Foundations of Physics, Vol. 17, No. 6, 1987

On the Concept of Time and the Origin of the Cosmological Temperature[1]

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Received December 11, 1986

Time arises in the theory of gravity through the semiclassical approximation of the gravitational part of the solution of the Wheeler–De Witt equation in the manner shown by Banks (SCAG). We generalize Banks' procedure by grafting a Born–Oppenheimer type approximation onto SCAG. This allows for the feedback of matter onto gravity, wherein the latter is driven by the (quantum) mean energy-momentum tensor of matter. The wave function is nonvanishing in classically forbidden configurations of gravity. In SCAG this is described by the evolution of matter in imaginary time. This is interpreted as an inverse temperature, and the norm of the matter wave function, no longer conserved for these gravitational configurations, is a partition function. A simple cosmological model is worked out to illustrate these ideas. In this model it is shown that the temperature of the matter which emerges into the classically permitted region is the inverse bounce time of the bounce executed by the system in the forbidden region (behind the horizon).

Time present and time past
are both perhaps present in time future.
And time future contained in time past.
If all time is eternally present
All time is unredeemable. — T. S. Eliot, "Burnt Norton," *Four Quartets*, 1943.

1. INTRODUCTION

Time does not appear in the Hamiltonian form of gravity. This is a consequence of the invariance of the action S under arbitrary space-time transformations, from which follows $\delta S/\delta g_{\mu\nu}=0$, where $g_{\mu\nu}$ is the space-time metric and $\delta S/\delta g_{\mu\nu}$ can be identified with the energy-momentum tensor of

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0015-9018/87/0600-0603$05.00/0

everything, gravity + matter; $\delta S/\delta g_{\mu\nu} = T^{\mu\nu} = 0$. The expression of this fact in the Schrödinger form of quantum mechanics is the Wheeler–de Witt equation $H\Psi = 0$, where H is the Hamiltonian of everything and Ψ the wave function. Perhaps, in a more familiar context, one may understand the vanishing of $T_{\mu\nu}$ from the electromagnetic analogy $SS/\delta A_{\mu} = 0$ which follows from gauge invariance. This result may be interpreted in terms of the total vanishing of electric current, that carried by matter and that carried by the field—explicitly called the displacement current by Maxwell. Each of these is, of course, separately conserved. And so it is with gravity.

From the above considerations it seems likely that the ultimate form of physics will retain this feature. One can hardly imagine at this point in scientific development that general covariance will be given up. Thus, even if the present theory of gravity is some phenomenological element of a fundamental theory—and it must be regarded as such because of its quantum illnesses—we may well expect $H_{\text{everything}} \Psi_{\text{everything}} = 0$. So the Schrödinger right-hand side is missing.

What then is time and how does it intrude into the seemingly static dynamics described by $H\Psi = 0$? The answer to this question has been recently found by Banks,[1] drawing, no doubt, on previous considerations of de Witt and Hawking.[2] (For simplicitly, I shall here concentrate on time as it comes up in cosmology, though Banks shows clearly how the idea of local time, such as a Towanaga local temporal displacement in field theory, arises from considerations of local gravitational fields). The conceptual answer of Banks is that time parametrizes how matter follows gravity. To say that the age of rocks on earth is 4×10^9 yrs really means that these rocks were created "when" the radius of the universe "was" such and so, corresponding to a particular value of the Robertson–Walker scale factor a. The latter is measured from the Big Bang where the scale of a is m_P^{-1} ($m_P \equiv$ Planck mass). (This is rather loose wording as the scale of a is determined by convention. One really means the scale of the visible universe.) The ultimate cosmological theory will give us the function $a(t)$; so, to state a value of t is tantamount to stipulating a. And it is a which figures in the gravitational metric. In this sense, time is calculated on gravitational configurations. One can only contemplate with admiration Mr. Eliot's remarkable poem, an expression of the poet's metaphysical sensing of what has to be.

How does one translate this idea into a mathematical formalism? The existence of a Hamiltonian implies an action, and action implies time, as can be inferred from its name. The energy eigenfunctions $\psi_E(x)$ are Fourier transforms in time of Feynman path integrals. In the semiclassical approximation, the paths are dominated by that of minimal action, and time is the classical time which parametrizes this classical path, the energy of which is E. Thus, if the semiclassical approximation to $\psi_E(x)$ is valid, the wave function "tells a story." Banks has shown how the semiclassical wave function for gravity provides a parametrization for the evolution of matter. The latter follows the former adiabatically, and since the former "tells a story," the latter follows the story. Banks shows that the way the story is followed is the time-dependent Schrödinger equation for matter.

In Banks' original version, he uses an approximation which is somewhat too crude, in that the gravitational field is not driven by the matter. I shall show below how to remedy this by grafting onto Bank's proposition a Born–Oppenheimer type approximation (I call this the BBO). I shall then show how BBO leads to Einstein's equation with gravity driven by the (quantum) mean energy momentum tensor of matter. The idea of all of this is that the mass scale of matter $\ll m_P$.

The word "semi" in semiclassical is not to be taken lightly. Much of quantum mechanics is still there. For one, if there are several classical paths between two points, such as in the presence of a reflecting barrier, the wave function is a superposition of these several contributions, and they interfere. For another, the wave function penetrates into classically forbidden regions (mathematically describable in terms of an imaginary or euclideanized time). Hartle and Hawking (for background and a review, see Hawking[2]), in their efforts to found quantum cosmology, have advocated the possibility that quantum gravity be formulated as a Euclidean functional integral. The classical stage of cosmology is then attained by analytic continuation to real times as one passes from the classically forbidden to the permitted region. In itself this is not such an audacious proposition; ordinary quantum mechanics can be so formulated. The center of the debate is not so much the issue of continuation in the complex time plane, but whether the universe can be described by a single wave function at all, a state which to some extent is characterized by the same properties of stability of a state of Minkowski quantum field theory. It is not my purpose here to enter into the polemics of this passionate question. Rather, I wish to explore some facets of the Hartle–Hawking hypothesis using Banks' construction.

In a simple model, a massless scalar field coupled conformally to gravity, I shall show how imaginary time in the forbidden region has the meaning of temperature. One recovers in this way the statistical thermodynamics of the model as previously worked out by Horwitz and Weil.[3] Indeed, it was in the course of discussion with these authors that it was found that their results were a consequence of a *single* Hartle–Hawking quantum state. The work presented here is based on the collaboration among the three of us.

That temperature does arise in single quantum states is, of course, well known since Hawking's seminal work on black hole radiation[2] and its extension to the de Sitter space by Gibbon and Hawking[2] (GH). The Horwitz–Weil (HW) phenomenon also occurs in a de Sitter type space, but it is distinct form that of GH. When it exists, HW prove it to be the more stable of the two. This is, however, a technical matter. François Englert has emphasized to me the generality of thermal phenomena in the presence of gravitational horizons as a fundamental fact of physics. We are here illustrating a general principle with the plaything of the present paper.

What makes this plaything a very poor one indeed is the other feature of semiclassical theory, interference. In the classically permitted region the pieces between the expanding and collapsing universe interfere in the wave function, hardly a respectable state of affairs. The question at hand once again transcends the consideration of our particular plaything. It is a matter of time reversal invariance, the presence of this collapsing universe. How does one get rid of it or at least of its interference with our own expanding universe? Casher and Englert[5] have suggested that a contracting universe be interpreted as an expanding anti-universe. Both are macroscopic and the interference will be washed out. Needless to say, much needs to be done to make this notion precise. To date, precious little has been done, and in this paper these deeper issues will not be joined.

The scientific contributions of Ilya Prigogine have greatly furthered our understanding of the arrow of time. The work presented here is an effort to understand time itself, a quantity which parametrizes sequences of events. In cosmology, the confrontation of the arrow of time, as defined on one hand by the mechanical notion of the expansion of the universe and on the other by the growth of entropy, is surely one of the deeper scientific questions of our day. Recently Prigogine has turned his ever active mind to this question, and we may well expect to find guidance from him, as we have on the other profound issues that he has elucidated in the past.

2. THE BBO APPROXIMATION

The why's and wherefore's of BBO are readily revealed by studying gravity in the cosmological context, i.e., by limiting oneself to mini-superspace, in technical jargon.

The classical theory of cosmology is based on the equation of energy balance

$$-H^2 + k/a^2 = -\frac{8\pi}{3} G\rho \tag{1}$$

Here a is the Robertson–Walker scale factor, ρ the energy density of matter, G the gravitational constant $\sim m_p^{-2} \sim 10^{-19} Gev^{-2}$, and H Hubble's constant $(=a^{-1}\, da/dt)$, t being the proper time. We shall usually employ η defined by

$$dt = a\, d\eta \tag{2}$$

The parameter $k = \pm 1$, 0 for closed, open, and flat spaces, respectively, thereby fixing the scale of a for the first two cases. A cosmological constant is included by allowing ρ to have a piece independent of a $(\equiv \rho_0)$, in which case we define

$$\frac{8\pi G}{3}\rho_0 \equiv \Lambda \tag{3}$$

Both G and Λ are supposed to contain the effects of matter through their renormalizations.

We rewrite (1):

$$H = H_G + H_M = 0 \tag{4}$$

$$\begin{aligned} H_G &= -\frac{m_P^2}{2}\left[\left(\frac{da}{d\eta}\right)^2 - ka^2\right] + \Lambda m_P^2 a^4 \\ H_M &= \frac{8\pi}{3}(\rho - \rho_0)\, a^4 \end{aligned} \tag{5}$$

We shall allow for the quantization of matter by replacing $(\rho - \rho_0)\, a^4$ by a general matter Hamiltonian

$$H_M = H_M(\phi, a) \tag{6}$$

where ϕ represents all matter fields; they are supposed to vary on mass scales $\ll m_P$.

Thus, H looks like the H of chemistry, gravity playing the role of nuclei and matter of electrons. The latter follow the former adiabatically. This is expressed through the Born–Oppenheimer type factorization of the wave function taken together with subsequent approximations. We write

$$H(a, \phi)\, \Psi(a, \phi) = [H_G(a) + H_M(\phi, a)]\, \Psi(a, \phi) = 0 \tag{7}$$

$$H_G = \frac{1}{2m_P^2}\frac{\partial^2}{\partial a^2} + \frac{km_P^2 a^2}{2} + \Lambda m_P^2 a^4 \equiv \frac{1}{2m_P^2}\frac{\partial^2}{\partial a^2} + V_G(a) \tag{8}$$

$$\Psi(a, \phi) \equiv \psi(a)\, \chi(a, \phi) \tag{9}$$

In (8), we have used the momentum corresponding to (4): $p =$

$\partial L/\partial(\partial a/\partial\eta) = -m_P^2(da/d\eta) \to i(\partial/\partial a)$. [Note that the gravitational kinetic energy is negative $(= -p^2/2m_P^2)$. This is what allows for inflation (as emphasized in Ref. 6).

Inserting (9) into (7) gives rise to

$$\frac{1}{2m_P^2}\left[\psi\frac{\partial^2}{\partial a^2}\chi + \chi\frac{\partial^2}{\partial a^2}\psi + 2\frac{\partial\chi}{\partial a}\frac{\partial\psi}{\partial a}\right] + \chi V_G\psi + \psi H_M\chi = 0 \tag{10}$$

In zeroth-order Born–Oppenheimer theory, the terms in $\partial\chi/\partial a$ and $\partial^2\chi/\partial a^2$ are neglected. Banks' innovation is to keep the term in $\partial\chi/\partial a$ but neglect $\partial^2\chi/\partial a^2$. Since the dependence of χ on a is accounted for in the Born–Oppenheimer series in $(m_{\text{light}}/m_{\text{heavy}})^{1/4}$, Banks has taken some of these corrections into account, but a systematic study has not yet been carried out. From now on, $\partial^2\chi/\partial a^2$ will be neglected. We multiply (10) by $\chi^\dagger$ and contract over ϕ to give

$$H_G(a)\psi + \langle H_M\rangle_a\psi + \frac{1}{2m_P^2}\left\langle\chi\left|\frac{\partial\chi}{\partial a}\right.\right\rangle\frac{\partial\psi}{\partial a} = 0 \tag{11}$$

Substitution back into (10) gives

$$[H_M - \langle H_M\rangle_a]\chi + \frac{1}{2m_P^2}\frac{\partial\ln\psi}{\partial a}\left[\frac{\partial\chi}{\partial a} - \left\langle\chi\left|\frac{\partial\chi}{\partial a}\right.\right\rangle\chi\right] = 0 \tag{12}$$

The equation we require for the semiclassical theory of gravity is that in which classical gravity is driven by the mean energy of matter. This is expressed by (11) provided we set the condition

$$\langle\chi\,|\,\partial\chi/\partial a\rangle = 0 \tag{13}$$

Inserting (13), we get

$$[H_G(a) + \langle H_M\rangle_a]\psi \equiv \left[\frac{1}{2m_P^2}\frac{\partial^2}{\partial a^2} + V_{\text{eff}}(a)\right]\psi = 0 \tag{14}$$

$$V_{\text{eff}} = V_G(a) + \langle\chi|\,H_M\,|\chi\rangle \tag{15}$$

and

$$[H_M - \langle H_M\rangle]\chi = -\frac{1}{m_P^2}\frac{\partial\ln\psi}{\partial a}\frac{\partial\chi}{\partial a} \tag{16}$$

The solutions of (16) automatically satisfy (13).

The next step is to require that $\psi(a)$ be the WKB solution of (14). In the absence of a reflecting barrier,

$$\psi = N^{-1/2} \exp[iS_{\rm cl}] \tag{17}$$

where $S_{\rm cl}$ is the classical action of gravity generated along the classical trajectory governed by the Hamiltonian $H_G + \langle H_M \rangle$. It is reckoned from some arbitrary point a_0. The quantity N is a normalization factor proportional to $|p(a)| = \sqrt{V_{\rm eff}(a)}$. It will be assumed throughout that the phase factor $\exp[iS_{\rm cl}(a)]$ varies rapidly compared to the variations of $V_{\rm eff}(a)$; thus, in what follows, the dependence of N on a will be neglected.

In the presence of a reflecting barrier, at say $a = a_0$, Eq. (17) is replaced by a sum of "in" and "out" waves

$$\psi = N^{-1/2} 2 \cos[S_{\rm cl}(a \,|\, a_0) + \pi/4], \qquad a > a_0 \tag{18}$$

where $2S_{\rm cl}(a \,|\, a_0)$ is the round trip action between a and a_0, $S_{\rm in}$ and $S_{\rm out}$ being equal, due to time reversal invariance. We have defined the classically permitted domain by $a > a_0$. For $a < a_0$, the WKB solution is

$$\psi = N^{-1/2} \exp\{-|S_{\rm cl}(a \,|\, a_0)|\} \tag{19}$$

where $S_{\rm cl}$ is still reckoned classically but with t replaced by it. The relative phases and constants in (18) and (19) are fixed, as written, by the Kramers connection formula. These formulas are elucidated in terms of temporal sequence by deriving them from the path integral. A sketch of this approach to obtain the WKB approximation is given in Appendix A. It is recommended that the reader be familiar with this technique for the understanding of Sect. 3, as well as to understand the meaning of time as a saddle-point parameter.

In what follows, ψ will be approximated by (17). In particular, it will be assumed that the interference term in (18) is washed out by some mechanism or other. Then

$$\frac{\partial \ln \psi}{\partial a} = i \frac{\partial S_{\rm cl}}{\partial a} = i p_G(a) = -i m_P^2 \dot{a} \tag{20}$$

so that (14) becomes

$$[H_M - \langle H_M \rangle] \chi = i \dot{a} \frac{\partial \chi}{\partial a} = i \frac{\partial \chi}{\partial t} \tag{21}$$

Define a new wave function

$$\tilde{\chi} = \chi \exp\left[-i \int^{t} \langle H_M \rangle \, dt' \right] \tag{22}$$

where dt' means $[(dt'/da')|_{\text{classical}}]\, da'$. Then

$$H_M \tilde{\chi} = i \frac{\partial \tilde{\chi}}{\partial t} \tag{23}$$

The full wave function Ψ is

$$\Psi = \psi(a)\, \chi(a, \phi) = \tilde{\psi}(a)\, \tilde{\chi}(a, \phi) \tag{24}$$

where

$$\tilde{\psi} = N^{-1/2} e^{iS_G(a)} \tag{25}$$

S_G is the classical action of the gravitational part alone. The form of the last equality in (24) is that proposed by Banks, but the meaning is different. The orbit on which the action is calculated is governed by $H_G + \langle H_M \rangle = 0$, whereas Banks calculates on the orbit $H_G = 0$. We include the back reaction of matter on the gravity.

The quantity $\tilde{\chi}$ is the usual Schrödinger wave function. In the classically permitted region its norm is constant, independent of a, hence of t. This in not true in the forbidden region and will be the heart of the matter of Sect. 3. This concludes the discussion of the origin of time and the derivation of Schrödinger equation. The crux of the matter is the adiabatic following of gravity by matter, which in turn feeds back to drive gravity. In cosmology, time begins at the Big Bang (provided one can handle the time-reversed universe).

If a reflection barrier (horizon) is present at, say $a = a_0$, then behind the barrier the "phase factor" of Eq. (19) is real and Schrödinger's equation is better likened to a diffusion equation. It is suggested that Im t be interpreted as an inverse temperature. Indeed, in the energy representation where

$$\tilde{\chi} = \sum C_n(t)\, \chi_n$$

and $H_M \chi_n = E_n \chi_n$, one sees that, if, at some time t_i, C_i were of the form $Ke^{-E_n t_i}$ (with K constant), then at every imaginary t it stays of this form $C_n = Ke^{-E_n(t+t_i)}$ We would then have

$$\langle H_M \rangle = \sum_n E_n e^{-2(t+t_i)E_n} \Big/ \sum_n e^{-2(t+t_i)E_n}$$

so that $2(t+t_i)$ would be an inverse temperature. So the problem is to find out whether or not there is a canonical distribution at some a. (The above consideration makes sense only if the time dependence of a in H_M is negligible).

In trying to solve this problem, we resorted to steepest-descent techniques and found that we were in fact redoing the original statistical analysis of HW on their model. Something like BBO arises but a rather surprising subtlety occurs which makes the evolution of temperature different from that of BBO. This is the subject of Sect. 3. The statistical analysis of HW is the subject of Appendix B.

3. TEMPERATURE

From (13) and its complex conjugate, the norm of χ is a fixed constant for all a. This implies that $\tilde{\chi}$, the Schrödinger function (23), shares the same property throughout the classical region (t real). But, for a in the forbidden region, the norm of $\tilde{\chi}$ varies. Here Im $t \neq 0$, and we have

$$\frac{\partial \langle \tilde{\chi} | \tilde{\chi} \rangle}{\partial \operatorname{Im} t} = 2\langle E_M(t) \rangle \tag{26}$$

Quantum physics of matter in its conventional form is drastically modified "behind the Big Bang." The product $\langle \tilde{\chi} | \tilde{\chi} \rangle$ appears as a sort of partition function, 2 Im t being something of an inverse temperature.

In BBO, the evolution of this "temperature" with a in the forbidden region would be found from the classical trajectory

$$H_G(a(t)) + \langle E_M(t) \rangle = 0, \qquad \operatorname{Im} t \neq 0 \tag{27}$$

As we have mentioned above, if matter were canonically distributed in energy at some t, it would remain so. Thus BBO defines an evolving canonical distribution in the forbidden region. The fundamental question, then, is whether matter is canonically distributed at some a. If so, is BBO a good approximation? Does (27) make good sense? How does one come upon such a thermal initial condition?

In grasping for answers, we explored the toy model of HW, set out in Appendix B. Though surely absurdly simple, it does have sufficient physics to allow one to formulate some conjectures, to wit: yes! for large systems matter *is* canonically distributed. The temperature *is* determined by a self-consistent equation. But it does not evolve in the BBO way, Eq. (27). In the HW model, E_M is kept fixed in the integration, equal to its value at the point a for which one is calculating the temperature, rather than varying as in (27). In general dynamic systems, this will not be true either; the variation of $\langle E_M \rangle$ in time will be due to the nonconformal coupling of matter to gravity. In all events, the BBO equation of evolution cannot be justified in the nonclassical region. I now display the argument for these assertions using the functional integral.

A convenient way to proceed is to investigate the normalization factor $\langle \tilde{\chi} | \tilde{\chi} \rangle$. For a general dynamic system described by coordinates (fields) x (our notation follows Appendix A), we have

$$\langle xt | x_0 t_0 \rangle = \sum_{n,\alpha} \psi_{n,\alpha}(x)\, \psi^{\dagger}_{n,\alpha}(x_0)\, e^{-iE_n(t-t_0)}$$
$$= \int \mathscr{D}(x)\, e^{iS(xt|x_0t_0)} \tag{28}$$

where α labels the degenerate states of E_n. For $x = x_0$, we take the Fourier transform to give

$$\sum_{\alpha} |\psi_{n,\alpha}(x)|^2 = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} dt\, e^{iE_n t} \int \mathscr{D}(x)\, e^{iS(xt|xt_0)} \tag{29}$$

(The notation is symbolic in that n is a continuous index, and $\delta(E - E_n)$ has been absorbed into the measure that defines $\sum_n$.) For our case, $E_n = 0$ and $x = a$, $\{\phi\}$, where $\{\phi\}$ means field configuration. Then

$$\sum_{\substack{\text{states of}\\ \text{zero energy}}} |\Psi(a, \{\phi\})|^2 = \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} dT \int \mathscr{D}'(a) \int \mathscr{D}'(\{\phi\})\, e^{iS(a,\{\phi\},T|a,\{\phi\},0)} \tag{30}$$

where S is the total action expended in the round trip from and to a, $\{\phi\}$ in time T. The prime is to remind us that the integral is not over the final (=initial) configuration (a, $\{\phi\}$). Script $\mathscr{D}$ without prime includes this last integration as well. In terms of it, the integral of (30) over $\{\phi\}$, which we denote by $P(a)$ (=probability to find the system at a), is[3]

$$P(a) = \int dT \int \mathscr{D}'(a)\, e^{iS_G(a,T|a,0)} \int \mathscr{D}(\{\phi\})\, e^{iS_M(a,\{\phi\},T|a,\{\phi\},0)} \tag{31}$$

There has been an interchange of orders of integration, which is in general not justified. Thus in the subsequent steepest-descent calculation over T

[3] In the present case, we are interested in the evolution of the vacuum state due to the variation of a. Thus, there is only one state contributing to the left-hand side of (30), the vacuum. We are thereby placing ourselves in the context of the Hartle–Hawking assumption of "the quantum state of the universe."

and $\mathscr{D}(a)$, the classical evolution of gravity will be determined by matter in the mean. In the correct order, each matter configuration would drive the classical gravity according to its particular foibles. Thus the interchange of orders is valid only if energy fluctuations are small, i.e., the matter system large. *This is how thermodynamics happens in cosmology.* It is hard to imagine any significance at all for a formulation in minisuperspace (i.e., cosmology in the usual sense of the mean as first formulated by Einstein) were this condition not fulfilled!

We now go to the semiclassical approximation of gravity by carrying out the steepest descents on a and T; thus

$$P(a) = |\tilde{\psi}(a)|^2 \langle \tilde{\chi} | \tilde{\chi} \rangle_a \tag{32}$$

where

$$\begin{aligned} |\tilde{\psi}(a)|^2 &= e^{iS_G(a, T^*(a) | a, 0)} \\ \langle \tilde{\chi} | \tilde{\chi} \rangle &= \int \mathscr{D}\{\phi\} \, e^{iS_M(\phi, a, T^* | \phi, a, 0)} \end{aligned} \tag{33}$$

and we recover the square of BBO [Eq. (25)]. It remains to discuss the nature of the classical closed orbit described in time $T^*(a)$.

Were there no classically forbidden region, both S_G and S_M would be real, and the classical time to go from a to a would vanish. Thus $P(a) = \text{const}$ (always neglecting the a dependence of N). If the wave function Ψ is calculated from some reference point a_0 in the WKB approximation, the above result merely reflects the fact that the phase factors on the round trip between a_0 and a cancel out in the computation of $|\Psi|^2$, the actions being equal and opposite.

In the classically forbidden region, S becomes imaginary and then $P(a)$ varies in a. Noting that Eq. (33) for $\langle \tilde{\chi} | \tilde{\chi} \rangle$ is indeed a partition function for T imaginary then yields

$$P(a) = e^{-S_G(a, T^*(a) | a, 0)} Z(T^*(a)) \tag{34}$$

where S_G is the gravitational action expended in the bounce time T^*, computed classically in the Euclideanized problem.

We apply this to the HW model, where the classically forbidden region is defined by $a_- \leqslant a \leqslant a_+$ corresponding to the two zeros of the effective potential:

$$V_{\text{eff}} = -a^2 + H^2 a^4 + \langle E_M \rangle$$

The bounce must be taken between a and a_- to have nonsingular energies.

(If it were taken between a and a_+, then $T^*(a_+)=0$, corresponding to infinite temperature.) The equation for $T^*(a)$ for the HW model is particularly simple if one makes the conformal transformation to the dimensionless field ϕ $(=a\Phi$; see Appendix B), in which case $\langle E_M \rangle$ is independent of a. We then have

$$H_G(a)+\langle E_M \rangle = \left(\frac{da}{d\eta}\right)^2 - a^2 + H^2 a^4 + \langle E_M \rangle = 0 \tag{35}$$

This first integral of the equation of motion gives the modification of the pure gravitational bounce of de Sitter space $(H_G(a)=0)$ due to the presence of matter energy $= E_M(T^*(a))$. This latter quantity, determined from the equation of state, is handled as a constant in (35), since S_M in Eq. (33) has no explicit dependence on a (due to the conformal invariance of the HW model). $T^*(a)$ is the period of the bounce as determined by the further integration of (35). In this way, the HW calculation is recovered, and indeed justified.

For $a<a_+$, we find that the temperature increases as one penetrates into the forbidden region. But, though the detailed calculation has not been made (numerical integration is necessary) it would appear that there is no self-consistent solution for a sufficiently small. This would be a most satisfying situation if borne out; semiclassical physics ought to be irrelevant at very small a.

The above result differs from that anticipated in the BBO approximation [discussion following Eq. (27)], in that we have found that, during the bounce dynamics which determine $P(a)$, the temperature $T^*(a)^{-1}$ does not vary in $\langle E_M \rangle$, whereas in BBO it does. The difference is significant.

In the classically permitted region, $T^*(a)$ has a real part which effectively vanishes corresponding to the part of the round trip to and from a_+ and an imaginary part from the complete bounce $a_- \leqslant a \leqslant a_+$. The contribution to $P(a)$ is unity from the former, due to the cancellation of the phases between $\Psi^\dagger$ and Ψ in this region, whereas the phases add in the bounce region. Thus $P(a)=P(a_+)$ for $a>a_+$, as in BBO. It is easy to see that the steepest-descent calculation and BBO agree in this region, the Schrödinger equation for $\tilde{\chi}$ emerging from the steepest-descent calculation. Thus $P(a)$ and $T^*(a)$ are constant for $a>a_+$.

When the matter is not conformally coupled, the dynamics is far more complicated because the dependence of S_M on a cannot be "gauged away." But the above calculation is rather convincing in enforcing one's expectations that the temperature will arise, in general, behind the horizon.

ACKNOWLEDGMENTS

The original work reported here is a collaboration among Gerry Horwitz, Daniel Weil, and myself. I wish to thank my two colleagues for the enjoyable, if sometimes difficult, hours spent together and to the Hebrew University of Jerusalem for providing for my fruitful stay there in November 1985. My profound gratitude to Tom Banks, who took the time one rainy day in Brussels to tell me about time, and to François Englert whose astute remarks are reflected throughout the text.

APPENDIX A: WKB THEORY FROM PATH INTEGRALS

We work with one degree of freedom. The two alternative forms of the Feynman propagator are

$$\begin{aligned} K(xt \mid x_0 t_0) &= \int \mathscr{D}(x')\, e^{iS(xt \mid x_0 t_0)} \\ &= \int dE\, \psi_E(x)\, \psi_E^\dagger(x_0)\, e^{-iE(t-t_0)} \end{aligned} \tag{A1}$$

where K is the half-sum of retarded and advanced propagators, $\psi_E(x)$ are the energy eigenfunctions, and S the action along the path which covers the intermediate points x': From (A2),

$$\psi_E(x)\, \psi_E^\dagger(x_0) = \int dt\, e^{iEt} \int \mathscr{D}(x')\, e^{iS(xt \mid x_0 t_0)} \tag{A2}$$

The path of minimal action is described by the classical action $S_{\text{cl}}(xt \mid x_0 t_0)$, and the semiclassical approximation uses the Gaussian fluctuation about this. The integral on t is then obtained by taking the stationary phase

$$E + \left.\frac{\partial S_{\text{cl}}}{\partial t}\right|_{t=t^*} = 0 \tag{A3}$$

the Hamilton–Jacobi equation. Whence

$$t^* = \int_{x_0}^{x} \frac{dx'}{\sqrt{2(E - V(x'))}} \tag{A4}$$

if there is only one saddle.

When there is a barrier, there are two saddles. Fix x_0 at the position of the barrier, $E = V(x_0)$ and $t_0^*(x_0) = 0$. The saddle times are then the "in" and "out" times from x to and from x_0:

$$t_\pm^* = \pm \int_{x_0}^{x} \frac{dx'}{\sqrt{2(E - V(x'))}} \tag{A5}$$

To determine the integration over t in (A2) requires analysis of the contour in the complex-t plane. The necessary details are beautifully illustrated in Sommerfeld's discussion of Bessel functions in *Partial Differential Equations.*[7] One begins and ends the contour in the upper half plane and steers it through the two real saddles, whose directions are at $\pm\pi/4$ with respect to the real axis, to give

$$\psi_E(x) = N^{\quad 1/2}(x) 2 \cos\left[\int_{x_0}^{x} p(x')\, dx' + \frac{\pi}{4}\right] \tag{A6}$$

where $p = \sqrt{2(E - V(x))}$, i.e., the phase is $Et_\pm x + S_{\text{cl}} \pm\ = \pm\ [\int_{x_0}^{x} p\, dx' + \pi/4]$. The quantity N comes from the Gaussian fluctuation. It is easy to to compute given the condition for the validity of the steepest-descent approximation in the first place, i.e., for slowly varying $V(x)$. Then $N = |p(x)|$.

In the classically forbidden region there are two imaginary saddles given by Eq. (A5) with the square root replaced by $i\sqrt{|E - V(x)|}$. For $(x - x_0)$ large, steepest descents once more can be validated for slowly varying $V(x)$. Since the end points are in the upper half plane, it is only the positive imaginary saddle that can be passed. This is parallel to the real axis and

$$\psi_E(x) = N^{-1/2} e^{-\int_{x_0}^{x} |p(x')|\, dx'} \tag{A7}$$

This establishes Kramers' connection formula.

The point of all of this is that the stationary wave function "tells a story" through the function $t^*(x)$ in semiclassical approximation. To follow the story is to follow time.

APPENDIX B: THE HORWITZ–WEIL PHENOMENOM

The model is that of a massless scalar field conformally coupled to gravity, in its mini-superspace mode only. The total action is

$$S = S_G + S_M$$

$$S_G = (3/4\pi) \int d^4x \sqrt{g}\, [R + 2\Lambda], \qquad m_P = 1 \tag{B1}$$

$$S_M = \tfrac{1}{2} \int d^4x \sqrt{g}\, [g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} - R/6\, \Phi^2]$$

The model simplifies on using a trick that generalizes a well-known procedure in this type of problem. When the cosmology is Euclidean (i.e., the time slice is flat), one defines $\tilde{g}_{\mu\nu} = g_{\mu\nu}/a$ and $\phi = a\Phi$ (multiplied by convenient constants); $\tilde{g}_{\mu\nu}$ then is the flat-space metric and the problem in mini-superspace looks like a Minkowski-type problem if the time variable η is used ($d\eta = dt/a$). One finds $L = \frac{1}{2}(-(\partial_\eta a)^2 + (\partial_\mu \phi)^2)$. In the present problem, because of the cosmological constant Λ, the time slices are closed spheres (in the absence of matter, de Sitter space). One proceeds analogously but with $\tilde{g}_{\mu\nu}$ having a space part which is that of a static spherically symmetric closed space. The constants are chosen to make $\tilde{g}_{\mu\nu}$ equal to the static Einstein metric corresponding to the value of Λ. With these transformations, the gravitational Lagrangian is

$$L_G = -\frac{1}{2}\left(\frac{\partial a}{\partial \eta}\right)^2 - V(a)$$
$$V(a) = \frac{1}{2}\, [-a^2 + H^2 a^4] \tag{B2}$$

The Einstein radius is determined by $dV/da|_{a=a_E} = 0$, $a = \sqrt{2}/H$, H being the Hubble constant of de Sitter space as seen from the equation of motion

$$d^2a/d\eta^2 + a - 2H^2 a^3 = 0 \tag{B3}$$

which in proper time reads $a\ddot{a} + \dot{a}^2 - 2H^2a^2 + 1 = 0$. It is easily checked that the zero-energy solution is $a = \cosh Ht/H$, hence $\dot{a}/a = H$ for $t \gg H^{-1}$.

The matter action under the rescaling becomes that of a massless scalar in the static sphere. This contains the usual kinetic energy plus corrections due to finite size and curvature. Horwitz and Weil postulate that this matter is in thermal equilibrium, defined by a microcanonical ensemble: $H_M + H_G = 0$. The entropy S_E is then given by

$$e^{S_E} = \mathrm{tr}\, \delta(H_M + H_G)$$
$$= \frac{1}{2\pi i}\, \mathrm{tr} \int_C d\beta' \exp[-\beta'(H_M + H_G)] \tag{B4}$$

where C is parallel to the imaginary axis with Re $\beta < 0$. Assuming that the trace and integral operations commute (as they do in flat-space thermodynamics), one has

$$e^{S_E} = \frac{1}{2\pi i} \int_C d\beta' \, \mathrm{tr}_G \, e^{\beta' H_G} e^{-\beta' F(\beta')} \tag{B5}$$

here tr_G denotes the trace over the gravitational field, and F is the free energy of matter in the Einstein space. This is known and is of the form

$$F(\beta) = -C\beta^{-4} + \text{corrections} \tag{B6}$$

The term in β^{-4} is the usual flat-space free energy and the corrections are due to the curature and finite size (Casimir effect). In flat-space thermodynamics, H_G would be a fixed energy, and a steepest descent integral on β' in (B5) would yield $e^{S_E} = e^{\beta(E+F)} \times O(\sqrt{N})$, where $E = -\partial\beta F/\partial\beta$. Large N justifies the procedure. Here, too, it is assumed that N is sufficiently large to validate the saddle-point integrations. Then gravity is handled in the semiclassical approximation, i.e., $\mathrm{tr}_G \, e^{-\beta' H_G} = \int \mathscr{D}(a)$ [Amplitude for a round trip in time $i\beta'$] $\simeq \int da_i \, e^{-S_G(a_i, \beta')}$, where S_G is the classical action expended in the bounce to and from a_i in time $i\beta'$. Thus (B5) becomes

$$e^{S_E} = \frac{1}{2\pi} \int d\beta' \int da_i \, e^{-S_G(a_i, \beta') - \beta' F(\beta')} \tag{B7}$$

The saddle point in β' is written, using $\partial\beta F/\partial\beta = -E_M$,

$$\left. \frac{\partial S_G}{\partial \beta'} + E_M(\beta') \right|_{\beta' = \beta} = 0 \tag{B8}$$

and the saddle condition on a_i gives

$$\left. \frac{\partial S_G}{\partial a} \right|_{\beta' = \beta} = a'(\beta) = 0 \tag{B9}$$

so the system bounces to and from rest.

Since the system is periodic in the bounce time β, the latter is identified with the inverse temperature of the system. The parameter β is determined self-consistently by integrating the saddle condition (B8):

$$\beta = \oint_{\text{Bounce}} \frac{da'}{\sqrt{a^2 - H^2 a^4 + E_M(\beta)}}$$

Horwitz and Weil have shown that nontrivial solutions exist for $\Lambda < \Lambda_c$, where $A_c \simeq 100$, i.e., $H \lesssim 10 m_P$. The de Sitter space is thus modified because of the nonvanishing of $\langle E_M \rangle$. For small Λ, HW recover the Hawking entropy associated with the horizon $(=\pi/H^2)$, the area of the horizon.

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