.11, descends to a diffeomorphism of $S(t_0)$ for any $t_0 > 0$, that is, homothetic taut contact circles on M^3 not only induce isomorphic pairs (S, X_c) and (S', X_c'), but also isomorphic pairs $(S(t_0), X_c)$ and $(S'(t_0), X_c')$ for each $t_0 > 0$. Pairs with different t_0 cannot be isomorphic.

We have thus associated to each homothety class a family of pairs $(S(t_0), X_c)$, parametrized by a real number t_0, and this family is a diffeomorphism invariant for the homothety class. We shall use this observation in Section 6 to distinguish homothety classes.

The Γ-invariance of $\widetilde{X}_c$ implies that each map φ_t of $\widetilde{S}$ commutes with every element of Γ. Thus, if we let G be the group $\pi_1(S(t_0))$, considered as a transformation group of $\widetilde{S}$ by the monodromy representation, we have a direct product decomposition

$$G \cong \langle \varphi_{t_0} \rangle \times \Gamma.$$

The vector field $\widetilde{X}_c$ is invariant under every element of G. An element of G leaves $\widetilde{\Omega}$ invariant if and only if it is in Γ, because we have $\varphi_t^* \widetilde{\Omega} = e^t \widetilde{\Omega}$ for all t.

Remark. In the proof of Theorem 3.11 we have seen that the global transversals j, piercing each orbit of X in S exactly once, form a single isotopy class. By mapping these under the projection $S \to S(t_0)$, we determine an isotopy class of transversals piercing each orbit of X in $S(t_0)$ exactly once. The isotopy classes of transversals in $S(t_0)$ correspond to the elements of $[M^3, S^1] = H^1(M^3; \mathbb{Z})$. Thus it is not immediate to determine the pair (S, X)

from the pair $(S(t_0), X)$, unless $H_1(M^3;\mathbb{Z})$ is a finite group. We deal with this delicate point by giving the following alternative description of the covering space S of $S(t_0)$:

$$S = \Gamma \setminus \widetilde{S},$$

where $\widetilde{S}$ is the universal cover of $S(t_0)$ and Γ is the isotropy subgroup of $\widetilde{\Omega}$ in the monodromy representation G of $\pi_1(S(t_0))$. Thus the determination of the possible forms $\widetilde{\Omega}$ implies the determination of the possible (non-compact) surfaces S. This shows that it is convenient to work at the level of the quadruples $(\widetilde{S}, \Gamma, \widetilde{X}_c, \widetilde{\Omega})$ rather than at the level of the quotients $S(t_0)$, although these will be needed to make the classification possible because they are compact.

In Section 4 we classify those compact complex surfaces of the form $M^3 \times S^1$ where the obvious circle action is by holomorphic maps. We obtain seven classes of such surfaces. If one such surface is $\langle \varphi_{t_0} \rangle \setminus S$, where $S = M^3 \times \mathbb{R}$ has complex structure constructed as in Theorem 3.6, then the special features stated in Corollary 3.3 and in Proposition 3.4 for the complex structure of S impose additional restrictions, and not all surfaces found in Section 4 are allowed. We use this idea in Section 5.1 to show that a surface $S(t_0)$ has to be in one of the following classes:

(a') Hopf surfaces,
(b') Properly elliptic surfaces of geometric type $\widetilde{SL}_2 \times E^1$,
(c') Hyperelliptic surfaces with Euler class $(0,0)$,
(c") Complex tori.

For each of these four classes, the surfaces $S(t_0)$ form a proper subclass. These subclasses are completely determined in Sections 5.2, 5.3, and 5.4.

Notice that the four classes above give a description $S(t_0) = G \setminus \widetilde{S}$, where $\widetilde{S}$ is known and either all possibilities for G are known (case of Hopf surfaces) or G is a subgroup of the isometry group of some standard homogeneous metric on $\widetilde{S}$. Then we apply the conditions:

- $\widetilde{X}_c$ is G-invariant and nowhere zero,
- $\widetilde{\Omega}$ is holomorphic symplectic, and $L_{\widetilde{X}_c} \widetilde{\Omega} = \widetilde{\Omega}$,
- $\widetilde{\Omega}$ is Γ-invariant,
- G is the direct product $\langle \varphi_{t_0} \rangle \times \Gamma$,

to determine the triple $(\Gamma, \widetilde{X}_c, \widetilde{\Omega})$ up to biholomorphism of $\widetilde{S}$. Surfaces of classes (b') and (c') are elliptic, and we prove in Section 4 that the elliptic fibres contain the orbits of X. This helps in the determination of X_c for these two classes.

At that point, the quadruples $(\widetilde{S}, \Gamma, \widetilde{X}_c, \widetilde{\Omega})$ are finally classified up to biholomorphism.

For example, for class (b') we get $\widetilde{S} = \widetilde{SL}_2 \times E^1$ with the standard compatible complex structure, $\pm X$ is the unit vector field along the E^1-factor, and Γ is a discrete subgroup of the obvious action of $\widetilde{SL}_2$ by left multiplication. Since not all discrete isometry groups of $\widetilde{SL}_2 \times E^1$ are equivalent to the direct product of a subgroup of $\widetilde{SL}_2$, acting by left multiplication, and an infinite cyclic group of translations along the E^1-factor, we see that not all properly elliptic surfaces of this geometric type are surfaces $S(t_0)$ arising from a taut contact circle.

In particular, the vector field $\widetilde{X}$ has a unique canonical form for class (b'). The same is true for classes (c') and (c"), but for class (a') the situation is more complicated, and we get a continuous family of non-equivalent canonical forms as well as a discrete set of additional canonical forms. The families of taut contact circles of Proposition 1.7 are obtained from the families of canonical forms for $\widetilde{X}$ in class (a').

After classifying the quadruples $(\widetilde{S}, \Gamma, \widetilde{X}_c, \widetilde{\Omega})$ up to biholomorphism, we obtain the list of all possible 3-manifolds M^3 by the procedure indicated above. Cases (a) and (b) of Theorem 1.2 correspond to surfaces $S(t_0)$ in the classes (a') and (b'), respectively. Case (c) of Theorem 1.2 corresponds to surfaces $S(t_0)$ in classes (c') and (c").

Returning to the example of class (b'), the hypersuface $H_0 = \widetilde{SL_2} \times \{0\}$ is a transversal for the unit vector field along the E^1-factor, and it is invariant under left multiplication by any subgroup Γ of $\widetilde{SL_2}$. Thus, we can write:

$$M^3 = \Gamma \setminus \widetilde{SL_2},$$

and these are the 3-manifolds arising in case (b) of Theorem 1.2. One arrives at cases (a) and (c) of Theorem 1.2 in the same way.

With the help of the invariant $(S(t_0), X_c)$, we completely classify homothety classes corresponding to case (a). For cases (b) and (c), we give a construction of all homothety classes, which we further discuss in Section 7, but the study of the moduli spaces in these two cases is left to a forthcoming paper [11].

Simultaneously with the description of all the homothety classes, we discuss Cartan structures. This is possible because we have a characterization, also in terms of holomorphic objects, of the homothety classes which contain a Cartan structure. We end this section with such a characterization.

Proposition 3.13 *Let a taut contact circle* (ω_1, ω_2) *be constructed as in Proposition* 3.10, *that is,* $\omega_1 + i\omega_2 = j^*(X_c \rfloor \Omega)$. *Then* (ω_1, ω_2) *is a Cartan structure if and only if* $JX = -2\,\mathrm{Im}\,X_c$ *is tangent to the immersion* j.

Proof. It suffices to check the condition locally. So let $X_c = z_1\partial_{z_1}$ and $\omega_1 + i\omega_2 = j^*(z_1 dz_2)$. Then

$$\begin{aligned}\omega_1 \wedge d\omega_2 &= j^*\left((-y_1\partial_{x_1} + x_1\partial_{y_1}) \rfloor (dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2)\right)\\ &= j^*\left(-2(\mathrm{Im}\,X_c) \rfloor (dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2)\right).\end{aligned}$$

Thus $\omega_1 \wedge d\omega_2 \equiv 0$ if and only if j is tangent to $-2\,\mathrm{Re}\,X_c$.

Since (ω_1, ω_2) is a taut contact circle, the identity $\omega_1 \wedge d\omega_2 \equiv 0$ is the only condition for it to be a Cartan structure.

Corollary 3.14 *Let* (ω_1, ω_2) *be a taut contact circle on a closed* 3-*manifold* M^3. *Then* (ω_1, ω_2) *is homothetic to a Cartan structure if and only if there is a compact hypersurface in* $M^3 \times \mathbb{R}$, *transverse to* ∂_t, *piercing each orbit of* ∂_t *exactly once, and tangent to* $J\partial_t$.

4 Complex surfaces

This and the following section rely heavily on the Enriques-Kodaira classification of compact complex surfaces in general, for which [1] is a standard reference, as well as on Wall's detailed study [27] of geometric structures (in the sense of Thurston) on these surfaces. We freely use some fundamental results on geometric manifolds. For this the survey by Scott [23] is a good introduction; the most relevant references for us are [25, 26, 27].

In the light of Theorem 1.4, the first step towards proving Theorem 1.2 is the following.

Theorem 4.1 *A compact complex surface W is diffeomorphic to a complex surface of the form $M^3 \times S^1$ on which the obvious smooth S^1-action is by holomorphic automorphisms, if and only if W is one of the following.*

(a) *A Hopf surface that is (topologically) of the form $(\Gamma \setminus S^3) \times S^1$ with Γ a discrete subgroup of $U(2)$.*

(b) *A properly elliptic surface of the form $(\Gamma \setminus (H \times E^1)) \times S^1$ or $(\Gamma \setminus \widetilde{SL}_2) \times S^1$, with Γ a discrete subgroup of the identity component of the isometry group of $H \times E^1$ or $\widetilde{SL}_2$, respectively, where H denotes the hyperbolic plane and E^1 the Euclidean line.*

(c) *One of the hyperelliptic surfaces (which are topologically T^2-bundles over T^2 with monodromy A, I, where $A \in SL_2\mathbb{Z}$ is periodic and I the identity matrix, $A \neq I$) with Euler class $(0,0)$. Up to diffeomorphism, there are four such surfaces.*

(d) *A complex torus, diffeomorphic to T^4.*

(e) *A primary Kodaira surface, which is topologically a T^2-bundle over T^2 with trivial monodromy and non-zero Euler class.*

(f) *A secondary Kodaira surface of the form $(\Gamma \setminus Nil^3) \times S^1$, where Γ is a discrete subgroup of the identity component of the isometry group of Nil^3 (the Heisenberg group).*

(g) *A ruled surface of genus 1 that is topologically $S^2 \times T^2$.*

Proof. All the manifolds listed in the theorem are of the form $M^3 \times S^1$ and can be endowed with a complex structure such that the obvious S^1-action is by holomorphic maps. This can be seen from [27], since all the manifolds in the theorem admit a geometric structure in the sense of Thurston and a complex structure compatible with the geometry.

We have to show that no other complex surfaces are possible. Note that there may be different decompositions of W as $M^3 \times S^1$, not necessarily compatible with the geometric structure, so the classification of the corresponding M^3 is not a straightforward consequence.

Observe that $W = M^3 \times S^1$ has to be minimal. For any rational curve C in W necessarily represents a class in $H_2(W)$ that lies in the image of $i_* : H_2(M^3) \to H_2(W)$. Hence C has self-intersection 0.

Also, W is clearly (real) parallelizable, so in particular its Euler number c_2 equals zero. Then from the Enriques-Kodaira classification (cf. [1]) we see that W is among the following:

(i) surfaces of class VII_0,
(ii) ruled surfaces of genus 1,
(iii) hyperelliptic surfaces,
(iv) primary and secondary Kodaira surfaces,
(v) tori,
(vi) minimal properly elliptic surfaces.

We now deal with these complex surfaces in turn.

(i) Surfaces of class VII_0 have first Betti number $b_1 = 1$. Hence, if $W = M^3 \times S^1$, then $b_1(M^3) = 0$. So $b_2(M^3) = 0$ by Poincaré duality and therefore $b_2(W) = 0$. By a famous result of Bogomolov ([2, 3]; see also [24] for an alternative proof of this result), such a surface is a Hopf surface or an Inoue surface.

If W is a Hopf surface, then by [16] it has to be as described in (a).

We now make the following observation.

Proposition 4.2 *No Inoue surface is diffeomorphic to a 4-manifold $M^3 \times S^1$.*

Proof. Let W be an Inoue surface. We show that the fundamental group $\pi_1(W)$ does not have a direct summand $\mathbb{Z}$.

There are three families of Inoue surfaces, S_M, S_N^+ and S_N^- (see [15]). The fundamental group $\pi_1(S_M)$ has generators g_0, g_1, g_2, g_3 and relations

$$g_i g_j = g_j g_i \text{ for } i,j = 1,2,3,$$

$$g_0 g_i g_0^{-1} = g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}} \text{ for } i = 1,2,3,$$

where $M = (m_{ij}) \in SL_3\mathbb{Z}$ is a unimodular matrix with eigenvalues $\alpha, \beta, \overline{\beta}$ such that $\alpha > 1$ and $\beta \neq \overline{\beta}$. An element of π_1 that yields a free generator of $H_1(S_M; \mathbb{Z})$ has to be of the form

$$g_1^{a_1} g_2^{a_2} g_3^{a_3} g_0$$

(or an inverse of that). The condition that this commutes with g_0 in π_1 is that (a_1, a_2, a_3) is an eigenvector of M with eigenvalue 1, which cannot happen.

This proves Proposition 4.2 for the surfaces S_M. The proof for the other two families of Inoue surfaces is similar.

(ii) Up to diffeomorphism, there are only two ruled surfaces of genus 1, the trivial and the non-trivial S^2-bundle over T^2. A standard argument shows that the latter has non-vanishing second Stiefel-Whitney class $w_2(W)$, and hence is not parallelizable.

(iii) Up to diffeomorphism, there are seven hyperelliptic surfaces. These are elliptic surfaces without exceptional fibres over an elliptic curve, hence topologically T^2-bundles over T^2, with $b_1(W) = 2$. The elliptic fibration is unique up to bundle isomorphism, even in the topological category [22] (that is, there is a unique fibration of this surface as a T^2-bundle over T^2, without any reference to the complex structure). The holomorphic S^1-action on W has to send fibres to fibres (see Lemma 4.3 below), so it projects to an S^1-action on the base torus. Hence the quotient of W under this S^1-action is either a T^2-bundle over S^1, if the projected S^1-action is non-trivial, or an S^1-bundle over T^2, if the S^1-action is along the fibres. The latter cannot happen, since then $b_1(W) = 3$ or 4. In the former case, the torus bundle is easily seen to have Euler class $(0,0)$ (see [22] for the definition of this class), and precisely four of the seven hyperelliptic surfaces meet this condition (cf. [25]).

We shall have to use this type of argument several times in the sequel, so we formulate it as a separate lemma.

Lemma 4.3 *Let $\pi : W \to \Sigma$ be a ruling or an elliptic fibration of a complex surface W over a complex curve Σ, and assume that W admits an S^1-action by holomorphic automorphisms. Then this S^1-action sends general fibres to general fibres and each exceptional fibre onto itself. So the S^1-action projects to an action on Σ, and this action is again by holomorphic automorphisms. Furthermore, if the action is free, there are no singular fibres, that is, the only exceptional fibres are multiple fibres.*

Remark. We use the term *ruling* in the sense of [1], meaning an analytic fibre bundle with fibre $\mathbb{C}P^1$. Other authors refer to this as *geometric* ruling.

Proof. Consider a general fibre F_0. Let $\varphi_t : W \to W$ be the action of $t \in S^1$. Then $F_t = \varphi_t(F_0)$ is a complex submanifold of W isotopic to F_0. Since all general fibres represent the same homology class with self-intersection 0, we have, for any general fibre F, either $F_t \cap F = \emptyset$ or $F_t = F$.

By a similar argument, the S^1-action has to send each of the finitely many isolated exceptional fibres (in the elliptic case) onto itself.

So the S^1-action projects to an S^1-action by holomorphic automorphisms of Σ, with at least as many fixed points as exceptional fibres.

Let X be the real vector field that induces the S^1-action on W. We have seen that X has to be tangential to the exceptional fibres. A holomorphic action cannot move any singular point of such a fibre to a non-singular point. Since X is nowhere zero if the action is free, we conclude that the only exceptional fibres are multiple fibres.

(iv) The primary Kodaira surfaces are listed as (e) in Theorem 4.1; it remains to show that among the secondary Kodaira surfaces only those listed in (f) are possible. From Wall [27, Lemma 7.2] we know that a secondary Kodaira surface (which has Kodaira dimension $\kappa = 0$ and $b_1(W) = 1$) admits a unique elliptic fibration, and this fibration has no singular fibres. Furthermore, W is modelled on the geometry $Nil^3 \times E^1$ and the base orbifold of the elliptic fibration is a sphere with three or four cone points (corresponding to multiple fibres) [25, 27], since the base orbifold has to be orientable and of orbifold characteristic 0 (the base cannot be a torus, for then there would be no singular fibres and we would have a primary Kodaira surface). As shown in Lemma 4.3, the projected S^1-action has fixed points at these cone points, hence is trivial. In other words, the S^1-action is along the fibres. This implies that the quotient M^3 under this S^1-action is a Seifert bundle over a sphere with three or four cone points, and it has to be of geometric type Nil^3, that is, a Seifert bundle with non-zero Euler class. This follows from the fact that W is a Seifert T^2-bundle with non-zero Euler class or from [27, Theorem 10.1], where it is shown that the geometric type of W is uniquely determined. Then $M^3 = \Gamma \setminus Nil^3$ with Γ as claimed in (f).

(v) The tori give class (d) in Theorem 4.1.

(vi) The minimal properly elliptic surfaces are surfaces with $\kappa = 1$. Topologically these are Seifert T^2-bundles over base orbifolds of negative orbifold characteristic. Again Wall shows that the elliptic fibration is unique and that the possible geometric types are $H \times \mathbb{C}$ and $\widetilde{SL}_2 \times E^1$. Lemma 4.3 shows that there are no singular fibres. Furthermore, the base orbifolds clearly admit no non-trivial S^1-action and so the circle action on the surface has to go along the fibres. Now argue as in (iv) to obtain class (b).

This completes the proof of Theorem 4.1.

5 Proof of Theorem 1.2 – Part II

In this section we use the notation of Section 3, and we follow the outline (except for Section 5.5) given in Section 3.3.

In Section 5.3 we describe in detail the complex structure on $\widetilde{SL}_2 \times E^1$ and its canonical holomorphic symplectic form $\widetilde{\Omega}_H$; in Section 5.4 we describe the complex structure on $\widetilde{E}_2 \times E^1$ and its canonical holomorphic symplectic form $\widetilde{\Omega}_E$.

The main result we are going to prove is the following.

Theorem 5.1 *A quadruple $(\widetilde{S}, \Gamma, \widetilde{X}_c, \widetilde{\Omega})$ is associated to a taut contact circle on a closed 3-manifold M^3, as described in Section 3.3, if and only if it is biholomorphic to a quadruple in any of the following four classes:*

(a') $\widetilde{S} = \mathbb{C}^2 - \{(0,0)\}$, $\widetilde{\Omega} = (constant) \cdot dz_1 \wedge dz_2$, *$\Gamma$ is any finite subgroup of $SU(2)$, $\widetilde{X}$ is given by Proposition 5.5 below. Then* $S(t_0) = (\langle \varphi_{t_0} \rangle \times \Gamma) \setminus (\mathbb{C}^2 - \{(0,0)\})$ *is a Hopf surface, and* $M^3 \cong \Gamma \setminus S^3$.

(b') $\widetilde{S} = \widetilde{SL}_2 \times E^1$, $\widetilde{\Omega} = (constant) \cdot \widetilde{\Omega}_H$, *$\varphi_t$ is translation by $-t$ in the E^1-direction, and Γ is any discrete, compact subgroup of $\widetilde{SL}_2$ acting by left multiplication. Then* $S(t_0) = (\Gamma \times \langle t_0 \rangle) \setminus (\widetilde{SL}_2 \times E^1)$ *is a properly elliptic surface and* $M^3 \cong \Gamma \setminus \widetilde{SL}_2$.

(c') $\widetilde{S} = \widetilde{E}_2 \times E^1$, $\widetilde{\Omega} = (constant) \cdot \widetilde{\Omega}_E$, *$\varphi_t$ is translation by $-t$ in the E^1-direction, and Γ is any discrete, cocompact subgroup of $\widetilde{E}_2$, acting by left multiplication, with not all elements of Γ translations. Then* $S(t_0) = (\Gamma \times \langle t_0 \rangle) \setminus (\widetilde{E}_2 \times E^1)$ *is a hyperelliptic surface with Euler class* $(0,0)$, *and* $M^3 \cong \Gamma \setminus \widetilde{E}_2$ *is a T^2-bundle over S^1 with non-trivial periodic monodromy.*

(c'') $\widetilde{S}, \widetilde{\Omega}, \varphi_t$ *are as in (c'), and Γ is any lattice of rank 3 in the translation part* $\mathbb{C} \times (2\pi i\mathbb{Z})$ *of $\widetilde{E}_2$. Then* $S(t_0) = (\Gamma + \{0\} \times (t_0\mathbb{Z})) \setminus (\widetilde{E}_2 \times E^1)$ *is a complex torus, and* $M^3 \cong \Gamma \setminus \widetilde{E}_2$ *is diffeomorphic to T^3, the T^2-bundle over S^1 with trivial monodromy.*

Notice that Theorem 1.2 is contained in Theorem 5.1.

The other important result is the complete classification up to biholomorphism of the surfaces $S(t_0)$ in class (a') of the above theorem, and of the corresponding pairs $(S(t_0), X_c)$. This is done in Section 5.2.

We now explain how the present section is organized. In Section 5.1 we use elementary topological arguments to show that ruled surfaces, properly elliptic surfaces of type $H \times \mathbb{C}$, and Kodaira surfaces can never be a surface $S(t_0)$ constructed from a taut contact circle. That is, we reduce the list of complex surfaces in Theorem 4.1 to Hopf surfaces, hyperelliptic surfaces, properly elliptic surfaces of type $\widetilde{SL}_2 \times E^1$, and complex tori. In Section 5.2 we find which Hopf surfaces can be constructed as $S(t_0)$, similarly for properly elliptic ones in Section 5.3, likewise for hyperelliptic ones and complex tori in Section 5.4. Except for certain Hopf surfaces and complex tori, all the surfaces determined in Section 5.1 are elliptic, and the arguments to reduce the diffeomorphism classification of these surfaces to a diffeomorphism classification of the corresponding 3-manifolds are already contained in Section 4, where we were able to describe the properties of the circle action even without knowing any details about the biholomorphism classification of these surfaces.

As for complex tori, they give $M^3 \cong T^3$ by a very short argument given at the beginning of Section 5.2 below.

Therefore, except for the non-elliptic Hopf surfaces, it would be enough to prove Theorem 5.1 up to diffeomorphism rather than biholomorphism. We give an outline of the topological arguments of such a proof in Section 5.5.

However, for the homothety classification of taut contact circles and to understand which 3-manifolds correspond to non-elliptic Hopf surfaces, we need to prove Theorem 5.1 up to biholomorphism. The Hopf case is by far the most difficult, due to the fact that not all Hopf surfaces are geometric. By comparison, the result in the other cases is a fairly straightforward consequence of the invariance conditions imposed by the geometry.

Notation. The following conventions will be used in Sections 5, 6, and 7. When dealing with Hopf surfaces, the coordinates on $\mathbb{C}^2$ will be denoted (z_1, z_2). When dealing with properly elliptic surfaces, the upper half plane in $\mathbb{C}$ will be denoted H and the coordinates in $H \times \mathbb{C}$ will be denoted $(z, \widetilde{w})$.The corresponding point $(z, e^{\widetilde{w}})$ in $H \times (\mathbb{C} - \{0\})$ will be denoted (z, w). Finally, in the context of hyperelliptic surfaces, the coordinates on $\mathbb{C} \times \mathbb{C}$ will be denoted (z, w).

5.1 Reduction of the problem

At several places in this section we use the fact, from Proposition 3.4, that the Reeb vector fields ξ_1, ξ_2 span a complex line complementary to the complex line which contains X.

First we show that $M^3 \times S^1$ cannot be a ruled surface. Suppose that it were, and let C be a holomorphic sphere from a ruling. By Lemma 4.3 the vector field X is either tangent to C or transverse to C. The former possibility is ruled out by the hairy ball theorem, hence X must be transverse to C. The same is true for all constant linear combinations of X and JX, because C is a holomorphic sphere. Since the Reeb vector fields ξ_1, ξ_2 determine a complementary complex line distribution, they give a parallelization of C, which is absurd.

This shows that the complex surfaces in class (g) of Theorem 4.1 do not arise from a 3-manifold admitting a taut contact circle.

Next we show that from the construction in Theorem 3.6 (or Section 3.3) no surface of geometric type $H \times \mathbb{C}$ can arise. This is one of the two possibilities in case (b) of Theorem 4.1.

The complex surfaces of geometric type $H \times \mathbb{C}$ are elliptic fibrations over an orbifold of negative orbifold characteristic. Such orbifolds do not admit a non-trivial S^1-action, so from Lemma 4.3 we conclude (assuming that the complex surface arose from the construction in Theorem 3.6) that X is tangent to the fibres, hence so is JX. Then ξ_1 and ξ_2 span a distribution transverse to the fibres. Topologically, the surfaces of geometric type $H \times \mathbb{C}$ are Seifert T^2-bundles with zero Euler class. This means that we can find a section, that is, an immersed surface (of negative Euler characteristic) transverse to all fibres. The Reeb vector fields would provide a parallelization of such a surface, which is impossible.

The third class of complex surfaces we want to consider in this section are the Kodaira surfaces (classes (e) and (f) in Theorem 4.1).

Let S be a primary Kodaira surface. Topologically, this surface fibres in non-isomorphic ways as a T^2-bundle over T^2 (see [22]), but the elliptic structure is unique [27, Lemma 7.2] and is as described in Theorem 4.1.

Suppose that S arose from the construction in Theorem 3.6. As shown in Lemma 4.3, the vector field X sends fibres to fibres, so it can induce either the trivial action or a free S^1-action on the base torus. In the latter case, the M^3 quotient would be a T^2-bundle over S^1 with non-trivial monodromy, and S would fibre holomorphically as a T^2-bundle with non-trivial monodromy and zero Euler class, which is impossible. Hence, X is everywhere tangent to the fibres of the elliptic fibration.

Consider a fibre $F_0 = S^1 \times S^1$, where we may assume that X is tangent to the first S^1-factor. We see that F_0 is transverse to M^3, so a priori F_0 intersects M^3 in a finite union of circles. However, since any orbit of X has to intersect each of these circles, and it intersects M^3 only once, we conclude that $F_0 \cap M^3 = S^1$. Since $Y = JX - fX$ is tangent both to F_0 and M^3, this intersection circle is in fact an orbit of Y. So the common kernel of ω_1 and ω_2 has closed orbits. Moreover, we see that it is these orbits that make M^3 into an S^1-bundle over T^2 (with non-zero Euler number e). Note that the Reeb vector fields are everywhere transverse to this fibration.

Let D be a 2-disc in the base torus T^2. Fix a section σ of the S^1-bundle over $T^2 - D$, and identify $\sigma(T^2 - D)$ with $T^2 - D$. Projecting ξ_1 along the fibres onto $T^2 - D$, we obtain a nowhere zero vector field Z on $T^2 - D$. Along the boundary of $T^2 - D$, we may view Z as a vector field tangent to D, and this vector field has rotation number zero with respect to the centre of D since T^2 has Euler characteristic zero.

Over D we have a section σ_1, and we write $D_1 = \sigma_1(D)$. In identifying the boundary tori of the trivial bundles $D_1 \times S^1$ and $(T^2 - D) \times S^1$, the boundary ∂D is identified (homologically) with $d_1 - e\tilde{c}$, where d_1 is the class of ∂D_1 and $\tilde{c}$ the class of the circle fibre in $\partial D_1 \times S^1$.

Clearly the Reeb vector field ξ_1 (regarded as a vector field tangent to D_1) has rotation number zero along ∂D_1 with respect to the centre of D_1, since it extends as a nowhere zero vector field over D_1.

As we move once around a fibre over a point in ∂D_1, the Reeb vector field makes m full turns with $m \neq 0$ (and m locally constant on ∂D_1, hence constant), since the fibre is an integral curve of the contact distribution $\ker \omega_1$, which forces the contact plane to keep rotating with positive angular velocity along this curve (because of the non-integrability of the contact distribution), and this also forces the rotation of ξ_1. This implies that Z has in fact rotation number $-me$ along ∂D with respect to the centre of D, a contradiction if $e \neq 0$.

We note this as a separate result (see also [9] for related statements and a more detailed account of the preceding argument).

Proposition 5.2 *Let M^3 be a non-trivial S^1-bundle over T^2. Then M^3 does not admit a contact form whose Reeb vector field is everywhere transverse to the fibration.*

Finally, we consider secondary Kodaira surfaces. We have seen in the proof of Theorem 4.1 that the S^1-action has to be along the fibres, so we would obtain a Seifert bundle M^3 with non-trivial Euler class over a good (in the sense of [23]) euclidean orbifold (a sphere with three or four cone points), and with Reeb vector fields transverse to the Seifert fibration. M^3 is finitely covered by an S^1-bundle $\widetilde{M}^3$ over T^2 with non-zero Euler class, and the contact forms on M^3 would lift to contact forms on $\widetilde{M}^3$ whose Reeb vector fields are transverse to the fibration, which is impossible by the proposition above.

5.2 Hopf surfaces

Before dealing with Hopf surfaces, we consider the case where $S(t_0) = M^3 \times S^1$ is a complex torus. Here $\widetilde{S} = \mathbb{C}^2$ and $\widetilde{X}_c = (1/2)(\widetilde{X} - iJ\widetilde{X})$ is holomorphic on $\mathbb{C}^2$ with fourfold periodic coefficient functions, hence constant. So X is a constant slope vector field, and the quotient of $S(t_0)$ under the circle action generated by X is necessarily a 3-torus.

This simple case contains the basic idea necessary to deal with the Hopf surfaces: If $S(t_0)$ is a Hopf surface then $\widetilde{X}_c$ is holomorphic on $\mathbb{C}^2 - \{0\}$ and extends to $\mathbb{C}^2$. Then $\widetilde{X}$ also extends because it is $2\mathrm{Re}(\widetilde{X}_c)$, and we shall use the flow structure of $\widetilde{X}$ to determine the Hopf surface and the 3-manifold.

A Hopf surface is a compact quotient of $\mathbb{C}^2 - \{0\}$ under a discrete group G of automorphisms. By Hartog's theorem, any automorphism of $\mathbb{C}^2 - \{0\}$

extends to a self-mapping of $\mathbb{C}^2$. This extension is also an automorphism. Thus we can view G as a group of automorphisms of $\mathbb{C}^2$, such that the elements of G not equal to the identity fix only the origin.

A list of the possible groups G can be found in [16]. Call a map $T : \mathbb{C}^2 \to \mathbb{C}^2$ a *contraction* if for any $(z_1,z_2) \in \mathbb{C}^2$ the sequence $T^\nu(z_1,z_2)$ converges to $(0,0)$ as ν goes to ∞, and this convergence is uniform in (z_1,z_2) ranging over compact sets. Then G is always a semidirect product $\langle T\rangle\Gamma$ of the infinite cyclic group $\langle T\rangle$ generated by a contraction T and a finite subgroup $\Gamma \subset U(2)$ which acts freely on S^3. The possible factorizations of G with an infinite cyclic factor are $\langle T\gamma\rangle\Gamma$ for any $\gamma \in \Gamma$, where the generator $T\gamma$ is always a contraction and where the factor Γ cannot be changed because it is the torsion part of G.

There are non-cyclic abelian subgroups of $U(2)$, such as $\{\pm 1\} \times \{\pm 1\}$, but if an abelian subgroup of $U(2)$ acts freely on S^3, then it has to be cyclic and of a special form. First of all, we can diagonalize the elements of Γ simultaneously, so Γ is conjugate to a subgroup of $U(1) \times U(1)$. Secondly, assuming without loss of generality that $\Gamma \subset U(1) \times U(1)$, the condition that Γ acts freely on S^3 implies that it acts freely on $S^1 \times \{0\}$ and on $\{0\} \times S^1$, and so the projections of $U(1) \times U(1)$ onto its factors inject Γ into $U(1)$. Thus Γ must be the cyclic subgroup of $U(1) \times U(1)$ generated by (e_1, e_2) with e_1 and e_2 primitive m^{th} roots of 1, where m is the order of Γ. Conversely, the cyclic group generated by any such pair of primitive roots of the same order acts freely on S^3.

If Γ is contained in $SU(2)$ then it automatically acts freely on S^3, because $SU(2)$ is the same as S^3 acting on itself by left multiplications. For each $m \geqq 1$, there is a unique cyclic subgroup of order m in $S(U(1) \times U(1))$, namely the group of all pairs $(\varepsilon, \varepsilon^{-1})$ where ε ranges over the m^{th} roots of 1. We shall denote this group by Γ_m.

For any factorization $G = \langle T\rangle\Gamma$, the holomorphic coordinates (z_1,z_2) (with domain and range all of $\mathbb{C}^2$) can be chosen so that with respect to them $\Gamma \subset U(2)$ and T is of one of the two following types.

Type (1): $T(z_1,z_2) = (\alpha z_1, \beta z_2)$, $0 < |\alpha| < 1$, $0 < |\beta| < 1$.

Type (2): $T(z_1,z_2) = (\alpha^n z_1 + \lambda z_2^n, \alpha z_2)$, $0 < |\alpha| < 1$, $\lambda \neq 0$.

It is also proved in [16] that if ψ is a holomorphic self-mapping of $\mathbb{C}^2$ which commutes with the contraction T, then ψ is as follows.

If T is of type (1), then ψ has to be linear.

If T is of type (2), then $\psi(z_1,z_2) = (\tilde{\alpha}^n z_1 + \tilde{\lambda} z_2^n, \tilde{\alpha} z_2)$.

Moreover, in type (2), ψ has finite order if and only if $\tilde{\lambda} = 0$ and $\tilde{\alpha}$ is a root of 1.

Suppose now that $S(t_0) \cong M^3 \times S^1$ is a Hopf surface. From the description given above of all possible factorizations of G with an infinite cyclic factor, it follows that the contraction T can be chosen so that $T = \varphi_{\pm t_0}$, for a suitable choice of sign, and then $G = \langle T\rangle \times \Gamma = \langle \varphi_{t_0}\rangle \times \Gamma$ is a direct product.

Moreover, the torsion group Γ equals the group $\pi_1(M^3)$ acting on $\widetilde{S}$ by the monodromy representation, and so $\pi_1(M^3)$ is finite and the universal cover $\widetilde{M}^3$ of M^3 is compact. Now every element of Γ commutes with T, and we apply the above given facts to find extra restrictions on T and Γ.

If T is of type (1) then Γ can be non-abelian only if $\alpha = \beta$, otherwise it is either trivial or cyclic as described above. If T is of type (2) then Γ has to be either trivial or the cyclic subgroup of $U(1) \times U(1)$ generated by (e_1^n, e_1) where e_1 is a primitive m^{th} root of 1 and $(n,m) = 1$.

Remark. Surfaces of type (1) are elliptic if and only if $\alpha^{n'} = \beta^{n''}$ for some positive integers n' and n'', and they are geometric if and only if $|\alpha| = |\beta|$. Surfaces of type (2) are neither elliptic nor geometric.

We give next an equivalence result for Hopf surfaces. This is valid for all Hopf surfaces, regardless of their being a surface $S(t_0)$ or not. It will be essential later for the distinction between homothety classes.

Lemma 5.3 *A Hopf surface of type* (1) *is never biholomorphic to a Hopf surface of type* (2)*. If a Hopf surface is of type* (1)*, then the unordered pair* $\{|\alpha|, |\beta|\}$ *is determined by the complex structure. If a Hopf surface is of type* (2)*, then the complex structure determines* $|\alpha|$ *and the integer* n.

Proof. Here we use the same method as the one employed by Kodaira-Spencer in the proof of Theorem 15.1 in [18]. Suppose there is a biholomorphism between the quotients

$$W = (\langle T\rangle\Gamma) \setminus (\mathbb{C}^2 - \{0\}) \quad \text{and} \quad W' = (\langle T'\rangle\Gamma') \setminus (\mathbb{C}^2 - \{0\}).$$

There is a lifting of this map to an automorphism Φ of $\mathbb{C}^2 - \{0\}$ which conjugates the group $\langle T\rangle\Gamma$ to the group $\langle T'\rangle\Gamma'$. (Conversely, if such a conjugating automorphism exists, then W is biholomorphic to W'.) This Φ extends to an automorphism of $\mathbb{C}^2$. Let Φ_0, T_0, T_0' denote the differentials at the origin $(0,0)$ of Φ, T, T', respectively. These are elements of $GL(2,\mathbb{C})$. We have

$$\Phi \circ T \circ \Phi^{-1} = (T')^{\pm 1}\gamma' \quad \text{for some} \quad \gamma' \in \Gamma'.$$

The left-hand side is a contraction, so the exponent on the right-hand side is $+1$. Taking differentials at the origin, we get

$$\Phi_0 \circ T_0 \circ \Phi_0^{-1} = T_0'\gamma'.$$

Suppose first that T is of type (2) and T' is of type (1), and let us derive a contradiction. Since T' and γ' are diagonalizable and commute, their product $T'\gamma'$ is diagonalizable and so T would be biholomorphically conjugate to a diagonalizable linear map. In particular, there would be two transverse complex curves through the origin invariant under T. If $n = 1$, then T_0 has the following Jordan canonical form

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

and it cannot be linearly conjugate to $T_0'\gamma' = T'\gamma'$.

If $n \geqq 2$, then T_0 is equal to

$$\begin{pmatrix} \alpha^n & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{with} \quad |\alpha^n| < |\alpha|,$$

and its only invariant lines are $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$. Therefore, if T has two transverse invariant curves through the origin, then these curves must be tangent to $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$ respectively. We see that $\mathbb{C} \times \{0\}$ is indeed T-invariant. A complex curve through the origin tangent to $\{0\} \times \mathbb{C}$ is locally a graph $z_1 = f(z_2)$ for some convergent power series $f(z_2)$. The image of this graph under T is the graph

$$z_1 = \alpha^n f\left(\frac{z_2}{\alpha}\right) + \lambda \left(\frac{z_2}{\alpha}\right)^n,$$

and it is trivial to see that it has at best a contact of order $n-1$ with $z_1 = f(z_2)$ (meaning that they differ by a non-zero multiple of z_2^n). Hence, a contraction of type (2) cannot be biholomorphically conjugate to any automorphism with two invariant tranverse complex curves through the origin. Thus T is not biholomorphically conjugate to $T'\gamma'$.

As the order of contact is a diffeomorphism invariant, we also conclude that the integer n is associated with T in a way that is invariant under biholomorphic conjugation. The same integer n is associated with $T\gamma$ for any $\gamma \in \Gamma$, and so the complex structure of W determines n.

Suppose now that T and T' are both of type (1), in which case $T = T_0$ and $T' = T_0'$. Then $\Phi_0 \circ T_0 \circ \Phi_0^{-1} = T_0'\gamma'$ implies that T and $T'\gamma'$ have the same eigenvalues. The eigenvalues of T are α and β, and those of $T'\gamma'$ are $\alpha'\varepsilon_1$ and $\beta'\varepsilon_2$ where $\varepsilon_1, \varepsilon_2$ are roots of 1. Hence either $|\alpha| = |\alpha'|$ and $|\beta| = |\beta'|$ or $|\alpha| = |\beta'|$ and $|\beta| = |\alpha'|$.

Suppose finally that T and T' are both of type (2), with the same integer n. Then $\Phi_0 \circ T_0 \circ \Phi_0^{-1} = T_0'\gamma'$ implies that the eigenvalues α^n, α of T_0 equal the eigenvalues $\alpha'^n e_1^{nk}, \alpha' e_1^k$ of $T_0'\gamma'$. We conclude $\alpha = \alpha' e_1^k$. A fortiori, we must have $|\alpha| = |\alpha'|$.

The lemma is now proved.

We now consider a Hopf surface obtained as $S(t_0)$ from a taut contact circle on M^3. Then, as explained above, G is the direct product $\langle \varphi_{t_0} \rangle \times \Gamma$, and $T = \varphi_{\pm t_0}$ is a contraction for a suitable choice of sign. We are now going to determine this sign and the form $\widetilde{\Omega}$.

The holomorphic 2-form $\widetilde{\Omega}$ is clearly invariant under $\pi_1(M^3) = \Gamma$. We can extend $\widetilde{\Omega}$ to the origin as a holomorphic 2-form, and then write

$$\widetilde{\Omega} = h(z_1, z_2)\, dz_1 \wedge dz_2,$$

where $h(z_1, z_2)$ is an entire function. We have $h(0,0) \neq 0$, for if $h(0,0) = 0$, then by the Weierstraß preparation theorem we would get zeros of $\widetilde{\Omega}$ in $\mathbb{C}^2 - \{0\}$.

We have $\varphi_t^* \widetilde{\Omega} = e^t \widetilde{\Omega}$ for all t, and the real part of $\widetilde{\Omega}$ is a symplectic form on $\mathbb{C}^2$. Therefore φ_t can be a contraction only for negative t. We conclude that $\varphi_{-t_0} = T$, and we have

$$T^* \widetilde{\Omega} = \varphi_{-t_0}^* \widetilde{\Omega} = e^{-t_0} \widetilde{\Omega}.$$

We can write $T(z_1, z_2) = (\alpha z_1 + \lambda z_2^n, \beta z_2)$, for both types (1) and (2), by allowing $\lambda = 0$ and imposing the condition $\lambda(\alpha - \beta^n) = 0$. Then the relation $T^* \widetilde{\Omega} = e^{-t_0} \widetilde{\Omega}$ translates to

$$\alpha\beta \cdot (h \circ T) = e^{-t_0} h$$

or

$$h \circ T = \frac{e^{-t_0}}{\alpha\beta} h.$$

Therefore, for all $(z_1, z_2) \in \mathbb{C}^2$,

$$\lim_{\nu \to \infty} \left(\frac{e^{-t_0}}{\alpha\beta} \right)^\nu h(z_1, z_2) = \lim_{\nu \to \infty} h(T^\nu(z_1, z_2)) = h(0,0) \neq 0,$$

and this convergence is possible only if $\alpha\beta = e^{-t_0}$. But then $h(z_1, z_2) = h(0,0)$ for all (z_1, z_2), and h is constant. Thus

$$\widetilde{\Omega} = c_0 dz_1 \wedge dz_2, \; c_0 \in \mathbb{C}.$$

Now the Γ-invariance of $\widetilde{\Omega}$ means that $\Gamma \subset SU(2)$. This guarantees that Γ acts freely on S^3, because $SU(2)$ is the same as S^3 acting on itself from the left.

If Γ is abelian (hence cyclic, as was observed above), then it has to be equal to Γ_m for some m.

If T is of type (2), then $\Gamma = \Gamma_m$ is generated by (e_1^n, e_1) where e_1 is a primitive m^{th} root of 1, and so $n+1$ must be a multiple of m. Equivalently, n is of the form $n = mq - 1$ for some positive integer q.

Now we can state a corollary of Lemma 5.3 for surfaces $S(t_0)$ which are Hopf surfaces.

Corollary 5.4 *The positive number t_0 is determined by the complex structure of the Hopf surface $S(t_0)$.*

Proof. We have just shown that $\alpha\beta = e^{-t_0}$. Then $e^{-t_0} = |\alpha||\beta|$, which for T of type (2) can be rewritten as $e^{-t_0} = |\alpha|^{n+1}$. Thus, by Lemma 5.3, the quantity e^{-t_0} is determined by the complex structure, and so is t_0.

We are now going to determine the vector field $\widetilde{X}_c$ by using the relations

$$\varphi_{-t_0} = T \quad \text{and} \quad L_{\widetilde{X}_c} \widetilde{\Omega} = \widetilde{\Omega},$$

together with the fact that each map φ_t commutes with each element of G.

If T is of type (1), then the φ_t must be linear, that is, there exists a constant matrix A_0 such that

$$\varphi_t = \exp(tA_0) \text{ for all } t.$$

Since $\varphi_{-t_0} = T$ is a linear diagonal map, the Jordan canonical form of A_0 has to be diagonal.

If Γ is abelian, then A_0 and all elements of G are simultaneously diagonalizable, and we can choose the linear coordinates (z_1, z_2) such as to keep the diagonal form of the elements of G and giving φ_t the following expression:

$$\varphi_t(z_1, z_2) = (e^{at} z_1, e^{bt} z_2),$$

where a and b are constants. It follows that

$$\widetilde{X}_c = az_1 \partial_{z_1} + bz_2 \partial_{z_2}.$$

If Γ is non-abelian, then A_0 has to be a scalar multiple of the identity, for otherwise the fact that each element of Γ commutes with φ_t would allow us to diagonalize all elements of Γ simultaneously with A_0, and Γ would be abelian. So for non-abelian Γ we get $a = b$ in the above formula for $\widetilde{X}_c$.

The relation $\widetilde{\Omega} = L_{\widetilde{X}_c} \widetilde{\Omega}$ is equivalent to $a + b = 1$. Also, since φ_{-t_0} is a contraction, the numbers $\mathrm{Re}(a)$ and $\mathrm{Re}(b)$ must be positive. Thus for T of type (1) we have

$$\widetilde{X}_c = az_1 \partial_{z_1} + (1 - a) z_2 \partial_{z_2} \quad \text{with} \quad 0 < \mathrm{Re}(a) < 1,$$

and if Γ is non-abelian, then $a = 1/2$.

If T is of type (2), then φ_t is a one-parameter group of mappings of the form $(\widetilde{\alpha}^n z_1 + \widetilde{\beta} z_2^n, \widetilde{\alpha} z_2)$. It follows that there are constants a and b_0 such that

$$\varphi_t(z_1, z_2) = (e^{nat} z_1 + b_0 t e^{nat} z_2^n, e^{at} z_2).$$

Thus

$$\widetilde{X}_c = (naz_1 + b_0 z_2^n) \partial_{z_1} + az_2 \partial_{z_2}$$

and the condition $L_{\widetilde{X}_c} \widetilde{\Omega} = \widetilde{\Omega}$ gives $a = 1/(n+1)$.

The relation $\varphi_{-t_0} = T$ implies $b_0 \neq 0$, for otherwise T would be of type (1).

We are now ready to state the biholomorphism classification of $(\Gamma, \widetilde{X}_c, \widetilde{\Omega})$ for surfaces $S(t_0)$ which are Hopf surfaces. We also have the biholomorphism classification of the surface $S(t_0)$ itself, and of the pair $(S(t_0), X_c)$.

Proposition 5.5 *If $S(t_0) = G \setminus (\mathbb{C}^2 - \{(0,0)\})$ is a Hopf surface constructed from a taut contact circle (ω_1, ω_2) on M^3, then, in suitable coordinates, $\widetilde{\Omega}$ equals $c_0 \cdot dz_1 \wedge dz_2$, for some constant c_0, and G equals the direct product*

$\langle \varphi_{-t_0} \rangle \times \Gamma$, *where* $t_0 > 0$, φ_t *is the flow of* $\widetilde{X} = 2\mathrm{Re}\,\widetilde{X}_c$, *and the possibilities for* Γ *and* $\widetilde{X}_c$ *are those given below.*

(1′) Γ *is a non-abelian subgroup of* $SU(2)$ *and*

$$\widetilde{X}_c = \frac{1}{2}(z_1\partial_{z_1} + z_2\partial_{z_2}).$$

(1″) Γ *is the cyclic subgroup* Γ_m *of* $S(U(1) \times U(1))$, *and*

$$\widetilde{X}_c = az_1\partial_{z_1} + (1-a)z_2\partial_{z_2} \quad \text{with} \quad 0 < Re(a) < 1.$$

(2) Γ *is* Γ_m *and*

$$\widetilde{X}_c = \left(\frac{n}{n+1}z_1 + b_0 z_2^n\right)\partial_{z_1} + \frac{1}{n+1}z_2\partial_{z_2},$$

where $b_0 \neq 0$ *and* $n = mq - 1$ *for some* $q \in \mathbb{N}$.

All these Hopf surfaces actually arise in such a construction.

In cases (1″) *and* (2), *the integer* $m \in \mathbb{N}$ *is determined by the homotopy type of the surface.*

For two such surfaces to be biholomorphic it is necessary that they be of the same type, (1′), (1″), *or* (2).

Two surfaces of type (1′) *are biholomorphic if and only if they have the same* Γ, *up to conjugation in* $SU(2)$, *and the positive number* t_0 *is the same for both. The number* t_0 *and the homotopy type of the surface determine together the complex structure, which in turn determines the vector field* X_c.

Two surfaces of type (1″) *are biholomorphic if and only if they have the same value for the pair* (m, t_0) *and the respective values for* a *are related by an element of the group generated by the rotation* $R : a \mapsto 1 - a$ *and the translation* $a \mapsto a + (2\pi i/mt_0)$. *Thus the moduli space of surfaces of type* (1″), *for fixed* (m, t_0), *is the quotient orbifold* Q_2 *of the slab* $\{0 < \mathrm{Re}(a) < 1\}$ *under this group of two generators. The vector field* X_c *determines the unordered pair* $\{a, 1-a\}$, *hence the moduli space of pairs* (S, X_c) *of type* (1″), *for fixed* (m, t_0), *is the quotient orbifold* Q_1 *of the slab* $\{0 < Re(a) < 1\}$ *under the rotation R. The canonical map* $Q_1 \to Q_2$ *has infinitely many sheets, and so every surface of type* (1″) *admits an infinite sequence of non-isomorphic vector fields* X_c.

Two surfaces of type (2) *are biholomorphic if and only if they have the same value for the triple* (m, q, t_0). *The vector field* X_c *is determined by the complex structure.*

Remarks. (1) The orbifold Q_i, $i = 1, 2$, is topologically a disk with i cone points of multiplicity 2.

(2) The surface is geometric if and only if it is of type (1′) or of type (1″) with $\mathrm{Re}(a) = 1/2$, independently of t_0. The surface is elliptic if and only if it is of type (1′) or of type (1″) with $a \in \mathbb{Q} + (2\pi i/t_0)\mathbb{Q}$.

Proof of Proposition 5.5. A good part of the proposition has already been proved above. What remains to be proved is the statements about moduli and the fact that all such Hopf surfaces do indeed arise from taut contact circles.

We first treat type (2), because this case is simpler.

The linear automorphism of $\mathbb{C}^2$ whose matrix is

$$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \quad c \neq 0,$$

conjugates every element of $U(1) \times U(1)$ to itself, and pushes forward the vector field

$$\left(\frac{n}{n+1} z_1 + b_0 z_2^n\right) \partial_{z_1} + \frac{1}{n+1} z_2 \partial_{z_2}$$

to the vector field

$$\left(\frac{n}{n+1} z_1 + (cb_0) z_2^n\right) \partial_{z_1} + \frac{1}{n+1} z_2 \partial_{z_2}.$$

We conclude that all pairs $(S(t_0), X_c)$ of type (2), with the same values for m, n, and t_0, are isomorphic to one another. We can restrict our attention to the value $b_0 = 1/(n+1)$ and so

$$\widetilde{X}_c = \frac{1}{n+1}((nz_1 + z_2^n)\partial_{z_1} + z_2 \partial_{z_2}),$$

but sometimes it will be convenient to let b_0 take on arbitrary values.

It is easy to see (cf. the proof of Lemma 5.3) that two pairs $(S(t_0), X_c)$ and $(S'(t_0'), X_c')$ of type $(1')$ are isomorphic if and only if $t_0 = t_0'$ and the corresponding torsion groups Γ and Γ' are conjugate in $GL(2, \mathbb{C})$. The list of such finite subgroups of $SU(2)$ is well-known (cf. [28, 20]), and it is the same list up to isomorphism or up to conjugation. Therefore the complex structure is determined in this case by the homotopy type and the number t_0.

Let now $(S(t_0), X_c)$ and $(S'(t_0), X_c')$ be pairs of type $(1'')$, with the same values for m and for t_0. Let a and a' denote the parameter values for $\widetilde{X}_c$ and $\widetilde{X}_c'$, respectively.

Again using the ideas in the proof of Lemma 5.3, we see that the surface $S(t_0)$ is biholomorphic to the surface $S'(t_0)$ if and only if there is an m^{th} root ε of 1 such that the matrices

$$\begin{pmatrix} e^{t_0 a} & 0 \\ 0 & e^{t_0(1-a)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon e^{t_0 a'} & 0 \\ 0 & \varepsilon^{-1} e^{t_0(1-a')} \end{pmatrix}$$

have the same eigenvalues. This condition is the same as

$$\text{either} \qquad \varepsilon e^{t_0 a'} = e^{t_0 a} \qquad \text{or} \qquad \varepsilon e^{t_0 a'} = e^{t_0(1-a)}.$$

The space of biholomorphism classes of surfaces of type $(1'')$, for fixed t_0 and m, is thus the quotient Q_2 of the slab $0 < \text{Re}(a) < 1$ under the group generated by the rotation R and the translation $a \mapsto a + (2\pi i/mt_0)$.

The pair $(S(t_0),X_c)$ is isomorphic to the pair $(S'(t_0),X_c')$ if and only if the unordered pair of eigenvalues at the origin is the same for both, that is, $a = a'$ or $a = 1 - a'$. The space of isomorphism classes of pairs $(S(t_0),X_c)$ of type $(1'')$, with t_0 and m fixed, is the quotient of $0 < \mathrm{Re}(a) < 1$ under the rotation R. In particular, every surface $S(t_0)$ of type $(1'')$ admits an infinite sequence of non-isomorphic vector fields X_c, because it determines a sequence of parameter values

$$a + 2\pi ik/mt_0, \quad k \text{ an integer},$$

which are pairwise non-equivalent under the rotation R.

Remark. For any Hopf surface $S(t_0)$ constructed from a taut contact circle, the vector field X_c determines t_0 as the common period of all the orbits of $2\,\mathrm{Re}(X_c)$. It should be noted that for surfaces of type $(1'')$ the complex structure does not determine X_c, but it does determine t_0.

It remains to show that all the Hopf surfaces in the proposition are actually surfaces $S(t_0)$ for some taut contact circle. Here we use Corollary 3.12 and the method indicated in Section 3.3 for finding the 3-manifold M^3.

The sphere $S^3 = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ is a $U(2)$-invariant transversal for all vector fields $\widetilde{X}$ of types $(1')$ and $(1'')$, and also for those of type (2), provided b_0 is sufficiently small (e.g. $b_0 = 1/(n+1)$).

Since $\widetilde{X}_c$ and $\widetilde{\Omega} = dz_1 \wedge dz_2$ are Γ-invariant, they descend to the surface $S = \Gamma \setminus (\mathbb{C}^2 - \{(0,0)\})$. Thus on S we have X_c and Ω satisfying condition (2) of Corollary 3.12. Since S^3 disconnects $\mathbb{C}^2 - \{(0,0)\}$, we conclude that all orbits of $\widetilde{X}$ in $\mathbb{C}^2 - \{(0,0)\}$ are open, and that (S,X_c) satisfies condition (1) of Corollary 3.12 with M^3 the orbit space $\Gamma \setminus S^3$ of X in S. By this corollary, we have that $(\langle\varphi_{t_0}\rangle \times \Gamma) \setminus (\mathbb{C}^2 - \{(0,0)\}) = \langle\varphi_{t_0}\rangle \setminus S$ arises from a taut contact circle on $\Gamma \setminus S^3$.

The proof of Proposition 5.5 is now complete

At this point we know that a Hopf surface is a surface $S(t_0)$ if and only if it is in class (a') of Theorem 5.1, and that the corresponding 3-manifolds are the left-quotients of the group $SU(2) = S^3$. Also, we have classified these Hopf surfaces and the corresponding pairs $(S(t_0),X_c)$ up to biholomorphism.

5.3 Properly elliptic surfaces

We first give a brief description of the geometry $\widetilde{SL_2} \times E^1$ (following [23] and [27]) and then go on to study the triples $(\Gamma,\widetilde{X}_c,\widetilde{\Omega})$ for surfaces $S(t_0)$ which are properly elliptic of geometric type $\widetilde{SL_2} \times E^1$.

Let H denote the upper half plane in $\mathbb{C}$, with the usual hyperbolic metric. The identity component of the isometry group of H is the group $PSL_2 = PSL_2\mathbb{R}$. If z is the usual coordinate on $\mathbb{C}$, then an element $\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in PSL_2$

acts on H by

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}(z) = \frac{rz+s}{tz+u},$$

and the isotropy group of this action is S^1.

One can identify PSL_2 with the unit tangent bundle STH by choosing any vector $v_0 \in STH$ at some fixed point of H and forming the map

$$PSL_2 \xrightarrow{\cong} STH \qquad A \longmapsto A_*(v_0).$$

Under this map, the natural metric on STH is pulled back to a left-invariant metric on PSL_2 which is independent of the choice of v_0. We further pull this metric back to $\widetilde{SL}_2$, and then give $\widetilde{SL}_2 \times E^1$ the product metric.

One can also identify $PSL_2 \times E^1$ with the bundle T_0H of non-zero tangent vectors on H via the map

$$PSL_2 \times E^1 \xrightarrow{\cong} T_0H \qquad (A, \lambda) \longmapsto e^\lambda A_*(v_0).$$

For the product metric on $PSL_2 \times E^1$ the fibres of T_0H are product cylinders $S^1 \times E^1$ where the S^1-factor has length 2π. Then $\widetilde{SL}_2 \times E^1$ is the universal cover of T_0H, with a special metric where each covering fibre is a Euclidean product $E^1 \times E^1$.

The bundle T_0H is biholomorphic to $H \times \mathbb{C}^*$, and we let (z, w) denote the usual coordinates on $\mathbb{C}^2$. We put this holomorphic structure on $PSL_2 \times E^1$. The universal cover of T_0H is biholomorphic to $H \times \mathbb{C}$ with coordinates $(z, \widetilde{w})$, again induced from $\mathbb{C}^2$. The relation between these two coordinate systems is $w = e^{\widetilde{w}}$. We consider $(z, \widetilde{w})$ as coordinates on $\widetilde{SL}_2 \times E^1$.

We use the notation λ, θ for the real and imaginary parts of $\widetilde{w}$, that is, we write $\widetilde{w} = \lambda + i\theta$. Notice that λ is the linear coordinate of the E^1-factor, and θ is the angular coordinate on the fibres of STH (resp. the linear coordinate on the fibres of the universal cover of STH).

The identity component of the full isometry group of $T_0H = PSL_2 \times E^1$ is $PSL_2 \times \mathbb{C}^*$. An element of PSL_2 acts on T_0H by

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}(z, w) = \left(\frac{rz+s}{tz+u}, \frac{w}{(tz+u)^2}\right).$$

An element $e^{\lambda+i\theta} \in \mathbb{C}^*$ acts on T_0H by multiplication on the fibres,

$$e^{\lambda+i\theta}(z, w) = (z, e^{\lambda+i\theta}w).$$

Consider now the following one-parameter subgroup of PSL_2,

$$\begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \qquad \tau \in \mathbb{R},$$

and let $\mathscr{A}(\tau)$ be its lifting to a one-parameter subgroup of $\widetilde{SL}_2$. The kernel of the projection homomorphism $\widetilde{SL}_2 \to PSL_2$ is the infinite cyclic group generated by $\mathscr{A}(\pi)$.

We now define an action of the direct product $\widetilde{SL}_2 \times \mathbb{C}$ on $\widetilde{SL}_2 \times E^1 = H \times \mathbb{C}$. Given an element $\widetilde{A} \in \widetilde{SL}_2$ that maps to $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ under the natural projection $\widetilde{SL}_2 \to PSL_2$, then

$$(\widetilde{A}, \widetilde{w}_0)(z, \widetilde{w}) = \left(\frac{rz+s}{tz+u}, \widetilde{w} + \widetilde{w}_0 - 2\log(tz+u) \right).$$

Notice that $\log(tz+u)$ is multi-valued on $PSL_2 \times H$, but it defines a single-valued function on $\widetilde{SL}_2 \times H$, normalized by the requirement that it equal 0 on (Identity, z). Its value on $(\mathscr{A}(\pi), z)$ is πi for all $z \in H$.

This defines a surjective homomorphism

$$\widetilde{SL}_2 \times \mathbb{C} \longrightarrow \mathrm{Isom}_0(\widetilde{SL}_2 \times E^1)$$

from $\widetilde{SL}_2 \times \mathbb{C}$ to the identity component of the full isometry group of $\widetilde{SL}_2 \times E^1$. It is obvious that

$$(\mathscr{A}(\pi), 0)(z, \widetilde{w}) = (z, \widetilde{w} - 2\pi i),$$

hence the action by $\widetilde{SL}_2 \times \mathbb{C}$ is not effective, its kernel being the infinite cyclic group generated by $\gamma_0 = (\mathscr{A}(\pi), 2\pi i)$. Therefore

$$\mathrm{Isom}_0(\widetilde{SL}_2 \times E^1) = (\widetilde{SL}_2 \times \mathbb{C})/\langle \gamma_0 \rangle = \widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{C}.$$

The isometry group $\mathrm{Isom}_0(\widetilde{SL}_2 \times E^1)$ contains the subgroup $\widetilde{SL}_2$ as the set of classes $[(\widetilde{A}, 0)]$ with $\widetilde{A} \in \widetilde{SL}_2$. It also contains the subgroup $\mathbb{C}$ as the set of classes [(Identity, $\widetilde{w}_0$)] with $\widetilde{w}_0 \in \mathbb{C}$.

We have an obvious projection homomorphism

$$\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{C} \longrightarrow PSL_2.$$

For Γ a discrete subgroup of $\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{C}$, we write Γ' for its image in PSL_2 under this projection (note that Γ' is again a discrete subgroup, cf. [27, p. 125]).

Complex surfaces of geometric type $\widetilde{SL}_2 \times E^1$ are quotients

$$W = G \backslash (\widetilde{SL}_2 \times E^1) = G \backslash (H \times \mathbb{C}),$$

where G is a cocompact discrete subgroup of $\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{C}$. These surfaces have a unique elliptic fibration [27, Lemma 7.2], where the fibres are given by $z =$ constant. The generic fibres are all isomorphic to the quotient of $\mathbb{C}$ under the lattice $G \cap \mathbb{C}$. The multiple fibres result from taking further quotients under finite groups of automorphisms of the generic fibre.

It is straightforward to see that $w\partial_w$ is a nowhere zero, holomorphic vector field on $PSL_2 \times E^1$ that is invariant under $PSL_2 \times \mathbb{C}^*$. Its lifting $\partial_{\widetilde{w}}$ to $\widetilde{SL}_2 \times E^1$ is invariant under $\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{C}$. Likewise, $\Omega_H = dz \wedge d(1/w)$ is a nowhere zero, holomorphic 2-form on $PSL_2 \times E^1$ that is invariant under PSL_2 and satisfies

$$w_0^* \Omega_H = w_0^{-1} \cdot \Omega_H \quad \text{for } w_0 \in \mathbb{C}^*,$$

where w_0^* denotes the pull-back under left multiplication by w_0.

The lifting of Ω_H to $\widetilde{SL}_2 \times E^1$ is

$$\widetilde{\Omega}_H = -e^{-\widetilde{w}} dz \wedge d\widetilde{w},$$

this is invariant under $\widetilde{SL}_2$ and satisfies

$$\widetilde{w}_0^* \widetilde{\Omega}_H = e^{-\widetilde{w}_0} \widetilde{\Omega}_H \quad \text{for all } \widetilde{w}_0 \in \mathbb{C}.$$

Notice that $\widetilde{w}_0 = 2\pi i$ can be identified with $\mathscr{A}(-\pi)$ in $\widetilde{SL}_2$, hence $\widetilde{\Omega}_H$ is invariant under an isometry of $\widetilde{SL}_2 \times E^1$ if and only if this isometry can be realized as left multiplication by an element in $\widetilde{SL}_2$. Indeed, a much stronger fact is true.

Lemma 5.6 *A diffeomorphism of $\widetilde{SL}_2 \times E^1$ preserves $\widetilde{\Omega}_H$ if and only if it is left multiplication by some element of $\widetilde{SL}_2$.*

Proof. Since $\widetilde{\Omega}_H$ determines the complex structure, such diffeomorphism has to be a biholomorphism of $H \times \mathbb{C}$. The Kobayashi pseudodistance (cf. [17]) degenerates along the $\mathbb{C}$-factors of $H \times \mathbb{C}$, thus a complex automorphism of $H \times \mathbb{C}$ has to take $\mathbb{C}$-fibres to $\mathbb{C}$-fibres. Also, any automorphism of $\mathbb{C}$ is of the form $\widetilde{w} \mapsto a\widetilde{w} + b$. Therefore the general complex automorphism of $H \times \mathbb{C}$ is of the form

$$\Phi(z, \widetilde{w}) = (A(z), a(z)\widetilde{w} + b(z)),$$

where $A \in PSL_2$ and $a(z), b(z)$ are entire functions, with $a(z)$ nowhere zero. Given this matrix $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ choose any sheet of the function $\log(tz + u)$ and write

$$\Phi(z, \widetilde{w}) = (A(z), a(z)\widetilde{w} - 2\log(tz + u) + b_0(z)),$$

where $b_0(z) = b(z) + 2\log(tz + u)$ is also an entire function. We compute

$$\Phi^* \widetilde{\Omega}_H = -a(z) \exp(-a(z)\widetilde{w} - b_0(z))\, dz \wedge d\widetilde{w}.$$

The identity $\Phi^* \widetilde{\Omega}_H = \widetilde{\Omega}_H$ is satisfied if and only if there exists a sheet of the function $\log a(z)$ for which we have

$$-a(z)\widetilde{w} - b_0(z) + \log a(z) \equiv -\widetilde{w}.$$

Since z and $\widetilde{w}$ are independent variables, we must have $a(z) \equiv 1$ and $b_0(z) = \log a(z) = 2\pi i k$, for some integer k. Thus

$$\Phi(z, \widetilde{w}) = (A(z), \widetilde{w} - 2\log(tz + u) + 2\pi i k)$$

equals left multiplication by an element of $\widetilde{SL}_2$.

The lemma is proved.

Let now $S(t_0) = M^3 \times S^1$ be a properly elliptic surface of geometric type $\widetilde{SL}_2 \times E^1$ arising from a taut contact circle on M^3. The elliptic fibration is given by

$$G \backslash (\widetilde{SL}_2 \times E^1) \longrightarrow \Gamma' \backslash H,$$

induced from the natural projection $H \times \mathbb{C} \to H$. Here $\Gamma \subset \widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{C}$ is the subgroup $\pi_1(M^3)$ of G, and $\Gamma' = G'$ is the image of G under the projection to PSL_2.

On $\widetilde{S} = \widetilde{SL}_2 \times E^1$ we have the form $\widetilde{\Omega}$ which is the lift of $\Omega = d(e^t(\omega_1 + i\omega_2))$ from $M^3 \times \mathbb{R}$, and the vector fields $\widetilde{X}$ and $\widetilde{X}_c$, the lifts of X and X_c, respectively. We observed in the last paragraph of Section 4 that the flow of X is along the fibres of the elliptic fibration. Since $\widetilde{X}$ has to be invariant under the fibre lattice $G \cap \mathbb{C}$, we find that

$$\widetilde{X}_c = a(z)\partial_{\widetilde{w}}$$

for some nowhere zero, holomorphic function $a(z)$ on H.

We can write $G = \Gamma \times \langle \varphi_{t_0} \rangle$, where φ_t denotes the flow of $\widetilde{X}$. Note that $\widetilde{\Omega}$ is invariant under Γ.

Because of the invariance of $\partial_{\widetilde{w}}$ under the full group $\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{C}$, we conclude that $a(z)$ is Γ-invariant, and so it defines a global holomorphic function on the base orbifold $\Sigma = \Gamma' \setminus H$ of the elliptic fibration of $S(t_0)$. This implies that $a(z)$ is a constant $a = a_0$, say.

Since $H \times \mathbb{C}$ is simply-connected, there is a holomorphic function $\phi(z, \widetilde{w})$ such that

$$\widetilde{\Omega} = e^{\phi}\widetilde{\Omega}_H.$$

Then the condition $L_{\widetilde{X}_c}\widetilde{\Omega} = \widetilde{\Omega}$ translates to

$$a_0\left(\frac{\partial \phi}{\partial \widetilde{w}} - 1\right) = 1,$$

whose solutions are

$$\phi = \phi_0(z) + \left(1 + \frac{1}{a_0}\right)\widetilde{w},$$

for any non-zero complex constant a_0 and any entire function $\phi_0(z)$ independent of $\widetilde{w}$. Therefore

$$\widetilde{\Omega} = -\exp(\phi_0(z) + (1/a_0)\widetilde{w})\, dz \wedge d\widetilde{w}.$$

Now define a holomorphic automorphism Ψ of $H \times \mathbb{C}$ by

$$\Psi(z, \widetilde{w}) = (z, -a_0(\phi_0(z) + \widetilde{w})).$$

This automorphism takes $\mathbb{C}$-factors to $\mathbb{C}$-factors, it pulls $\widetilde{X} = a_0\partial_{\widetilde{w}}$ back to $-\partial_{\widetilde{w}}$, and it satisfies

$$\Psi^*\widetilde{\Omega} = -\exp(\phi_0(z) + (1/a_0)(-a_0)(\phi_0(z) + \widetilde{w}))\, dz \wedge (-a_0)d\widetilde{w} = -a_0\widetilde{\Omega}_H.$$

Although Ψ need not be an isometry, it conjugates Γ into another group of isometries. In fact, since Γ preserves $\widetilde{\Omega}$, the group $\Psi^{-1} \circ \Gamma \circ \Psi$ preserves $(-1/a_0)\Psi^*\widetilde{\Omega} = \widetilde{\Omega}_H$ and by Lemma 5.6 it is a subgroup of $\widetilde{SL}_2$.

Hence, up to biholomorphism of the universal cover $\widetilde{S} = H \times \mathbb{C}$, we may assume

$$\widetilde{X}_c = -\partial_{\widetilde{w}}, \quad \widetilde{\Omega} = \widetilde{\Omega}_H, \quad \text{and} \quad \Gamma \subset \widetilde{SL}_2.$$

Note that this allows to identify $\langle \varphi_{t_0} \rangle$ with the subgroup $\langle t_0 \rangle$ of E^1.

As a Γ-invariant transversal for $\widetilde{X}$ we can use the hypersurface $\widetilde{SL}_2 \times \{0\}$. Therefore

$$M^3 \cong \Gamma \setminus (\text{orbit space of } \widetilde{X}) = \Gamma \setminus \widetilde{SL}_2,$$

which is a Seifert fibred manifold over the base orbifold $\Sigma = \Gamma' \setminus H$.

For any discrete, cocompact subgroup $\Gamma \subset \widetilde{SL}_2$, the complex surface $S = \Gamma \setminus (\widetilde{SL}_2 \times E^1)$ satisfies conditions (1) and (2) of Corollary 3.12, with X_c and Ω induced from $-\partial_{\widetilde{w}}$ and $\widetilde{\Omega}_H$, respectively, and so the corresponding surfaces $S(t_0)$ do arise from a taut contact circle on $M^3 = \Gamma \setminus \widetilde{SL}_2$.

We have now proved that a properly elliptic surface is a surface $S(t_0)$ if and only if it is in class (b') of Theorem 5.1, and that the corresponding 3-manifolds and triples $(\Gamma, \widetilde{X}_c, \widetilde{\Omega})$ are as stated in that theorem.

The group G now has the following description:

$$G = \Gamma \times \langle t_0 \rangle \subset \widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{C}.$$

The fibre lattice $G \cap \mathbb{C}$ has generators t_0 and $2\pi i r_0$, for some positive integer r_0 determined by the condition $\Gamma \cap \mathbb{C} = 2\pi i \langle r_0 \rangle$.

It is easy to see that the only orthogonal bases of the fibre lattice, positively oriented with respect to the complex structure, are the following:

$$\{t_0, 2\pi i r_0\}, \{-t_0, -2\pi i r_0\}, \{2\pi i r_0, -t_0\}, \{-2\pi i r_0, t_0\}.$$

The complex structure of $S(t_0)$ determines the elliptic fibration and so it determines the complex structure of the generic fibre, which implies that the fibre lattice is determined up to multiplication by a non-zero complex constant. Thus the following unordered pair of ratios is determined by $S(t_0)$:

$$\frac{2\pi i r_0}{t_0}, \quad \frac{-t_0}{2\pi i r_0} = \frac{t_0}{2\pi r_0} i.$$

This means that the complex structure of $S(t_0)$ determines the unordered pair

$$\frac{r_0}{t_0'}, \quad \frac{t_0'}{r_0},$$

where $t_0' = t_0/2\pi$. If a homothety class is given on M^3 up to diffeomorphism, then the complex structure of $S(t_0)$ is determined as a function of t_0, and the above unordered pair is also known as a function of t_0. Thus the positive integer r_0 is determined when the homothety class on M^3 is known up to diffeomorphism.

Remark. It follows from [21] (cf. [27, p. 141]) that there are strong restictions on the possible values of r_0, in particular, r_0 has to divide the Euler characteristic of the 2-manifold Σ_0 which covers the orbifold Σ.

5.4 Hyperelliptic surfaces and complex tori

We begin with a description of the geometry of $\widetilde{E}_2 \times E^1$ and the complex structure compatible with this geometry.

We write an element of the group of Euclidean motions of E^2 as

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \theta\right),$$

where the action on E^2 is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}.$$

We obtain the universal cover $\widetilde{E}_2$ by allowing any real value for θ, hence, we may regard $\widetilde{E}_2$ as $\mathbb{R}^3$ with multiplication

$$\begin{aligned} &\left(\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \theta_0\right) \cdot \left(\begin{pmatrix} u \\ v \end{pmatrix}, \theta\right) \\ &\quad = \left(\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \theta_0 + \theta\right). \end{aligned}$$

From this description it is obvious that the standard metric on $\mathbb{R}^3$ yields the left-invariant metric on $\widetilde{E}_2$ under this identification.

We write an element of $\widetilde{E}_2 \times E^1$ as

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \theta, \lambda\right),$$

and we give $\widetilde{E}_2 \times E^1$ the complex structure pulled back from $\mathbb{C}^2$ by identifying such an element with

$$(z, w) = (u + iv, \lambda + i\theta) \in \mathbb{C}^2.$$

The identity component of the group of isometries of $\widetilde{E}_2 \times E^1$ that preserve this complex structure is the semidirect product of translations $\mathbb{R}^4$ and unitary maps $U(2)$. If such an isometry fixes the 3-dimensional space $\{\lambda = 0\}$ (we shall call such an isometry a *complex isometry of* $\widetilde{E}_2$), it has to be of the form

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \theta, \lambda\right) \longmapsto \left(\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \theta + \theta_1, \lambda\right),$$

or

$$(z, w) \longmapsto (e^{i\theta_0} z + z_0, w + i\theta_1),$$

and such an isometry can be interpreted as an element of $\widetilde{E}_2$ acting by left multiplication if and only if $\theta_0 \equiv \theta_1 \bmod 2\pi$.

Now consider the holomorphic 2-form

$$\widetilde{\Omega}_E = e^{-w} dz \wedge dw$$

on $\widetilde{E}_2 \times E^1$, and observe that a complex isometry of $\widetilde{E}_2$ pulls back $\widetilde{\Omega}_E$ to $e^{i(\theta_0-\theta_1)}\widetilde{\Omega}_E$, so it preserves $\widetilde{\Omega}_E$ if and only if this isometry is an element of $\widetilde{E}_2$. Also notice that translations in λ-direction send $\widetilde{\Omega}_E$ to positive multiples of itself.

Now let $S(t_0) = M^3 \times S^1$ be a complex torus or a hyperelliptic surface (with Euler class $(0,0)$) that arises from a taut contact circle on M^3.

A hyperelliptic surface with Euler class $(0,0)$ can be described as follows (cf. [1, p. 148], [25], and [27, pp. 141–142], but note that in our description as compared with Wall's, the rôles of z and w are exchanged; we have opted for this change to get $\widetilde{\Omega}_E$ into the same form as $\widetilde{\Omega}_H$, but whereas the elliptic fibration in the properly elliptic case was given by $(z,w) \mapsto z$, in the hyperelliptic case it is given by $(z,w) \mapsto w$). Let first ε be a primitive m^{th} root of 1, where $m = 2,3,4,6$, and take the quotient $L \setminus \mathbb{C}^2$, where $L \subset \mathbb{C} \times \{0\}$ is a lattice of rank 2 invariant under multiplication by ε. Then one takes the further quotient under a lattice of rank 2 in $\{0\} \times \mathbb{C} \subset \mathbb{C}^2$, thus creating a quotient which is the product of two elliptic curves, where we think of the first factor as the fibre and of the second factor as the base. Finally, one takes the quotient under an action of a cyclic group of order 2,3,4, or 6, respectively, whose generator acts by

$$(z,w) \longmapsto (\varepsilon z, w + w_0),$$

where w_0 is an element (of order m) of the translation group of the base. (Note in particular that hyperelliptic surfaces with Euler class $(0,0)$ have no translational monodromy).

If $S(t_0)$ is a hyperelliptic surface, then it is easy to see that invariance under all the generators of the deck transformation group G forces $\widetilde{X}_c$ to be of the form $\widetilde{X}_c = a_0\partial_w$, for some non-zero complex constant a_0.

Analogously to the previous section, we have

$$\widetilde{\Omega} = e^{\phi}\widetilde{\Omega}_E,$$

and as there we conclude from $L_{\widetilde{X}_c}\widetilde{\Omega} = \widetilde{\Omega}$ that

$$\widetilde{\Omega} = \exp(\phi_0(z) + (1/a_0)w)\, dz \wedge dw.$$

Consider now the automorphism Ψ of $\mathbb{C}^2$ defined by

$$\Psi(z,w) = (z, -a_0(\phi_0(z) + w)).$$

This map pulls $\widetilde{X}_c$ back to $-\partial_w$. It also pulls $\widetilde{\Omega}$ back to $-a_0\widetilde{\Omega}_E$.

Any transformation $\gamma \in G$ is of the form

$$\gamma(z,w) = (az + z_0, w + w_1),$$

with $|a| = 1$. The conjugate of this map by Ψ is

$$\Psi^{-1} \circ \gamma \circ \Psi(z, w) = (az + z_0, w + f(z)),$$

where $f(z) = \phi_0(z) - \phi_0(az + z_0) - (w_1/a_0)$ is an entire function independent of w.

If γ is in Γ, then it preserves $\widetilde{\Omega}$ and so $\Psi^{-1} \circ \gamma \circ \Psi$ must preserve the pullback $\Psi^* \widetilde{\Omega} = -a_0 \widetilde{\Omega}_E$, and this means that

$$ae^{-w-f(z)} \equiv e^{-w},$$

hence $f(z) \equiv \log a$ for some choice of $\log a$. Thus the conjugate group $\Psi^{-1} \circ \Gamma \circ \Psi$ is contained in $\widetilde{E}_2$.

We conclude that, up to biholomorphism of $\widetilde{S} = \mathbb{C}^2$, we may assume

$$\widetilde{X}_c = -\partial_w, \quad \widetilde{\Omega} = \widetilde{\Omega}_E, \quad \text{and} \quad \Gamma \subset \widetilde{E}_2.$$

If $S(t_0)$ is a complex torus, then invariance of $\widetilde{X}_c$ under G is equivalent to $\widetilde{X}_c = b_0 \partial_z + a_0 \partial_w$, for some complex constants a_0, b_0, but we may have $b_0 \neq 0$ and even $a_0 = 0$. Let first Ψ_1 be a linear automorphism of $\mathbb{C}^2$ pulling $\widetilde{X}_c$ back to $-\partial_w$. The identity

$$L_{-\partial_w} \Psi_1^* \widetilde{\Omega} = \Psi_1^* \widetilde{\Omega}$$

implies

$$\Psi_1^* \widetilde{\Omega} = e^{\phi_0(z) - w} dz \wedge dw.$$

Now $\Psi_2(z, w) = (z, w + \phi_0(z))$ pulls $-\partial_w$ back to itself and pulls $\Psi_1^* \widetilde{\Omega}$ back to $\widetilde{\Omega}_E$.

The group G now consist only of translations. As Ψ_1 is linear, every element $\gamma \in \Psi_1^{-1} \circ G \circ \Psi_1$ is a translation

$$\gamma(z, w) = (z + z_0, w + w_1).$$

Then the conjugate of this map by Ψ_2 is

$$\Psi_2^{-1} \circ \gamma \circ \Psi_2(z, w) = (z + z_0, w + f(z)),$$

where $f(z) = \phi_0(z) - \phi_0(z + z_0) + w_1$. This conjugate map preserves $\widetilde{\Omega}_E$ if and only if $f(z) \equiv 2\pi i k$ for some integer k, i.e., if and only if it is in the translation part $\mathbb{C} \times (2\pi i \mathbb{Z})$ of $\widetilde{E}_2$.

We conclude that, up to biholomorphism of $\mathbb{C}^2$, a complex torus $S(t_0)$ has

$$\widetilde{X}_c = -\partial_w, \quad \widetilde{\Omega} = \widetilde{\Omega}_E, \quad \text{and} \quad \Gamma \subset \mathbb{C} \times (2\pi i \mathbb{Z}).$$

The hypersurface $\widetilde{E}_2 \times \{0\}$ is a Γ-invariant transversal for $-\partial_w$, therefore

$$M^3 \cong \Gamma \setminus \widetilde{E}_2.$$

Given any discrete, cocompact subgroup $\Gamma \subset \widetilde{E}_2$, the surface $S = \Gamma \setminus (\widetilde{E}_2 \times E^1)$ admits a pair (X_c, Ω) (induced from $-\partial_w$ and $\widetilde{\Omega}_E$) satisfying the conditions of Corollary 3.12, thus S arises from a taut contact circle on the orbit space of $X = 2\,\mathrm{Re}\,X_c$ in S, which is $\Gamma \setminus \widetilde{E}_2$.

We have now proved that a hyperelliptic surface or a complex torus is a surface $S(t_0)$ if and only if it is in classes (c') or (c"), respectively, of Theorem 5.1, and that the corresponding 3-manifolds and triples $(\Gamma, \widetilde{X}_c, \widetilde{\Omega})$ are as stated in that theorem.

Since in Section 5.1 we proved that a surface $S(t_0)$ has to be a Hopf surface, a properly elliptic surface of geometric type $\widetilde{SL}_2 \times E^1$, a hyperelliptic surface with Euler class $(0,0)$, or a complex torus, we have completed the proof of Theorem 5.1 and consequently of Theorem 1.2.

5.5 Topological arguments

In this section we give a brief sketch of the topological arguments that can be used to prove Theorem 5.1 up to diffeomorphism.

The key fact is that a complex surface $W = S(t_0) \cong M^3 \times S^1$ that arises from a taut contact circle on M^3 has trivial Chern classes (cf. Proposition 3.4).

Given a complex surface W with covering surface $\widetilde{W}$ and deck transformation group G, one can define a complex line bundle $\mathscr{L}$ over W by forming the quotient

$$\widetilde{W} \times \mathbb{C}/\sim,$$

$$(x, z) \sim (\gamma x, \mu(\gamma)^{-1} z), \quad \gamma \in G,$$

where $\mu \in \mathrm{Hom}(G, \mathbb{C}^*)$. Note that $\widetilde{W} \to W$ is the associated principal G-bundle of $\mathscr{L} \to W$.

Now a standard argument (cf. [9]) shows that under the (weak) assumption that $\pi_2(\widetilde{W}) = 0$, the condition $c_1(\mathscr{L}) = 0$ implies that $\mu|TG^{ab} = 0$, where TG^{ab} denotes the torsion part of the abelianization G^{ab} of G.

One then applies this topological lemma to the canonical line bundle of Hopf surfaces and properly elliptic surfaces of type $\widetilde{SL}_2 \times E^1$, arising as $S(t_0)$ from taut contact circles. In both instances we have seen that there is a nowhere zero, holomorphic 2-form $\widetilde{\Omega}$ on $\widetilde{W}$ that is preserved by an isometry γ of $\widetilde{W}$ if and only if $\gamma \in SU(2)$ in the Hopf case and $\gamma \in \widetilde{SL}_2$ in the properly elliptic case. This is sufficient to conclude the proof in the former case, in the latter one also has to take care of $2g$ free generators of G coming from the base Σ of the elliptic fibration (where g is the genus of Σ). However, these differ from elements in $\widetilde{SL}_2$ only by a translational component along the fibres of the elliptic fibration, and this component may be changed arbitrarily (and in particular be set to zero) without changing the diffeomorphism type of W. This follows from the work of Ue, since he shows that the diffeomorphism type is determined by the fundamental group $\pi_1 = G$ of W, and the relations in G are not affected by changing the translational part of any of the free generators coming from the base.

6 Homothety classes

In this section we apply the results of Section 3, together with Theorem 5.1 and Proposition 5.5, to give a list of all pairs (M^3, homothety class), up to diffeomorphism, and to show that the list has no repetitions when M^3 is a left-quotient of S^3. Thus we deduce Theorem 1.6 and Proposition 1.7. We also prove Corollary 1.9.

We keep the notation of Section 5.

Before we start, recall the definition of the standard Cartan structure on $\Gamma \backslash \mathscr{G}$, given in Section 2. This is induced by any left-invariant Cartan structure on $\mathscr{G}$.

For $\mathscr{G} = S^3$, with the ordinary inclusion $j : S^3 \longrightarrow \mathbb{C}^2$, the complex 1-form $j^*(z_1 dz_2 - z_2 dz_1)$ is left-invariant (see below) and its real and imaginary part form a Cartan structure. In the present section and the next, we take the standard Cartan structure on $\Gamma \backslash S^3$ to be induced by (the real and imaginary part of) $z_1 dz_2 - z_2 dz_1$.

Similarly, we take the standard Cartan structure on $\Gamma \backslash \widetilde{SL}_2$ (resp. on $\Gamma \backslash \widetilde{E}_2$) to be induced by $e^{-\widetilde{w}} dz$ (resp. $e^{-w} dz$).

Let first $S(t_0)$ be a Hopf surface. We can take S^3, with ordinary inclusion j, as the transversal for $\widetilde{X}$ in $\mathbb{C}^2 - \{(0,0)\}$.

We now follow the terminology of Proposition 5.5. If $S(t_0)$ is of type ($1'$), we consider

$$\omega_1 + i\omega_2 = j^*(\widetilde{X}_c \rfloor \widetilde{\Omega}),$$

where for $\widetilde{\Omega}$ we take $2\, dz_1 \wedge dz_2$ and

$$\widetilde{X}_c = \tfrac{1}{2}(z_1 \partial_{z_1} + z_2 \partial_{z_2}),$$

whence

$$\omega_1 + i\omega_2 = j^*(z_1 dz_2 - z_2 dz_1),$$

that is,

$$\begin{aligned} \omega_1 &= j^*(x_1 dx_2 - x_2 dx_1 + y_2 dy_1 - y_1 dy_2), \\ \omega_2 &= j^*(x_1 dy_2 - y_2 dx_1 + y_1 dx_2 - x_2 dy_1), \end{aligned}$$

which are two of the standard Maurer-Cartan forms on S^3. Thus, to surfaces of type ($1'$) there corresponds the standard Cartan structure on $\Gamma \backslash S^3$.

If $S(t_0)$ is of type ($1''$), we use the same j as above, for $\widetilde{\Omega}$ we take $dz_1 \wedge dz_2$, and

$$\widetilde{X}_c = a z_1 \partial_{z_1} + (1 - a) z_2 \partial_{z_2}, \quad \text{where } 0 < \mathrm{Re}(a) < 1.$$

Two pairs $(S(t_0), X_c)$ with the same t_0 and Γ are isomorphic if and only if they have the same unordered pair $\{a, 1 - a\}$. Since diffeomorphic homothety classes give isomorphic pairs $(S(t_0), X_c)$ for each t_0, we conclude that the moduli space of homothety classes giving surfaces of type ($1''$), for fixed $\Gamma = \Gamma_m$, is the orbifold quotient Q_1 of the slab $0 < \mathrm{Re}(a) < 1$ under the rotation $R : a \mapsto 1 - a$. By slight abuse of language, we call them homothety classes of type $(1'')$.

The contact circle is given by

$$\omega_1 + i\omega_2 = j^*(az_1 dz_2 + (a-1)z_2 dz_1).$$

Such a circle descends to the lens spaces $L(m, m-1)$, including $S^3 = L(1,0)$.

By Corollary 3.14, the homothety class contains a Cartan structure if and only if there is a transversal in $M^3 \times \mathbb{R}$ tangent to $J\partial_t$. As Γ_m is finite, this lifts to a compact transversal tangent to $-2\,\mathrm{Im}(\widetilde{X}_c)$ in $\mathbb{C}^2 - \{0\}$.

The flow of $-2\,\mathrm{Im}(\widetilde{X}_c)$ is given by

$$\psi_t(z_1, z_2) = (e^{iat} z_1, e^{i(1-a)t} z_2),$$

and if $\mathrm{Im}(a)$ is non-zero, then every point with at least one coordinate non-zero goes to infinity as t goes to one of $\{+\infty, -\infty\}$. Hence, in this case no compact real submanifold of $\mathbb{C}^2 - \{0\}$ tangent to $-2\,\mathrm{Im}(\widetilde{X}_c)$ can exist.

If a is real, then $-2\,\mathrm{Im}(\widetilde{X}_c)$ is tangent to S^3.

Therefore, the homothety class on $L(m, m-1)$ given by

$$az_1 dz_2 + (a-1)z_2 dz_1$$

contains a Cartan structure if and only if a is real. The quotient of the interval $0 < a < 1$ under R is a ray in Q_1 going from the cone point to infinity, and it is the part of the moduli space coming from Cartan structures on $L(m, m-1)$.

Every point a on the slab $0 < \mathrm{Re}(a) < 1$ can be moved to the interval $(0,1)$. This gives a homotopy of any homothety class of type $(1'')$ to one containing a Cartan structure. If we actually move a to the point $1/2$, then the homotopy ends in the homothety class of the standard Cartan structure.

Let now $S(t_0)$ be a Hopf surface of type (2). In this case M^3 is a lens space $L(m, m-1) = \Gamma_m \setminus S^3$. Fix some $q \in \mathbb{N}$ and let $n = mq - 1$. Let j be as before, and we take $(n+1)dz_1 \wedge dz_2$ for $\widetilde{\Omega}$, and

$$\widetilde{X}_c = \left(\frac{n}{n+1} z_1 + b_0 z_2^n\right) \partial_{z_1} + \frac{1}{n+1} z_2 \partial_{z_2},$$

where $b_0 \neq 0$ is small enough so that $2\,\mathrm{Re}(\widetilde{X}_c)$ is transverse to $j(S^3)$. Then, following the proof of Theorem 3.11, we get a taut contact circle on S^3 by setting

$$\omega_1 + i\omega_2 = j^*(nz_1 dz_2 - z_2 dz_1 + (n+1)b_0 z_2^n dz_2),$$

and this contact circle descends to $L(m, m-1)$ because of $n+1 \equiv 0 \bmod m$. The resulting homothety classes will be called homothety classes of type (2).

For given m, n, and t_0, all values of b_0 give the same pair $(S(t_0), X_c)$ up to isomorphism. Therefore there is a one-to-one correspondence between homothety classes, up to diffeomorphism, and pairs (m, n). Here m is a homotopy invariant of M^3, while $n = mq - 1$ is geometric.

The flow of $-2\,\mathrm{Im}(\widetilde{X}_c)$ is

$$\psi_t(z_1,z_2) = (\exp(int/(n+1))z_1 + ib_0 t \exp(int/(n+1))z_2^n, \exp(it/(n+1))z_2).$$

We see that a point with z_2 non-zero goes to infinity as t goes to infinity. So there is no compact real hypersurface of $\mathbb{C}^2 - \{0\}$ tangent to $-2\,\mathrm{Im}(\widetilde{X}_c)$ and the homothety class does not contain any Cartan structure. We can move b_0 until it reaches the value zero, while keeping it small so that S^3 is always transverse to $2\,\mathrm{Re}(\widetilde{X}_c)$. This is a homotopy of the homothety class into one containing a Cartan structure. It is an interesting homotopy because it is a 'jump' homotopy: While $b_0 \neq 0$, the homothety class is being deformed into equivalent ones, but for $b_0 = 0$ it is a new one. This jump phenomenon has been pointed out by Kodaira-Spencer in [18, pp. 435–436] for the case $n = 1$, where it prevents the construction of a 'good' moduli space of Hopf surfaces in the case where the contraction T is linear with double eigenvalue. For this reason, we define the moduli space on the lens spaces $L(m, m-1)$ only for homothety classes of type $(1'')$. The homothety classes of type (2), up to diffeomorphism, simply form a discrete set.

The diffeomorphism from the homothety class given by $b_0 \neq 0$ to the one with $b_0 = 1/(n+1)$, is induced by the linear map of $\mathbb{C}^2$ given by the matrix

$$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix},$$

where $c = 1/((n+1)b_0)$ goes to infinity as b_0 goes to zero.

As we pointed out in the proof of Proposition 5.5, the vector field $2\,\mathrm{Re}(\widetilde{X}_c)$ is transverse to S^3 for $b_0 = 1/(n+1)$. This particular choice gives

$$\begin{aligned} \omega_1 + i\omega_2 &= j^*(nz_1 dz_2 - z_2 dz_1 + z_2^n dz_2) \\ &= j^*(z_2^n \exp(z_1/z_2^n)\, d(z_2 \exp(-z_1/z_2^n))). \end{aligned}$$

Therefore the integral curves in S^3 of the common kernel $\ker\omega_1 \cap \ker\omega_2$ can be described by the equations

$$\begin{aligned} |z_1|^2 + |z_2|^2 &= 1, \\ z_2 \exp(-z_1/z_2^n) &= \text{constant}. \end{aligned}$$

Since surfaces of different types $(1')$, $(1'')$, (2) are not biholomorphic, we also have that homothety classes of different type are not diffeomorphic.

Observe that in all cases we can homotope the taut contact circle (ω_1, ω_2) on $\Gamma \setminus S^3$ to the standard Cartan structure: In case $(1'')$ we let a go to $1/2$, in case (2) we let b_0 go to 0 and n to 1 (for the purpose of a homotopy, n need not be an integer). Moreover, we noticed in the proof of Proposition 5.5 that the isomorphism type of Γ (as an abstract group) determines Γ up to conjugation in $SU(2)$, hence, there is a unique standard Cartan structure (up to diffeomorphism) on $M^3 = \Gamma \setminus S^3$, even if we only fix the diffeomorphism type of M^3 and not a specific subgroup $\Gamma \subset SU(2)$.

We have thus proved Theorem 1.6 for the left-quotients of S^3, as well as the following statement on moduli, of which Proposition 1.7 is the part for S^3.

Proposition 6.1 *The moduli space of taut contact circles of type* $(1'')$ *on the lens spaces* $L(m, m-1)$, *including* S^3, *up to homothety and diffeomorphism, is the orbifold* Q_1. *The homothety classes of type* (2), *again up to diffeomorphism, form a countable, discrete set* $\{P_n\}$, *where* n *ranges over the positive integers of the form* $n = mq - 1$, *and* P_n *is the homothety class induced by the complex form* $nz_1 dz_2 - z_2 dz_1 + z_2^n dz_2$. *The homothety classes containing Cartan structures correspond to the part of* Q_1 *coming from the real interval* $0 < a < 1$, *where* $1/2$ *gives the cone point and represents the homothety class of the standard Cartan structure.*

Next, suppose that $S(t_0)$ is a properly elliptic surface. Then $\widetilde{X}_c = -\partial_{\widetilde{w}}$ and we use $H \times (i\mathbb{R})$ as transversal for $\widetilde{X} = -\partial_\lambda$. Notice that $-2\,\mathrm{Im}(\widetilde{X}_c) = -\partial_\theta$ is tangent to this transversal. We let j be the inclusion of $H \times (i\mathbb{R})$ into $H \times \mathbb{C}$, and

$$\omega_1 + i\omega_2 = j^*(\widetilde{X}_c \rfloor \widetilde{\Omega}_H) = j^*(-e^{-\widetilde{w}} dz)$$

is invariant under the left action of $\widetilde{SL}_2$, because $\widetilde{X}_c$ and $\widetilde{\Omega}_H$ are. Thus we get the standard Cartan structure on $\Gamma \setminus \widetilde{SL}_2$.

The argument in the case $\mathscr{G} = \widetilde{E}_2$ is completely analogous.

This concludes the proof of Theorem 1.6.

To deduce Corollary 1.9, we argue as follows. We have seen that any taut contact circle is, up to homotopy, covered by the standard Cartan structure (ω_1, ω_2) on $\mathscr{G}$. For $\mathscr{G} = S^3$ or $\widetilde{E}_2$, it is easy to see that $\mathscr{D}_1 = \ker \omega_1$ is the standard tight contact structure on S^3 or $\mathbb{R}^3$, respectively. Since the condition of not being tight (hence, being overtwisted in the sense of [5]) is defined by the existence of an embedded 2-disc on which the 1-dimensional foliation induced by the contact structure has a closed leaf and a singular point inside this leaf, a contact structure covering an overtwisted contact structure is clearly overtwisted. Therefore, if the contact structure on the cover is tight, so is the contact structure on the quotient.

For $\mathscr{G} = \widetilde{SL}_2$, explicit symplectic fillings of the contact structures on left-quotients of $\mathscr{G}$ that come from the standard Cartan structure have been constructed in [7], by [5] this implies that these structures are tight.

The fact that contact structures coming from taut contact circles are tight might be very important for a complete homotopy classification of taut contact circles. For instance, it can be shown by elementary means that any taut contact circle on T^3 is homotopic to one of the form

$$\begin{aligned}\omega_1 &= \cos(r_0\theta_3)d\theta_1 + \sin(r_0\theta_3)d\theta_2,\\ \omega_2 &= -\sin(r_0\theta_3)d\theta_1 + \cos(r_0\theta_3)d\theta_2,\end{aligned}$$

where the θ_i are the angle coordinates corresponding to the three S^1-factors and $r_0 \in \mathbb{N}$. Recent work of Eliashberg-Hofer-Salamon and Giroux suggests that every tight contact structure on T^3 is of this form and that structures corresponding to different values of r_0 are not homotopic. This would imply that we have a countably infinite family of taut contact circles on T^3 and that every tight contact structure on T^3 can be realized as part of a taut contact circle.

Similar considerations apply to the other T^2-bundles over S^1 with periodic monodromy.

We conclude by pointing out that it is possible to give a more detailed description of the set of of homothety classes, up to diffeomorphism, in the cases $\mathscr{G} = \widetilde{SL}_2$ and $\mathscr{G} = \widetilde{E}_2$ as well, and it is also possible to describe the complex structure of these sets when such structure exists. For instance, the following proposition shows that the problem in the $\widetilde{SL}_2$ case reduces essentially to a problem in Teichmüller theory.

Proposition 6.2 *If the discrete, cocompact subgroups $\Gamma_{(1)}, \Gamma_{(2)}$ of $\widetilde{SL}_2$ have the same image $\Gamma'_{(1)} = \Gamma'_{(2)}$ in PSL_2, and if they yield isomorphic pairs $(S_{\Gamma_{(1)}}, X_c)$ and $(S_{\Gamma_{(2)}}, X_c)$, then $\Gamma_{(1)}$ and $\Gamma_{(2)}$ are conjugate in $\widetilde{SL}_2$.*

This proposition can be proved using the methods of Section 5.3. We plan to discuss the moduli problem in greater depth in a forthcoming paper [11].

7 Cartan structures

The purpose of this section is twofold. First we give a construction of all manifolds in Theorem 1.2 from (real) surfaces with a Riemannian metric. The construction actually yields a 3-manifold with a certain Cartan structure induced from the Liouville-Cartan pair of that real surface. Then we characterize these particular Cartan structures among the class of all Cartan structures, and we also discuss the set of Cartan structures within a homothety class.

Let Σ_0 be a closed, oriented surface of any genus, with a Riemannian metric. We do not require the curvature of this metric to be constant. Let $\mathscr{F}$ be a finite group of orientation-preserving isometries of Σ_0, and denote by $d\mathscr{F}$ the set of the differentials of the elements of $\mathscr{F}$. Then $d\mathscr{F}$ is a finite group of isometries of $ST\Sigma_0$, where it acts freely.

The quotient manifold $d\mathscr{F} \setminus ST\Sigma_0$ has a canonical Seifert fibration over the 2-dimensional orbifold $\Sigma = \mathscr{F} \setminus \Sigma_0$.

It is obvious that if $z_0 \in \Sigma_0$ has non-trivial isotropy group (necessarily cyclic of order q, say), then the fibre of $ST\Sigma_0$ over z_0 becomes a multiple fibre of multiplicity q in $d\mathscr{F} \setminus ST\Sigma_0$.

Remarks. (1) The manifold $d\mathscr{F} \setminus ST\Sigma_0$ is what in [23] is called the unit tangent bundle of the orbifold $\mathscr{F} \setminus \Sigma_0$.

(2) Suppose that Σ_0 is as above, only non-orientable. Let $\overline{\Sigma}_0 \to \Sigma_0$ be the canonical orientable covering surface, and let $F\Sigma_0 \to \Sigma_0$ be the bundle of orthonormal frames (whose fibre is the disjoint union of two circles). There is a natural map $F\Sigma_0 \to ST\overline{\Sigma}_0$ which is a diffeomorphism. For any group $\mathscr{F}$ of isometries of Σ_0, there is a naturally defined isomorphic group $\overline{\mathscr{F}}$ of isometries of $\overline{\Sigma}_0$, and a natural diffeomorphism $d\mathscr{F} \setminus F\Sigma_0 \cong d\overline{\mathscr{F}} \setminus ST\overline{\Sigma}_0$.

Sometimes we can also find 3-manifolds M^3 having a Seifert fibration over Σ and a covering map

$$M^3 \longrightarrow d\mathscr{F} \setminus ST\Sigma_0,$$

which takes fibres to fibres, with base map the identity $\Sigma \to \Sigma$. If r_0 is the number of sheets of this covering, then each fibre of M^3 wraps r_0 times around the fibre of $d\mathscr{F} \setminus ST\Sigma_0$ over the same basepoint, for some positive integer r_0.

Notice that this is equivalent to giving a manifold M^3 with a covering map $M^3 \to d\mathscr{F} \setminus ST\Sigma_0$ such that the inverse image of each Seifert fibre of $d\mathscr{F} \setminus ST\Sigma_0$ is a single circle, and then let these inverse images in M^3 define the Seifert fibration of M^3 over Σ. That is, the projection from M^3 to Σ is the composition of the covering map followed by the projection of $d\mathscr{F} \setminus ST\Sigma_0$ onto Σ.

Theorem 7.1 *A compact* 3*-manifold arises from the construction explained above, where the surface* Σ_0 *has genus* 0, 1, *or greater than* 1, *if and only if* M^3 *is a left-quotient of* S^3, $\widetilde{E}_2$, *or* $\widetilde{SL}_2$, *respectively.*

Remark. The greater part of this theorem is essentially implicit in [21]. However, the subsequent discussion in the present section refers to details in the proof of this theorem, so it seems convenient to present a complete proof, which also allows us to fix notation.

Proof. Let $\mathscr{G}$ be one of these three Lie groups, and let $\widetilde{\Sigma}$ be the corresponding simply-connected, 2-dimensional space form.

Suppose $M^3 = \Gamma \setminus \mathscr{G}$ for some discrete, cocompact subgroup Γ of $\mathscr{G}$. Let Γ' be the image of Γ under the obvious projection

$$\mathscr{G} \longrightarrow \mathrm{Isom}_0(\widetilde{\Sigma})$$

of $\mathscr{G}$ onto the group of orientation-preserving isometries of $\widetilde{\Sigma}$.

If Γ' is discrete, then $\Gamma \setminus \mathscr{G}$ has a canonical Seifert fibration with base the 2-dimensional orbifold $\Sigma = \Gamma' \setminus \widetilde{\Sigma}$. If $\mathscr{G}$ is S^3 or $\widetilde{SL}_2$, then Γ' is always discrete, but if $\mathscr{G}$ is $\widetilde{E}_2$ this is not always the case. We proceed now assuming that Γ' is discrete and, since the statement of the theorem is topological, we shall only need to check that in the case $\mathscr{G} = \widetilde{E}_2$ we already get all topological types of left-quotients if we only use subgroups Γ with discrete image Γ'.

We can always find a description $\Sigma = \mathscr{F} \setminus \Sigma_0$, where Σ_0 is a closed, oriented 2-manifold (no cone points), and $\mathscr{F}$ is a finite group of orientation-preserving isometries of Σ_0, for some metric on Σ_0 with the corresponding constant curvature. Equivalently, we have $\Sigma_0 = \Gamma_0' \setminus \widetilde{\Sigma}$, where Γ_0' is a normal subgroup of Γ' with finite index, acting freely on $\widetilde{\Sigma}$. The genus of Σ_0 is 0 if $\mathscr{G} = S^3$, 1 if $\mathscr{G} = \widetilde{E}_2$, and greater than 1 if $\mathscr{G} = \widetilde{SL}_2$.

Let now $\widetilde{\Gamma}_0$ be the inverse image of Γ_0' under the projection of $\mathscr{G}$ onto $\mathrm{Isom}_0(\widetilde{\Sigma})$. The quotient $\widetilde{\Gamma}_0 \setminus \mathscr{G}$ is the unit tangent bundle $ST\Sigma_0$. Define also $\widetilde{\Gamma} \subset \mathscr{G}$ as the inverse image of Γ'. Then we have an isomorphism of Seifert spaces over Σ,

$$\widetilde{\Gamma} \backslash \mathscr{G} \cong d\mathscr{F} \backslash ST\Sigma_0.$$

Since Γ and $\widetilde{\Gamma}$ both project to the same discrete group Γ', the quotients $\Gamma \backslash \mathscr{G}$ and $\widetilde{\Gamma} \backslash \mathscr{G}$ have Seifert fibrations over the same orbifold $\Sigma = \Gamma' \backslash \widetilde{\Sigma}$.

The inclusion $\Gamma \subset \widetilde{\Gamma}$ induces a covering map

$$M^3 = \Gamma \backslash \mathscr{G} \longrightarrow \widetilde{\Gamma} \backslash \mathscr{G},$$

which takes fibres to fibres, with base map the identity $\Sigma \to \Sigma$.

Since $\widetilde{\Gamma} \backslash \mathscr{G}$ is $d\mathscr{F} \backslash ST\Sigma_0$ as a Seifert space over Σ, this proves that $M^3 = \Gamma \backslash \mathscr{G}$ arises from the construction explained above, provided Γ' is discrete. We must now check that every left-quotient of $\widetilde{E}_2$ is obtained, up to diffeomorphism, from such a special Γ.

For T^3 this is obvious: take a lattice of rank 2 in E^2 and form its inverse image in $\widetilde{E}_2$. This produces a lattice of rank 3 and the quotient is a 3-torus.

Let M^3 be a torus bundle over S^1 with non-trivial periodic monodromy of order $m = 2, 3, 4, 6$. To construct such M^3, let first $\varepsilon = \exp(2\pi i/m)$. Take an ε-invariant lattice $L \subset \mathbb{C}$ of rank 2. For $m = 2$, any lattice in $\mathbb{C}$ of rank 2 is valid. For $m = 3, 4, 6$, the lattice L equals $\langle 1, \varepsilon \rangle$ up to multiplication by a non-zero complex number. This defines a special complex torus $\Sigma_0 = L \backslash \mathbb{C}$ with a flat metric, such that multiplication by ε in $\mathbb{C}$ determines an isometry of Σ_0 of order m, which we also denote ε. Then M^3 can be described as the quotient of $\mathbb{R}^3 = \mathbb{C} \times (i\mathbb{R})$ under the lattice

$$L + (0, 2\pi i)\mathbb{Z}$$

and the transformation

$$(z, i\theta) \mapsto (\varepsilon z, i\theta + 2\pi i/m),$$

which is the lift to the universal cover of $ST\Sigma_0$ of the differential $d\varepsilon$. It follows that M^3 is really $\langle d\varepsilon \rangle \backslash ST\Sigma_0$. Alternatively, the image Γ', of the group at hand, is the group generated by the translations of L and the rotation ε and is discrete.

To complete the proof, we must prove the converse statement, that is, if M^3 admits a covering map $M^3 \to d\mathscr{F} \backslash ST\Sigma_0$ such that the composite map

$$M^3 \to d\mathscr{F} \backslash ST\Sigma_0 \to \Sigma$$

is a Seifert fibration, then M^3 is a left-quotient of $\mathscr{G}$.

It is well-known that any metric on a closed, orientable surface Σ_0 is conformal to a metric of constant curvature, so that a finite group of isometries for the original metric still is a group of isometries for the new metric. So in the construction we already get all possible topological types M^3 if we restrict ourselves to metrics on Σ_0 of constant curvature.

If $\Sigma_0 = S^2$, then $d\mathscr{F} \backslash ST\Sigma_0$ is a left-quotient of $SO(3)$, hence a left-quotient $\Gamma \backslash S^3$ whose covering spaces are all left-quotients of S^3 under subgroups of Γ. Similarly for Σ_0 a Euclidean or hyperbolic surface.

We then get $M^3 = \Gamma \setminus \mathscr{G}$, where the image Γ' of Γ in $\mathrm{Isom}_0(\widetilde{\Sigma})$ has the orbifold $\mathscr{F} \setminus \Sigma_0$ as quotient $\Gamma' \setminus \widetilde{\Sigma}$. Hence Γ' is discrete.

This concludes the proof of the theorem.

Remarks. (1) Notice that we have proved more: any covering space of the manifold $d\mathscr{F} \setminus ST\Sigma_0$ is actually a left-quotient of $\mathscr{G}$. Thus given any covering map $M^3 \to d\mathscr{F}_1 \setminus ST\Sigma_0$ we can change $\mathscr{F}_1$ into a smaller group $\mathscr{F}$ and factor that covering map through a new covering $M^3 \to d\mathscr{F} \setminus ST\Sigma_0$ where inverse images of Seifert fibres are connected.

(2) From the construction it is also clear why the Seifert fibrations of left-quotients of $\widetilde{SL_2}$ have non-zero Euler class: the Euler class of the Seifert fibration $M^3 \to \Sigma$ is a factor of the Euler class of the Seifert fibration $ST\Sigma_0 \to \Sigma_0$, which is a negative integer.

Definition 7.2 *A* **Liouville-Cartan structure** *is a Cartan structure obtained as follows: take a closed, oriented surface Σ_0 with a Riemannian metric; take a finite group $\mathscr{F}$ of orientation-preserving isometries of Σ_0; finally take a covering map $M^3 \to d\mathscr{F} \setminus ST\Sigma_0$. Since the Liouville-Cartan pair of $ST\Sigma_0$ is invariant under $d\gamma$ for any orientation-preserving isometry γ, it descends to $d\mathscr{F} \setminus ST\Sigma_0$ and then is pulled back to M^3 via the covering map.*

Remark. Let Σ_0 be non-orientable, but otherwise as above. The bundle $F\Sigma_0$ of orthonormal frames has two tautological 1-forms θ^1, θ^2, which form a Cartan structure. We have already pointed out a natural diffeomorphism $d\mathscr{F} \setminus F\Sigma_0 \cong d\overline{\mathscr{F}} \setminus ST\overline{\Sigma}_0$, where $\overline{\Sigma}_0$ is the oriented covering surface of Σ_0. This natural diffeomorphism takes the Cartan structure on $d\mathscr{F} \setminus F\Sigma_0$ induced by (θ^1, θ^2) to the Liouville-Cartan pair on $d\overline{\mathscr{F}} \setminus ST\overline{\Sigma}_0$.

Given a diffeomorphism of M^3, we can use it to change the map from M^3 to $d\mathscr{F} \setminus ST\Sigma_0$. Hence the pullback of a Liouville-Cartan structure under any diffeomorphism is again a Liouville-Cartan structure.

Notice that we already obtain all Liouville-Cartan structures if we restrict ourselves to coverings $M^3 \to d\mathscr{F} \setminus ST\Sigma_0$ where the inverse images of Seifert fibres are connected.

The principle found in the proof of Theorem 7.1 is that there is a relation, going both ways, between Liouville-Cartan structures arising from 2-dimensional space forms and discrete, cocompact groups Γ whose image Γ' is discrete. Let us examine this relation more carefully.

Suppose $\Gamma \subset \mathscr{G}$ is a discrete, cocompact subgroup whose image Γ' is discrete. We have at least one pair $(\Sigma_0, \mathscr{F})$ for which there is a covering

$$\Gamma \setminus \mathscr{G} \longrightarrow d\mathscr{F} \setminus ST\Sigma_0,$$

constructed as in the proof of Theorem 7.1. Because of the invariance of the Liouville-Cartan pair under the maps $d\gamma$, the construction in Definition 7.2, applied to Σ_0, defines a Cartan structure on $\Gamma \setminus \mathscr{G}$ which only depends on Γ, and not on the particular choice of normal subgroup $\Gamma'_0 \subset \Gamma'$. This also means that different pairs $(\Sigma_0, \mathscr{F})$ can yield the same Cartan structure, as the orbifold Σ can have several descriptions $\Sigma = \mathscr{F} \setminus \Sigma_0$.

In sum, we have a Liouville-Cartan structure associated with every group Γ whose image Γ' is discrete. Recall from Section 6 that the standard Cartan structure is the one induced from the standard complex 1-form $z_1 dz_2 - z_2 dz_1$ for S^3, $e^{-\widetilde{w}}dz$ for $\widetilde{SL}_2$, and $e^{-w}dz$ for $\widetilde{E}_2$, respectively. Then the real and imaginary parts of the standard form are the lifts to $\mathscr{G}$ of the Liouville-Cartan forms on the unit tangent bundle of the corresponding simply-connected space form $\widetilde{\Sigma}$. We conclude that the Liouville-Cartan structures, arising form surfaces with metrics of constant curvature, are precisely the standard Cartan structures for the groups Γ whose image Γ' is discrete.

For a first description of the other Liouville-Cartan structures, the use of homothety classes is most convenient.

Proposition 7.3 *Given any homothety class of taut contact circles, the set of Cartan structures in this class is, if not empty, of the homotopy type of S^1 and its inclusion into the whole homothety class is a homotopy equivalence.*

If the homothety class contains one Liouville-Cartan structure, then all Cartan structures in this class are Liouville-Cartan structures, and they are in bijection (up to constant rotation) with the $\mathscr{F}$-invariant metrics within some conformal class, where $\mathscr{F}$ is a finite group acting on a compact real surface.

The homothety classes containing Liouville-Cartan structures are the standard homothety classes for the groups Γ whose image Γ', in the corresponding isometry group, is discrete.

Proof. If (ω_1, ω_2) is a Cartan structure, then so are the rotates by any constant angle. In fact, there is a unique vector field Y such that $Y \rfloor d\omega_1 = -\omega_2$ and $Y \rfloor d\omega_2 = \omega_1$, and then the constant rotates are the pullbacks of the original pair under the maps of the flow of Y. In particular, the constant rotates of a Liouville-Cartan structure are diffeomorphic Liouville-Cartan structures.

It is trivial to check that given a positive smooth function v, the pair $(v\omega_1, v\omega_2)$ is also a Cartan structure if and only if v is constant along the integral curves of the common kernel $\ker \omega_1 \cap \ker \omega_2$. The set of such positive functions v is a convex cone. Thus the set of Cartan structures within a homothety class is, if not empty, connected and of the homotopy type of S^1, and the inclusion of this set into the whole homothety class is a homotopy equivalence.

As to the size of the set of Cartan structures within a homothety class, this depends on the 1-dimensional foliation tangent to the common kernel. Clearly the curves of this foliation are all closed for Liouville-Cartan structures, forming a Seifert fibration over the orbifold $\Sigma = \mathscr{F} \setminus \Sigma_0$. Then the positive functions v constant along the fibres are the pullbacks of positive, $\mathscr{F}$-invariant, smooth functions on Σ_0. It is rather obvious that multiplying a Liouville-Cartan structure by such a function amounts to the same as passing in Σ_0 from an $\mathscr{F}$-invariant Riemannian metric to another $\mathscr{F}$-invariant metric conformal to it. So if a Cartan structure is a Liouville-Cartan structure, then all Cartan structures homothetic to it are also (constant rotates of) Liouville-Cartan structures, and this set is in natural bijection (up to the constant rotation) with a conformal class of $\mathscr{F}$-invariant metrics on Σ_0. Since any such conformal class admits a constant curvature representative, we also have that the homothety classes containing a Liouville-Cartan structure are precisely the standard homothety classes for the groups Γ whose image Γ' is discrete.

This proves the proposition.

We can give an even more explicit description of which homothety classes contain Liouville-Cartan structures. This is done in the following theorem.

Theorem 7.4 *If M^3 is a non-abelian quotient of S^3, a left-quotient of $\widetilde{SL}_2$, or a torus bundle over S^1 with non-trivial periodic monodromy, then every homothety class on M^3 contains a Liouville-Cartan structure. In particular, the common kernel has all of its orbits closed and all Cartan structures on M^3 are Liouville-Cartan structures.*

If M^3 is the torus T^3, the homothety classes containing Liouville-Cartan structures are those with all the integral curves of the common kernel closed.

If M^3 is a lens space $L(m, m-1)$, including S^3, then the Cartan structures with all the integral curves of the common kernel closed are the Cartan structures for which the Hopf surface $S = M^3 \times S^1$ is elliptic. Their homothety classes form the rational part $\langle R\rangle \setminus (\mathbb{Q} \cap (0,1))$ of the moduli space of homothety classes of type $(1'')$, and only the point $1/2$ represents a homothety class containing Liouville-Cartan structures.

In all cases, every taut contact circle on M^3 is homotopic to a Liouville-Cartan structure.

Proof. On $\Gamma \setminus S^3$, with Γ non-abelian, any homothety class contains a standard Cartan structure and every Cartan structure is a Liouville-Cartan structure.

On a lens space $L(m, m-1)$ we have the Cartan structures induced by

$$a z_1 dz_2 - (1-a) z_2 dz_1,$$

where $a \in (0,1)$. For $a = 1/2$ we get the standard Cartan structure, which is the Liouville-Cartan structure when we consider the cyclic group Γ_m as lift of a cyclic group of rotations of S^2 around its poles, and any Cartan structure in this homothety class is a Liouville-Cartan structure up to constant rotation. The homothety classes of type $(1'')$ on $L(m, m-1)$ with $a \neq 1/2$ do not contain any standard Cartan structure and so they do not contain Liouville-Cartan structures; yet their common kernel, defined as $\ker\omega_1 \cap \ker\omega_2$ for any taut contact circle (ω_1, ω_2) in the homothety class, has all integral curves closed precisely if a is rational. Thus on these lens spaces we have infinitely many Cartan structures with all integral curves of the common kernel closed, but which do not contain any Liouville-Cartan structures in their homothety classes.

In the case of $\Gamma \subset \widetilde{SL}_2$, the image Γ' in PSL_2 is always discrete, and we have seen in Section 6 that every homothety class contains a standard Cartan structure. So in this case every homothety class contains a Liouville-Cartan structure, and indeed all Cartan structures are (constant rotates of) Liouville-Cartan structures.

Consider now Cartan structures on the torus T^3. We have seen in Section 6 that standard Cartan structures exist in every homothety class on T^3. The possible complex tori $S = T^3 \times S^1$ arising from taut contact circles correspond to the lattices in $\mathbb{C}^2$ of the form

$$\langle (1,0), (z_0,0), (z_1, 2\pi i r_1), (0, t_0) \rangle,$$

where z_0 is on the upper half plane, z_1 is arbitrary, and r_1 is any positive integer. The homothety class is induced by the standard complex form $e^{-w}dz$. We then get a Cartan structure on T^3 if we construct T^3 as the quotient of $\mathbb{R}^3 = \mathbb{C} \times (i\mathbb{R})$ under the lattice

$$\Gamma = \langle (1,0), (z_0,0), (z_1, 2\pi i r_1) \rangle,$$

whose image in E_2 is

$$\Gamma' = \langle 1, z_0, z_1 \rangle,$$

which is discrete if and only if $qz_1 \in \langle 1, z_0 \rangle$ for some positive integer q.

The standard form is $\omega_1 + i\omega_2 = e^{-w}dz$, and so for any homothety class on T^3, the common kernel $\ker \omega_1 \cap \ker \omega_2$ is spanned by ∂_θ, where θ is the imaginary part of w. Thus the integral curves of the common kernel are closed if and only if $qz_1 \in \langle 1, z_0 \rangle$ for some positive integer q. Thus for the torus T^3 the homothety classes containing Liouville-Cartan structures are those for which the integral curves of the common kernel are closed.

We take q minimal, that is, equal to the order of z_1 modulo $\langle 1, z_0 \rangle$. We consider the 1-dimensional complex torus

$$\Sigma_0 = \langle 1, z_0, z_1 \rangle \setminus \mathbb{C},$$

then M^3 is the 3-torus $T^3 = \Sigma_0 \times S^1$ which covers $ST\Sigma_0$ by wrapping the S^1-factors $r_0 = qr_1$ times around the fibres of $ST\Sigma_0$.

Let now $m = 2, 3, 4, 6$. We want to describe the Liouville-Cartan structures on the torus bundle M^3 over S^1 with monodromy of period m. Let ε, $L \subset \mathbb{C}$ and $\Sigma_0 = L \setminus \mathbb{C}$ be as in the proof of Theorem 7.1, so that ε is also considered as an isometry of Σ_0. The hyperelliptic surfaces that can arise from taut contact circles are the quotients of $\mathbb{C}^2$ under the group generated by the lattice

$$L + (0, 2\pi i r_0)\mathbb{Z} + (0, t_0)\mathbb{Z},$$

and the transformation

$$(z, w) \mapsto (\varepsilon z, w + w_0),$$

where w_0 is a translation of $\langle t_0, 2\pi i r_0 \rangle \setminus \mathbb{C}$ of order exactly m.

Notice that the torus which covers M^3 has $z_1 = 0$ in the preceding description. The image Γ' is generated by L and ε and is discrete. At this point we have finished proving Theorem 7.4, but we shall take this opportunity to describe in more detail the homothety classes on M^3, and how this manifold relates to the unit tangent bundle of a 2-torus.

The condition $\log \varepsilon \equiv w_0 \pmod{2\pi i}$ must be satisfied for the transformation to belong to $\widetilde{E}_2$. This is equivalent to $w_0 = 2\pi i p/m$ for some integer p satisfying the congruence $p \equiv 1 \pmod m$. Then the condition that the translation of $\langle t_0, 2\pi i r_0\rangle \backslash \mathbb{C}$ induced by w_0 be of order m requires that $p = p_0 r_0$ for some integer p_0, and that p_0 be coprime with m. All this amounts to solving the congruence $p_0 r_0 \equiv 1 \pmod m$. Given r_0, there exists a p_0 satisfying the congruence if and only if $\gcd(r_0, m) = 1$, and then p_0 is unique modulo m, which means that w_0 is uniquely determined, as a translation of $\langle t_0, 2\pi i r_0\rangle \backslash \mathbb{C}$, by r_0. The values for r_0 admissible here are not arbitrary as for T^3, they are the integers coprime with m.

The transformation can be rewritten as follows,

$$(z, w) \mapsto (\varepsilon z, w + 2\pi i q + 2\pi i/m),$$

for some integer q. We obtain M^3 as the quotient of $\mathbb{R}^3 = \mathbb{C} \times (i\mathbb{R})$ under the lattice

$$L + (0, 2\pi i r_0)\mathbb{Z}$$

and the transformation

$$(z, i\theta) \mapsto (\varepsilon z, i\theta + 2\pi i q + 2\pi i/m),$$

which is a lift to the universal cover of $ST\Sigma_0$ of the differential $d\varepsilon$. For fixed m and L, denote the resulting quotient by $M^3(r_0)$. Then there is an obvious r_0-sheeted covering map

$$M^3(r_0) \longrightarrow M^3(1),$$

and a natural isomorphism between $M^3(1)$ and $\langle d\varepsilon\rangle \backslash ST\Sigma_0$. The composition of the maps

$$M^3(r_0) \longrightarrow M^3(1) \cong \langle d\varepsilon\rangle \backslash ST\Sigma_0 \longrightarrow \langle \varepsilon\rangle \backslash \Sigma_0$$

endows $M^3(r_0)$ with a Seifert fibration over the orbifold $\Sigma = \langle \varepsilon\rangle \backslash \Sigma_0$.

8 Taut contact spheres

In this section we prove Theorem 1.10. Let M^3 be a closed 3-manifold and $(\omega_1, \omega_2, \omega_3)$ a taut contact 2-sphere on M^3. Set $\Omega_i = d(e^t \omega_i)$, $i = 1, 2, 3$, on $M^3 \times \mathbb{R}$. The Ω_i are symplectic forms on $M^3 \times \mathbb{R}$ that satisfy

$$\Omega_1^2 = \Omega_2^2 = \Omega_3^2 \ (\neq 0),$$

$$\Omega_i \wedge \Omega_j \equiv 0 \ \text{ for } \ i \neq j.$$

Such a triple $(\Omega_1, \Omega_2, \Omega_3)$ was called a *conformal symplectic triple* in [6]. It is shown there, by a straightforward extension of the argument in Section 3 of the present paper, that such a conformal symplectic triple induces a hyperkähler structure $(J_1, J_2, J_3; g)$ on $M^3 \times \mathbb{R}$ with

$$g = \Omega_i(\cdot, J_i \cdot),$$

where the J_i are ∂_t-invariant as in Corollary 3.3. Indeed, J_k is the complex structure constructed from the taut contact circle (ω_i, ω_j), where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Since the complex structures J_i and the non-degenerate 2-forms $e^{-t}\Omega_i$ descend to $M^3 \times S^1$, so does the Riemannian metric $e^{-t}g$. This implies that $M^3 \times S^1$ is a hyperhermitian manifold. By Boyer [4], $(M^3 \times S^1, e^{-t}g)$ is conformally equivalent to a complex torus with its flat metric, a $K3$ surface with a hyperkähler Yau metric, or a coordinate quaternionic Hopf surface with its standard conformally flat metric.

Clearly $K3$ surfaces cannot arise from this construction since they are not diffeomorphic to a 4-manifold of the form $M^3 \times S^1$ (the $K3$ surfaces have non-zero Euler class c_2).

If M^3 admits a taut contact circle and $M^3 \times S^1$ is diffeomorphic to a Hopf surface, then we have seen that M^3 actually admits a taut contact 2-sphere.

Thus it remains to show that a complex torus does not arise from this construction, in other words, that T^3 does not admit a taut contact 2-sphere.

Assuming that it did, we lift Ω_i, J_i, $e^{-t}g$ from $M^3 \times \mathbb{R}^1$ to the universal cover $\mathbb{C}^2$. Boyer's theorem tells us that $e^{-t}g$ is equal to the flat metric g_0 up to some conformal factor $\lambda : \mathbb{C}^2 \to \mathbb{R}^+$. However, since both $\Omega_i^0 = g_0(J_i \cdot, \cdot)$ and

$$\Omega_i = g(J_i \cdot, \cdot) = \lambda e^t g_0(J_i \cdot, \cdot) = \lambda e^t \Omega_i^0$$

are symplectic forms on $M^3 \times \mathbb{R}$, we see that $\lambda = e^{-t}\lambda_0$ for some positive constant λ_0. For

$$d\Omega_i = d(\lambda e^t \Omega_i^0) = d(\lambda e^t) \wedge \Omega_i^0$$

is identically zero if and only if λe^t is a constant, since (in the tangent space at any point of $M^3 \times \mathbb{R}$) there is a non-degenerate 2-dimensional subspace with respect to the symplectic form Ω_i^0 in the (at least) 3-dimensional space $\ker d(\lambda e^t)$.

Hence, $g = \lambda_0 g_0$ descends to $M^3 \times S^1$ (since $M^3 \times S^1$ is the quotient of $\mathbb{C}^2$ under a lattice Γ of translations), but so does $e^{-t}g$, which is absurd.

This completes the proof of Theorem 1.10.

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