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Nonlocality and Gleason's Lemma. Part I. Deterministic Theories

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J. S. Bell's classic 1966 review paper on the foundations of quantum mechanics led directly to the Bell nonlocality theorem. It is not widely appreciated that the review paper contained the basic ingredients needed for a nonlocality result which holds in certain situations where the Bell inequality is not violated. We present in this paper a systematic formulation and evaluation of an argument due to Stairs in 1983, which establishes a nonlocality result based on the Bell–Kochen–Specker "paradox" in quantum mechanics.

1. The classic 1966 review paper on the foundations of quantum mechanics (QM) by J. S. Bell[3] is perhaps best known for two things. First, the paper shows how the existence of the 1952 Bohm hidden variable (h.v.) theory is consistent with, and reveals the shortcomings of, von Neumann's formal 1932 "no-go" result for h.v. interpretations of QM. Second, it raises the question (which Bell himself was to answer elsewhere) whether the manifest nonlocality of the Bohm theory is not a feature of *all* allowed h.v. theories that reproduce the quantum statistics.

What Bell did not realize at the time was that there is a direct link between the nonlocality issue and another aspect of the h.v. question that he examined in his review paper. Following a suggestion of Jauch, Bell had drawn attention to the implications of important algebraic work due to

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[3] See Ref. 1. We are pleased to have this opportunity to dedicate our paper to John Bell for the leading role he has played—and hopefully will continue to play—in clarifying the foundations of quantum mechanics.

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Gleason[2] in 1957, which concerned the possible measures defined on the closed subspaces of a Hilbert space. He demonstrated that a class of h.v. theories—those now commonly called "noncontextualist"—are rendered strictly inconsistent by Gleason's work, when such theories are applied to quantum systems whose associated Hilbert spaces have dimension three or greater. Furthermore, Bell provided a new, relatively simple proof of this result which does not depend on Gleason's work, an achievement repeated in a more striking fashion in independent work by Kochen and Specker[3] in 1967. This no-go result has frequently been referred to as the Kochen–Specker "paradox" (although the Bell–Kochen–Specker paradox is perhaps a fairer term).

Now, it was shown in 1983 in a lengthy argument by Heywood and Redhead[4] that a new Bell–Kochen–Specker (BKS) paradox arises in the case of pairs of distant spin-1 systems for a certain kind of *consistent*, *local* h.v. theory. The Heywood–Redhead argument was not entirely algebraic (as was the original BKS paradox); it required taking into account the probabilistic (correlation) predictions associated with the quantal state of the spin-1 systems. Then, in a later 1983 paper on quantum logic, Stairs[4] demonstrated how the Heywood–Redhead result can be generalized and the proof greatly simplified. The aim of the present paper is to present a more systematic version of the nonlocality result based on Stairs' construction, and to compare its implications with those of the well-known Bell nonlocality theorem in QM. To do this, we must first fix our terms.

2. In any h.v. theory, the existence is postulated for any quantum system of an apparently uncontrollable parameter (or parameters), whose (joint) value λ_t at any time t determines, or partially determines, the outcome of a measurement of any observable of the system were it performed at t. In the case of partial determination, features (controllable or otherwise) of the measurement device may also play a causal role. In a deterministic h.v. theory, once all the values of the relevant parameters are fixed at t, the predicted outcome of a measurement of any observable A—we shall denote it by $[A]^t$—is uniquely determined. (We leave open the separate issue whether λ_t evolves deterministically or stochastically with t for the system in the absence of measurements.) An ensemble of such systems in some pure quantal state ψ_t will correspond to a mixture of such uncontrollable λ_t states, their distribution being related to ψ_t. The averages $\overline{[A]}^t$ defined by that distribution are assumed to coincide with the QM averages $(\psi_t, \hat{A}\psi_t)$ for all A, where $\hat{A}$ is the self-adjoint operator associated with A.

[4] See Ref. 5. Of his proof, Stairs writes (p. 581): "...apparently a version of it was first offered by Simon Kochen, though his version never appeared in print."

A *noncontextualist* deterministic h.v. theory is one in which (i) there is a one-to-one correspondence between the set of all *bona fide* observables of the quantum system and some subset (not necessarily proper) of the set of self-adjoint operators on the Hilbert space $\mathscr{H}$ associated with it, (ii) for all A, $[A]^t$ is determined exclusively by λ_t, independently of the measurement context, and (iii) for any observables A, B where $B=f(A)$, then $[B]^t=f([A]^t)$.[5] It is such theories, when $\mathscr{H}$ has dimension three or greater, that first Bell and then Kochen and Specker showed to be incompatible with Gleason's 1957 work. Now assumption (ii) specifically implies that $[A]^t$ does not depend on whether A is being measured simultaneously with some compatible observable B, or with some other such observable C. When B and C are themselves incompatible, $[A]^t$ is thus insensitive to the choice between mutually exclusive measurement contexts. In what Shimony[(7)] was later to call a "judo-like manoeuvre," Bell[(1)] invoked the Bohrian principle of the totality of the quantum phenomenon to cast doubt on assumption (ii), and hence deflate the impact of the BKS paradox in its original version.

Notice that for the observables A, B, C to have the compatibility relations attributed to them above ($[\hat{A}, \hat{B}]=[\hat{A}, \hat{C}]=0$; $[\hat{B}, \hat{C}]\neq 0$), it is necessary that A be degenerate, or as it is sometimes said, A must be *nonmaximal.* It is demonstrable that such "noncontextualist" h.v. theories which assign well-defined values $[A]^t$ to only the maximal (nondegenerate) observables A are internally consistent; indeed there are models of such theories.[6] Much less trivially, it is also known[(9)] that the extension of such h.v. theories to *locally maximal* observables of composite systems is likewise consistent. (For a composite system consisting say of two systems 1 and 2 with respective Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$, an observable is said to be locally maximal if it corresponds to an operator of the type $\hat{A}\otimes\hat{1}$ or $\hat{1}\otimes\hat{B}$ on $\mathscr{H}_1\otimes\mathscr{H}_2$, where $\hat{A}$ ($\hat{B}$) is nondegenerate in $\mathscr{H}_1$ ($\mathscr{H}_2$), and $\hat{1}$ denotes the identity operators in both $\mathscr{H}_1$ and $\mathscr{H}_2$.) It is this result, incidentally, which assures the consistency of the assumptions involved in introducing a local (and hence noncontextualist) h.v. theory of correlated spin-1/2 systems in Bell's celebrated 1964 derivation of the Bell inequality.[7]

Finally, we must mention that part of Gleason's 1957 work which is relevant to this paper. Gleason introduced the notion of a *frame function of weight W* for a separable Hilbert space $\mathscr{H}$. This is a real-valued function defined on (the surface of the) unit sphere of $\mathscr{H}$, such that if $\{\phi_1\}$ is any orthonormal basis of $\mathscr{H}$, then $\sum_i f(\phi_i)=W$. Gleason showed that any

[5] Such a h.v. theory is called one of the "zeroth kind" in Ref. 6.
[6] See, e.g., Ref. 8.
[7] This point is discussed at greater length in Ref. 10.

nonnegative frame function in $\mathbb{R}^3$ is continuous; we shall refer to this result as *Gleason's Lemma.*[8]

3. Consider a composite quantum system composed of two separated subsystems 1, 2 (with zero interaction potential), which have interacted in the past, the composite system being in the "hidden" state λ_t consistent with its pure quantal state ψ_t at time t. We make the following assumptions, *which are restricted throughout to locally maximal observables* for the system:

I (Separability). Any observables $A\otimes 1$, $1\otimes B$ are single-valued functions of λ_t, with respective values $[A\otimes 1]^t$, $[1\otimes B]^t$. These values correspond to the outcomes of (any kinds) of measurements designed to measure separately A on system 1, and B on system 2, were they performed at t.

II (Locality). $[A\otimes 1]^t$ is insensitive to the choice between measurements at t of any mutually incompatible observables (maximal or otherwise) on 2. A similar condition holds for $[1\otimes B]^t$ and distant measurements at system 1. (It is not required that λ_t factorizes into a product of $\lambda_{1,t}$, $\lambda_{2,t}$ states.)

III (Excluded Joint Events). Let $P^{\psi_t}(A=a, B=b)$ denote the joint probability in QM of obtaining the pair of values a, b in measurements of A, B performed at t on systems 1, 2, respectively, for the pure state ψ_t of the composite system. Then if

$$P^{\psi_t}(A=a, B=b)=0$$

we have either $[A\otimes 1]^t\neq a$, or $[1\otimes B]^t\neq b$, or both.

Now it can easily be shown from Assumptions I, III that $[A\otimes 1]^t$ and $[1\otimes B]^t$ always belong to the spectra of $\hat{A}$ and $\hat{B}$, respectively (we omit the proof).[9] This is of course a necessary condition for the h.v. theory to be consistent with QM.

We are now in a position to prove the main result.

Theorem. A nonstochastic h.v. theory satisfying the above assumptions is inconsistent with the QM description of pairs of distant spin-1 systems prepared in the singlet state of total spin zero.

[8] This result is contained in the proof of the theorem in Sec. 2.8 in Ref. 2. It is straightforward to show that the result holds for frame functions defined on any Hilbert spaces of dimension three or greater; see Ref. 11, which also contains a discussion of the main theorem in Ref. 4.

[9] We thank Arthur Elby for pointing out that the "spectrum rule" is not required as an independent assumption.

Proof. (Stairs 1983). Consider a triad θ of mutually orthogonal directions $\{x, y, z\}$, and the locally maximal observable $H^\theta \otimes 1$, where

$$\hat{H}^\theta = h_x \hat{P}_{|S_x=0\rangle} + h_y \hat{P}_{|S_y=0\rangle} + h_z \hat{P}_{|S_z=0\rangle}$$

on $\mathscr{H}_1$ and h_x, h_y, h_z are distinct real numbers. $\hat{P}_{|\cdot\rangle}$ is the projection operator $|\cdot\rangle\langle\cdot|$ on $\mathscr{H}_1$, S_x the component of spin in the x direction, etc. Following Heywood–Redhead and Stairs, we could choose H^θ to be the spin Hamiltonian operator $aS_x^2 + bS_y^2 + cS_z^2$, so that $h_x = b + c$, $h_y = a + c$, $h_z = a + b$.[10] Consider also the locally maximal observable $1 \otimes S_{\hat{n}}$, the component of spin in the direction defined by the unit vector $\hat{n}$, for system 2.

For all θ, $\hat{n}$, t, the predictions $[H^\theta \otimes 1]^t$, $[1 \otimes S_{\hat{n}}]^t$ are well defined (Assumption I), and $[H^\theta \otimes 1]^t \in \{h_x, h_y, h_z\}$, $[1 \otimes S_{\hat{n}}] \in \{-1, 0, 1\}$. Moreover, from Assumption II, $[H^\theta \otimes 1]^t$ is unaffected by the choice of measurement being performed at t on 2.

We assume that the composite system is in the singlet state at t, which may be written (ignoring spatial degrees of freedom) as

$$|\psi\rangle = -\frac{1}{\sqrt{3}}[|S_x=0\rangle \otimes |S_x=0\rangle - |S_y=0\rangle \otimes |S_y=0\rangle + |S_z=0\rangle \otimes |S_z=0\rangle]$$

Let us furthermore suppose that $[H^\theta \otimes 1]^t = h_x$, say. Since we have

$$P^\psi(H^\theta = h_x, S_x = \pm 1) = 0$$

and

$$P^\psi(H^\theta = h_x, S_{y,z} = 0) = 0$$

we conclude from Assumption III that $[1 \otimes S_x]^t = 0$, and $[1 \otimes S_{y,z}]^t \in \{1, -1\}$. Thus, to each of the spatial directions x, y, z in θ, we can assign a number ($[1 \otimes S_{\hat{n}}]^t$, $\hat{n} = x, y, z$) from the set $\{-1, 0, 1\}$, such that one and only one direction is assigned 0. It is clear that this generic result for θ does not depend on the original choice above $[H^\theta \otimes 1]^t = h_x$. Furthermore, the rotational symmetry of ψ implies that the result holds for all triads θ.

Thus, the value assignment $[1 \otimes S_{\hat{n}}]^t$ generated by λ_t induces a map from the surface of the unit sphere in $\mathbb{R}^3$ onto $\{-1, 0, 1\}$, with the

[10] See Ref. 11, Secs. 1.7 and 1.8. A proposal concerning how the spin Hamiltonian might be measured in the case of an orthohelium atom in its lowest orbital state was provided in Ref. 3.

property that for any orthogonal triple of points on the sphere, one and only one point is mapped to 0. Similarly, $\hat{n} \mapsto ([1 \otimes S_{\hat{n}}]^t)^2$ generates a map from the unit sphere in $\mathbb{R}^3$ onto $\{0, 1\}$ with the same property. This map constitutes a nonnegative, discontinuous frame function of weight 2 in $\mathbb{R}^3$, which is excluded by Gleason's Lemma.

4. The advantages of the above proof in relation to that of Heywood and Redhead (apart from its considerable gain in simplicity) are twofold. First, it refers exclusively to locally maximal observables. This means that it is applicable to the entire range of "contextualist" h.v. theories which differ only concerning their treatment of *locally nonmaximal* observables. In particular, the proof does not require (as does Heywood and Redhead's) adoption of van Fraassen's[(12)] suggestion that to each such "observable," or rather its associated operator, there corresponds an uncountable infinity of distinct measurable physical magnitudes, in violation of assumption (i) in Sec. 2 above.

Secondly, it is not required that $[A]^t$, for every observable A, be interpreted to represent an objective element of reality (premeasurement value) associated with A, which the measurement process somehow faithfully reveals at t. Such a view is decidedly at odds with Bohm's[11] 1952 h.v. theory, and Bell[(15)] has urged its relaxation in the hidden variables program generally.

In fact, the "faithful measurement" hypothesis is clearly incompatible with the possibility, which we alluded to in Sec. 2, that $[A]^t$ for some if not all observables A is determined by the *pair* λ_t, μ_t, where μ_t is the initial "hidden" state of the measurement device employed to measure A at t. We shall refer to theories which incorporate this possibility as (nonstochastic) *Bohm-type theories*, since they capture an essential feature of Bohm's 1952 model. Now it has recently been shown by Elby[(16)] that the quasi-algebraic proof of nonlocality above can be generalized to *stochastic* h.v. theories, including the stochastic version of a Bohm-type theory. It turns out as a consequence of the singlet state correlations that a local stochastic h.v. theory collapses to a deterministic one for the observables in question (analogously with the situation for spin-1/2 systems), at least if the Assumptions in Sec. 3 above are taken to hold for all possible λ_t. Furthermore, local μ-state dependency must disappear for the particular measurement events under consideration. (For details we refer the reader to Elby's paper.[(16)])

5. We turn finally to a comparison of the quasi-algebraic nonlocality proof with the Bell nonlocality theorem for nonstochastic h.v. theories.

[11] See Ref. 13. For a review of more recent developments, see Ref. [14].

(a) Following the precedent set in Bell's seminal 1964 paper,[17] subsequent derivations of a deterministic Bell-type theorem by other authors have usually referred to local h.v. theories whose predictions depend only on the state λ_t of the object system, as in Assumption I in Sec. 3. What is demonstrated is that certain predictions of such a theory for pairs of spin-1/2 particles in the quantal singlet state must violate QM. These predictions concern the upper bound on the absolute value of a certain linear combination of correlation coefficients associated with selected pairs of (locally maximal) spin components on the two systems.

However, in 1971 Bell[18] generalized his theorem to Bohm-type deterministic h.v. theories. In deriving an expression for each of the required correlation coefficients, Bell averaged over the uncontrollable μ-parameters associated with the spin measuring devices, and considered products of averaged spin components of the form $\overline{[S_{\hat{n}}\otimes 1]^t}\cdot\overline{[1\otimes S_{\hat{n}'}]^t}$—written simply as $\overline{[S_{\hat{n}}]^t}\cdot\overline{[S_{\hat{n}'}]^t}$—associated with each of the possible λ_t states of the particles. He required only that $\overline{[S_{\hat{n}}]^t}$ be independent of $\hat{n}'$, the "setting" of the distant apparatus, and similarly for $\overline{[S_{\hat{n}'}]^t}$ and $\hat{n}$ (a consequence of the assumption that the initial distributions of the μ-states for the two devices are independent). We call this *average locality*. In then integrating over the λ_t states, Bell showed that such average locality is sufficient in the Bohm-type theory to derive a QM-violating inequality. However, in the course of his derivation, Bell implicitly assumed that $\overline{[S_{\hat{n}}\otimes S_{\hat{n}'}]^t}$ is equal to $\overline{[S_{\hat{n}}]^t}\cdot\overline{[S_{\hat{n}'}]^t}$. Such factorizability is not a consequence of average locality. In fact, attention to a recent detailed analysis[19] of the Bell experiment involving Stern–Gerlach devices reveals that factorizability of the Bell averages is violated in the Bohm theory, although average locality is not. Thus Bell's 1971 argument, as it stands, is incomplete.

It is straightforward to show that both factorizability and average locality are consequences of (i) the standard assumption that $[S_{\hat{n}}\otimes S_{\hat{n}'}]^t = [S_{\hat{n}}]^t\cdot[S_{\hat{n}'}]^t$, and (ii) the joint assumption that $[S_{\hat{n}}]^t$ is (the value of) a function whose only arguments are λ_t and the hidden μ-state of the *local* device, the form of the function itself being independent of $\hat{n}'$ (and similarly for $[S_{\hat{n}'}]$). Bohm's theory actually violates both parts of (ii), an assumption which Bell clearly expects to hold in a local Bohm-type theory. A similar assumption is required in the quasi-algebraic result for Bohm-type theories in the spin-1 case. Thus we may conclude that despite the apparent simplicity of his 1971 theorem, Bell's proof of the nonlocality of Bohm-type theories relies on assumptions no weaker than those in the corresponding quasi-algebraic proof.

(b) There is a sense, however, in which the quasi-algebraic proof fills a gap left by the Bell theorem. The latter makes essential use of pairs of spin components for the spin-1/2 particles whose correlation coefficient in QM do not reach the extremal values of ± 1. If consideration in the Bell argument is limited to pairs of spin components that are "perfectly" correlated, no violation of the QM predictions follows.

The situation with correlated spin-1 systems is somewhat different. Although the observables $S_{\hat{n}} \otimes 1$, $1 \otimes S_{\hat{n}}$ are again perfectly correlated in the singlet state (such correlations also being consistent with locality), pairs of the type $H^{\theta} \otimes 1$, $1 \otimes S_{\hat{n}}$ $(\hat{n} \in \theta)$ are not. Yet, as we have seen, QM predicts zero joint probability for certain outcomes related to measurement of these latter observables. (A similar situation does not arise with spin-1/2 systems for observables not perfectly correlated.) As before, no Bell-type violation of QM can be demonstrated by consideration of such correlations alone for spin-1 systems, in a local h.v. theory. The theorem in Sec. 3 thus demonstrates the unavoidable existence of nonlocality in nonstochastic h.v. theories (including those of the Bohm variety) related to certain kinds of correlations in the quantum world that evade Bell's nonlocality result.[12]

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[12] A proof of nonlocality for *pairs of pairs* of spin-1/2 systems, which relies on perfect correlations alone, but which makes no appeal to the Gleason result, has recently been found by Greenberger *et al.* in Ref. 20. A generalization of this result is found in the paper by Clifton *et al.* in this set of issues.

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