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Quantum Mechanics without Probability Amplitudes

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First steps are taken toward a formulation of quantum mechanics which avoids the use of probability amplitudes and is expressed entirely in terms of observable probabilities. Quantum states are represented not by state vectors or density matrices but by "probability tables," which contain only the probabilities of the outcomes of certain special measurements. The rule for computing transition probabilities, normally given by the squared modulus of the inner product of two state vectors, is re-expressed in terms of probability tables. The new version of the rule is surprisingly simple, especially when one considers that the notion of complex phases, so crucial in the evaluation of inner products, is entirely absent from the representation of states used here.

1. INTRODUCTION

What is the origin of the structure of quantum mechanics? This is one of a number of deep questions John Wheeler has brought to our attention in recent years.[(1)] It can also be phrased this way: Why is nature so constructed that we find it convenient to describe it in terms of complex probability amplitudes? Perhaps some insight can be gained into this question if we consider what quantum mechanics looks like when it is *not* expressed in terms of probability amplitudes. If, as Professor Wheeler has argued, the origin of quantum mechanics' structure is to be sought in a theory of observation and observers and meaning, then we would do well to focus our attention not on amplitudes but on quantities which are more directly observable. My aim in doing the work presented here has been to express as much as possible of the physical content of quantum mechanics purely

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in terms of *probabilities*, which are directly observable, rather than in terms of amplitudes. This aim is only partially fulfilled in the present paper: We express some but not all of the essential features of quantum mechanics in terms of probabilities. But this is only intended to be a beginning, and there is no obvious barrier in the path.

It is obviously *possible* to devise a formulation of quantum mechanics without probability amplitudes. One is never forced to use any quantities in one's theory other than the raw results of measurements. However, there is no reason to expect such a formulation to be anything other than extremely ugly. After all, probability amplitudes were invented for a reason. They are not as directly observable as probabilities, but they make the theory simple. I hope to demonstrate here that one *can* construct a reasonably pretty formulation using only probabilities. It may not be quite as simple as the usual formulation, but it is not much more complicated.

The key element in this no-amplitude formulation is the existence of what can be called "mutually unbiased" measurements. Loosely speaking, two measurements are mutually unbiased, in the sense used here, if they are as noncommutative as two measurements can possibly be. For example, in the case of a spin-1/2 particle, the measurements S_x and S_y are mutually unbiased: if the outcome of the measurement S_x can be predicted with certainty, then the two possible outcomes of S_y are equally likely.

In order to construct the kind of formulation of quantum mechanics we want, we first need a way of expressing any quantum state in terms of probabilities. For any given system, we will select once and for all a particular set of special "reference measurements" in terms of which all the states will be defined. For each state, we imagine setting up a table as follows: each column of the table corresponds to one of the special reference measurements, and the entries in a given column are the probabilities of the outcomes of that particular measurement. Such a table of probabilities will be sufficient to define the state. Exactly how the reference measurements are chosen will be explained in detail later; in the simplest cases they are a set of mutually unbiased measurements. For example, we will take S_x, S_y, and S_z to be a set of reference measurements for spin-1/2 particles.

We next need a way of computing the probability of a transition from one state to another when an appropriate measurement is performed. In the usual formulation this probability, for pure states, is equal to the squared modulus of the inner product of the two states vectors. In our formulation we have no state vectors but only probability tables. One might think it would be difficult to extract from these tables information which normally comes from the inner product, since the latter depends crucially on the complex phases of the state vectors' components. Surprisingly, it

turns out not to be particularly difficult. Equally surprisingly, one finds that the amount of work involved in computing these transition probabilities depends on whether the number of outcomes of a complete measurement on the system is prime or composite.

One would like to go further and find out how to express in terms of probability tables the ordinary Schrödinger evolution of a quantum state, which is normally given by a family of unitary operators. This work has not yet been done. Also, we restrict our attention here to systems with finite-dimensional state spaces, although I think an extension to the infinite-dimensional case ought to be possible.

In Sect. 2 we first ask how many mutually unbiased measurements—that is, measurements all unbiased with respect to each other—we would *like* to be able to find for an N-dimensional state space, and we see that there do exist that many if N is prime. We discuss the representation of states by probability tables in Sec. 3, and the computation of transition probabilities in Sect. 4. Finally, in Sect. 5 we mention some of the mathematical coincidences that make our formulation possible. In order to save space, I will not include the proofs of all the results presented here.

2. MUTUALLY UNBIASED MEASUREMENTS

The concept of mutually unbiased measurements can be motivated by the following practical question: What is the best way to ascertain the quantum state of a given ensemble? The general problem of state determination has been treated in Refs. 2 and 3. The idea of using mutually unbiased measurements has been developed by Ivanovic.[3]

Consider, for example, an ensemble of neutral spin-1/2 particles. We do not know the state of the ensemble—it may be pure or mixed—and would like to find it by making a series of measurements on the ensemble. We are interested only in the *spin* state and not the translational state; so we are looking for a 2×2 density matrix. Suppose we begin by measuring the z-component of spin of a large number of the particles. We find that 60% of them yield the value $+1/2$ and 40% yield the value $-1/2$. So far we do not have enough information to determine the state. It takes three independent real numbers to define a 2×2 density matrix, and so far we have obtained only one, namely, the probability of the outcome $S_z = +1/2$. To get the second number we need to make a different measurement, say S_y, on another subensemble, and to get the third number we need to make a third measurement, say S_x, on a third subensemble. Once these measurements have been made, we will have enough information to determine the state.

It is helpful to analyze this spin-1/2 problem in more detail. A general 2×2 density matrix can be written in the form

$$\rho = b|s\rangle\langle s| + (1-b)I/2 \tag{1}$$

where I is the identity matrix, $|s\rangle$ is a pure spin state, and b is a real number such that $0 \leqslant b \leqslant 1$. The quantity b can be thought of as the degree of purity of the state. Each state ρ can be represented by a point in the unit sphere in three dimensions: the distance of the point from the center is given by b, and the direction from the center is given by the unique unit vector $\mathbf{n}$ such that $|s\rangle$ is an eigenstate of $\mathbf{n} \cdot \boldsymbol{\sigma}$ with eigenvalue $+1$. If a particle in the state ρ is subjected to a measurement of, say, S_z (where the z-axis is taken to be vertical), then the probability of the outcome $S_z = +1/2$ turns out to be equal to the height of the point representing ρ, measured from the level of the bottom of the sphere and measured in units of the sphere's diameter. Thus, if we perform the measurement S_z on a large subensemble and find that two-thirds of the particles yield the outcome $S_z = +1/2$, we can conclude that the ensemble's "state-point" lies on a horizontal plane two-thirds of the way from the bottom of the sphere to the top.

This way of representing states makes it easy to see how we can determine the state of an ensemble from three different measurements performed on subensembles. Each measurement restricts ρ to a plane slice of the sphere. As long as the three axes along which the spin component is measured are not coplanar, the intersection of the three planes gives us the point representing the actual state of the ensemble.

Thus, almost any choice of the three measurements will allow a determination of the ensemble's state. However, if there is any *error* in our estimates of the probabilities—there will always be statistical error if the ensemble is finite—then not all triples of measurements are equally good. For example, if the three axes are not coplanar but make very small angles with each other, then each measurement gives practically the same information; the state-point is pinned down well in one dimension but hardly at all in the other two dimensions. Without going into a formal argument—because this is not the main point of the paper—I would like to suggest that it is best, in some reasonable sense of "best," to choose three axes that are mutually perpendicular. In that way each different measurement gives us as much new information as possible. Thus, the set of measurements consisting of S_x, S_y, and S_z is, I think, an optimal set for determining the density matrix.

Let us now try to generalize these ideas to systems whose pure-state space has not just two dimensions but N dimensions. In that case the most

general state is given by an $N \times N$ density matrix, which requires $N^2 - 1$ independent real numbers for its specification. Any single complete measurement—we will consider here only complete measurements, i.e., without degeneracy—has N possible outcomes, so it can gives us $N - 1$ independent probabilities. Therefore, just on the basis of counting the number of variables, we expect that the number of different measurements we need in order to determine the density matrix completely is $(N^2 - 1)/(N - 1)$, which equals $N + 1$. For example, when N equals 2, as for a spin-1/2 particle, we need to make three different measurements, as we have already seen.

We are now faced with the question: What is the best choice of the $N + 1$ measurements? As in the spin-1/2 case, we would like the measurements to be as different from each other as possible. I conjecture that the kind of difference we would most desire is expressed by the property of being *mutually unbiased*, which we now define. Let B and B' be two orthonormal bases for the N-dimensional state space. We will say these two bases are mutually unbiased if for every vector $|v\rangle$ in B and every vector $|v'\rangle$ in B', $|\langle v|v'\rangle|^2 = 1/N$. Two *measurements* are defined to be mutually unbiased if the bases composed of their eigenstates are mutually unbiased.

We would thus like to find, if possible, $N + 1$ bases for an N-dimensional state space, all unbiased with respect to each other. If such bases can be found, then the measurements to which they correspond will be as different from each other as possible, and will constitute, I think, an optimal set of measurements for ascertaining the state of an ensemble. (I assume that every basis does correspond to a realizable measurement; i.e., there is no superselection rule.) In two dimensions, we have seen that it is possible to find $N + 1 = 3$ mutually unbiased bases. Here they are explicitly, in a particular representation:

$$B^0 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \qquad B^1 = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\},$$

$$B^2 = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \right\} \tag{2}$$

We now ask: Is it possible to find $N + 1$ mutually unbiased bases in N dimensions?

I do not know the complete answer to this question. I do know that if N is a *prime* number, the answer is yes. The existence of the desired bases in that case has been demonstrated by Ivanovic[3] by explicit construction. Let B_l^{nm} be the lth component of the mth vector in the nth basis. (Here $l = 1,..., N$; $m = 1,..., N$; and $n = 0,..., N$.) For prime values of N greater than

2, one can show by direct calculation that the following bases are all unbiased with respect to each other:

$$B_l^{0m} = \delta_{ml}$$

$$B_l^{Nm} = \frac{1}{\sqrt{N}} \exp\left[\frac{2\pi i}{N} ml\right] \tag{3}$$

$$B_l^{nm} = \frac{1}{\sqrt{N}} \exp\left[\frac{2\pi i}{N} n(m+l)^2\right], \qquad \text{for} \quad n = 1,..., N-1$$

This is obviously not the only possible set of $N+1$ mutually unbiased bases. One can, of course, freely change the ordering of the vectors in each basis or the labelling of the bases themselves, and one can perform a simultaneous rotation on all the bases. Nevertheless, for the rest of this paper we will work only with the particular set of bases given above. This is done in the same spirit as choosing a coordinate system.

Here are some interesting mathematical questions concerning mutually unbiased bases to which I do not know the answers. (1) When N is prime, can one find *more* than $N+1$ mutually unbiased bases? I suspect not. (2) For a general composite N, what is the maximum number of mutually unbiased bases one can find? On the basis of trial and error, I would guess that the maximum number is less than $N+1$ for all composite values of N, i.e., that the desired bases can be found if and only if N is prime. For our purpose, it turns out that the mutually unbiased bases for prime N are all we will need.

Each of the above bases is supposed to correspond to a realizable measurement. It would be interesting to try to figure out how one might actually go about performing these measurements. In the case of spin-1/2 particles, it is easy: one measures S_x, S_y, and S_z by appropriately rotating the spin of each particle before it enters the inhomogeneous field of a Stern–Gerlach magnet. For systems with larger N the problem is considerably more difficult, and I have not yet found any reasonably practical solution.

3. THE REPRESENTATION OF QUANTUM STATES IN TERMS OF PROBABILITIES

3.1. Prime-Dimensional State Space

Let ρ be an $N \times N$ density matrix where N is prime. A system in the state ρ, when subjected to the nth measurement of Eq. (2) or (3), will yield the mth outcome with probability

$$p_{nm} = \sum_{k,l=1}^{N} B_k^{nm*} \rho_{kl} B_l^{nm} \tag{4}$$

From the set of all these p_{nm}'s, with $n = 0,..., N$ and $m = 1,..., N$, one can reconstruct the density matrix ρ as follows:

$$\begin{aligned} \rho_{kk} &= p_{0k} \\ \rho_{kl} &= \sum_{n=1}^{N} \sum_{m=1}^{N} p_{nm} B_k^{nm} B_l^{nm*}, \qquad \text{for} \quad k \neq l \end{aligned} \tag{5}$$

Equation (5) can be verified by substituting for the p_{nm}'s the expressions given in Eq. (4) and for the B^{nm}'s the expressions given in Eqs. (2) and (3).

Thus, for prime N, we can take the measurements defined by Eqs. (2) and (3) as our reference measurements, with respect to which any quantum state can be represented uniquely by a table of probabilities. Equations (4) and (5) provide a way of going back and forth between the representation of a state by a density matrix and its representation by a probability table. To give an example, in the case $N = 3$, the state

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

is represented by the table

		measurement			
		0	1	2	3
outcome	1	1	1/3	1/3	1/3
	2	0	1/3	1/3	1/3
	3	0	1/3	1/3	1/3

(7)

Notice that the numbers in each column add to 1, as they must since they are the probabilities of the three possible outcomes of a given reference measurement. Notice also that because this particular state is an eigenstate of measurement #0, it has equal probabilities of all the possible outcomes of all the other measurements; this follows from the fact that the four reference measurements are mutually unbiased.

Does every conceivable probability table correspond to a possible quantum state? The answer is no. There are certain restrictions that every realizable probability table must satisfy. Are these restrictions equalities or inequalities? Answer: The only equalities that must be satisfied are that the numbers in each column add to 1. Otherwise the restrictions are inequalities. These inequalities can be summarized in one requirement: The

probability table p must be such that the matrix ρ computed from it via Eq. (5) has no negative eigenvalues.

It would be good to find a more direct statement of the inequalities which does not involve ρ. As a first step in this effort, we state here *one* of the required inequalities: the sum of the squares of all the elements of the table must be less than or equal to 2. This fact follows from the requirement of non-negative eigenvalues, but by itself it is not enough to guarantee the non-negativity. Incidentally, one can show that for any probability table which does correspond to an actual state, the sum of the squares *equals* 2 if and only if the state is pure.

3.2. Composite-Dimensional State Space

When N is not prime, the construction given in Eq. (3) does not yield a set of mutually unbiased bases. We therefore need to generalize our means of choosing a set of reference measurements.

Our approach will be to treat every system having a composite-dimensional state space as if it were a composite system. That is, only systems with prime N will be treated as truly fundamental and indivisible. It may well be that nature is actually structured in that way. Our approach is certainly most appealing if that is the case, and it does not seem improbable that it is. However, even if there do exist fundamental systems with composite N, it is still possible to *treat* them as if they had components. In the case of a spin-3/2 particle, for example, the particle's four-dimensional state space could be factorized into two two-dimensional factor spaces, such that the operator "sign of S_z" acts only on one of these spaces, and the operator "magnitude of S_z" acts only on the other. These two smaller spaces could be thought of as belonging to two subsystems, which together constitute the original spin-3/2 particle. In this spirit, we will regard any system for which N has f prime factors as a composite system consisting of f primitive subsystems.

The reference measurements for composite systems are chosen as follows. Imagine performing on every primitive subsystem a reference measurement for that subsystem, chosen from the set specified in Sect. 3.1. The performance of all these measurements, i.e., one on each subsystem, is counted as a reference measurement on the whole system, and every reference measurement is constructed in this way. Thus, if $N = N_1 N_2 \cdots N_f$, where each N_i is prime, the total number of reference measurements on the whole system is $(N_1+1)(N_2+1)\cdots(N_f+1)$, since there are N_i+1 reference measurements for the ith subsystem.

Consider for example a system consisting of an electron and a proton; as always we consider only the spin state. For this system we have

$N_1 = N_2 = 2$ and $N = 4$. A typical reference measurement would consist of performing a measurement of, say, S_y on the electron and S_x on the proton. There are altogether nine different reference measurements, each of which has four possible outcomes, so we can imagine constructing a probability table having nine columns and four rows. As we will see shortly, such a table contains a complete description of the system's state.

For a general composite system, each of the reference measurements has eigenstates of the form

$$v_{l_1 \cdots l_f} = B^{n_1 m_1}_{l_1} B^{n_2 m_2}_{l_2} \cdots B^{n_f m_f}_{l_f} \tag{8}$$

Here each n_i has a value from 0 to N_i, and the ordered set $(n_1,..., n_f)$ labels the measurement: it says we are performing the n_1th reference measurement on the first subsystem, the n_2th on the second, and so on. Each m_i has a value from 1 to N, and the ordered set $(m_1,..., m_f)$ labels the particular outcome, or particular eigenvector, of the measurement. The l_i's label the components of this eigenvector. The components of the density matrix of the composite system can be labelled as

$$\rho_{k_1 \cdots k_f; l_1 \cdots l_f}, \qquad \text{where} \quad k_i, l_i = 1,..., N_i \tag{9}$$

When the measurement $(n_1,..., n_f)$ is performed on the state ρ, the probability of the outcome $(m_1,..., m_f)$ is

$$p_{n_1 \cdots n_f; m_1 \cdots m_f} = \sum_{k_1,..., k_f} \sum_{l_1,..., l_f} (B^{n_1 m_1}_{k_1} \cdots B^{n_f m_f}_{k_f})^* \, \rho_{k_1 \cdots k_f; l_1 \cdots l_f} (B^{n_1 m_1}_{l_1} \cdots B^{n_f m_f}_{l_f}) \tag{10}$$

Equation (10) tells us how to go from the density matrix of a state to its probability table. As in the case of prime N, we would like to have a prescription for going the other direction, i.e., for recovering the density matrix ρ from the p's. The prescription turns out to be very similar to that given in Eq. (5):

$$\rho_{k_1 \cdots k_f; l_1 \cdots l_f} = \sum_{n_1=1}^{N_1} \cdots \sum_{n_f=1}^{N_f} \sum_{m_1=1}^{N_1} \cdots \sum_{m_f=1}^{N_f} p_{n_1 \cdots n_f; m_1 \cdots m_f} \times (B^{n_1 m_1}_{k_1} \cdots B^{n_f m_f}_{k_f})(B^{n_1 m_1}_{l_1} \cdots B^{n_f m_f}_{l_f})^* \tag{11}$$

if $k_i \neq l_i$ for each $i = 1,..., f$. For each value of i for which $k_i = l_i$, one eliminates the sum over n_i and replaces n_i in the summand by zero. Equation (11) is our guarantee that the probability table contains all the information one needs to determine the state.

Again we need to address the question: what restrictions must one impose on a probability table in order to ensure that it corresponds to a

genuine quantum state? As before, there will be some equalities and some inequalities. Here are the equalities that must be satisfied:

(i) The probabilities of the outcomes of reference measurements performed on any *subset* of the f primitive subsystems cannot depend on the choice of measurements on the *other* subsystems. For example, the reduced probability table obtained by summing $p_{n_1 \cdots n_f; m_1 \cdots m_f}$ over m_i cannot depend on n_i.

(ii) For each measurement $(n_1, \ldots, n_f)$, the sum over *all* possible outcomes, $\sum_{m_1,\ldots,m_f} p_{n_1 \cdots n_f; m_1 \cdots m_f}$, must equal 1.

One can show quite generally, i.e., not necessarily in the context of quantum mechanics, that a probability table satisfying the above two restrictions contains exactly $N^2 - 1$ independent real numbers. This is precisely the number needed to specify a general quantum state.

There are still some *inequalities* that any legitimate probability table must satisfy. As before, these can be summarized by saying that the matrix ρ defined by Eq. (11) must have no negative eigenvalues.

3.3. Systems of Identical Particles

When a system contains identical particles, it is no longer true that one needs $N^2 - 1$ numbers to specify a state of the system. Consider, for example, a system of two electrons. If they were not identical, the spin state of this system would be described by an ordinary 4×4 density matrix. However, because of the indistinguishability, odd and even spin states must be correlated to spatial states of different parity, so they cannot be coherently superposed. This means that the density matrix for the spin, written in the usual singlet-triplet basis, must be block diagonal, containing a 3×3 block and a 1×1 block. (We do not consider here the description of the spatial part of the state.) The number of independent numbers in such a density matrix is not $4^2 - 1 = 15$, but only 9.

In such a case, our description of states in terms of probability tables is modified only slightly. We require the probability table to be invariant under the interchange of identical particles; in other respects the description is the same as before. For example, in the two-electron case, we require that $p_{n_1 n_2; m_1 m_2} = p_{n_2 n_1; m_2 m_1}$; one can convince oneself that this restriction does indeed cut the number of independent variables from 15 to 9. Quite generally, one can show that for any system containing identical particles, every unsymmetrized but otherwise legitimate probability table yields, upon symmetrization, a table corresponding to a legitimate symmetrized density matrix. This would obviously be true if we were requiring that *no* measurement show any difference among the indistinguishable par-

ticles. But we are only requiring that none of the *reference* measurements show any difference among the particles. This restriction turns out to be enough.

4. CALCULATION OF TRANSITION PROBABILITIES

Suppose a system in state S is subjected to a measurement one of whose eigenstates is a different state S'. What is the probability $P(S \to S')$ that the outcome corresponding to S' will occur? This probability is normally obtained by taking the trace of $\rho\rho'$, where ρ is the density matrix of the original state S and ρ' is the density matrix of the pure state S'. How do we express this same transition probability in terms of probability tables?

If the system is primitive, the answer could hardly be simpler. Let p_{nm} and p'_{nm} be the probability tables for the two states S and S'. The transition probability $P(S \to S')$ turns out to be given by

$$P(S \to S') = \left(\sum_{n=0}^{N} \sum_{m=1}^{N} p_{nm} p'_{nm} \right) - 1 \tag{12}$$

The remarkable simplicity of this result is the main advantage of using *mutually unbiased* measurements to define states, rather than some equally complete but different set of measurements. The advantage is analogous to that gained by using an orthogonal coordinate system, as opposed to an oblique coordinate system, to analyze a problem in, say, classical mechanics.

For composite systems, the calculation of transition probabilities is somewhat more complicated, but it is still reasonably simple considering that we are working only with probabilities and not with amplitudes. To simplify the notation let us define the symbol $[p, p']$ to mean the sum of products of corresponding terms of the probability tables p and p'. That is,

$$[p, p'] = \sum_{n_1=0}^{N_1} \cdots \sum_{n_f=0}^{N_f} \sum_{m_1=1}^{N_1} \cdots \sum_{m_f=1}^{N_f} p_{n_1 \cdots n_f; m_1 \cdots m_f} p'_{n_1 \cdots n_f; m_1 \cdots m_f} \tag{13}$$

The quantity $[p, p']$ has the following interpretation: it is the sum, over all reference measurements, of the probability that the primed and unprimed states will yield exactly the same outcome. We can call $[p, p']$ the "coincidence sum" for the two states.

Let us denote by ${}^i p$ the reduced probability table obtained from p by summing over m_i. It gives the probabilities of the outcomes of all measurements which do not involve the ith primitive subsystem. For exam-

ple, ${}^1p_{n_2\cdots n_f;m_2\cdots m_f} = \sum_{m_1} p_{n_1\cdots n_f;m_1\cdots m_f}$ is the probability of getting the outcome $(m_2,\ldots, m_f)$ when the measurement $(n_2,\ldots, n_f)$ is performed on subsystems 2 through f. As we have mentioned before, this probability is independent of what measurement is made on subsystem 1. Likewise, ${}^{ij}p$ is the table obtained from p by summing over m_i and m_j, and the analogous symbols with more indices are defined similarly.

The transition probability between two states S and S' of a general system, with probability tables p and p', turns out to be given by the following combination of coincidence sums:

$$P(S \to S') = \{p, p'\} \equiv [p, p'] - \sum_{i=1}^{f} [{}^ip, {}^ip'] + \sum_{\substack{i,j=1 \\ i<j}}^{f} [{}^{ij}p, {}^{ij}p'] - \cdots + (-1)^f \tag{14}$$

The last term comes from $(-1)^f [{}^{1,\ldots,f}p, {}^{1,\ldots,f}p']$, but in this expression each of the "tables" in the brackets consists of the single number 1, so the bracket also equals 1. Equation (14) is the main result of this paper. Notice that Eq. (12) is a special case of Eq. (14).

We have been thinking of the state S' as a pure state, so that it could be an eigenstate of a measurement. It happens, not surprisingly, that $\{p, p'\}$ is always equal to the trace of $\rho\rho'$, even when both states are mixed.

5. DISCUSSION

We have seen how to represent a general quantum state (of a system with a finite-dimensional state space) by a "probability table," and how to compute the transistion probability between two states from their probability tables. As a by-product, we have obtained a prescription, Eq. (11), for finding the density matrix of an arbitrary system directly from measurable probabilities.

It is interesting to list some of the mathematical facts that have allowed us to write down at least this much of quantum mechanics purely in terms of probabilities.

(i) The number of real numbers needed to specify a quantum state, $N^2 - 1$, divided by the number of numbers obtainable from a given measurement, $N - 1$, happens always to be an integer. This means that of the data collected from the $N + 1$ reference measurements, N being prime, every available number has to be used. There are no redundant variables in the data. Thus our probability tables were "just the right size" for specifying a state.

(ii) It happens that for an N-dimensional complex vector space, with N prime, there *are* $N+1$ mutually unbiased bases, exactly as many as we needed.

(iii) For composite systems, it was not surprising that for every legitimate probability table, the results of measurements on *part* of the system did not depend on the choice of measurement performed on the rest of the system. This property is, after all, the reason we cannot use the violation of Bell's inequality to transmit information faster than light.[4] We noted that this requirement, along with the obvious requirement that the sum of the probabilities of all the outcomes of a measurement be equal to 1, reduced the number of independent numbers a table could contain down to N^2-1, which is precisely the number needed to specify a state. This agreement between "numbers obtainable" and "numbers needed" allowed us to apply to the treatment of *composite* systems the results we had obtained for primitive systems.

Let me pursue this last point a little further. Consider a hypothetical world where, as in the quantum mechanics of our world, one needs to perform several different measurements on an ensemble in order to determine its state. Let $g(N)$ be the number of different measurements needed if each measurement has N possible outcomes. (For quantum mechanics, $g(N)=N+1$.) Suppose further that, as in quantum mechanics, none of the $(N-1)\,g(N)$ independent probabilities obtained from these measurements is redundant; it takes exactly that many numbers to determine the state.

In such a world, let Z_1 and Z_2 be two systems, having respectively N_1 and N_2 possible outcomes of a complete measurement. Let M_1 be a set of $g(N_1)$ measurements sufficient for the determination of the state of Z_1, and let M_2 be an analogous set for Z_2. Suppose we now *insist* that, as in quantum mechanics, one can obtain a complete determination of the state of the composite system Z_1+Z_2, with no redundant numbers,[2] by performing on an ensemble of such systems every possible combination of a measurement from M_1 and a measurement from M_2. Assuming as usual that the choice of measurement on one subsystem does not affect the other, one can show

[2] Given the fact that the choice of measurement on one subsystem cannot influence another subsystem—let us call this "the causality condition"—an experimenter using our method of determining an ensemble's state knows from the outset that not all the probabilities he obtains are going to be independent of each other. But this lack of independence, arising from the causality condition, is not what we are calling "redundancy." Rather, we imagine that the experimenter will select from his data a maximal set of numbers which *are* logically independent, even given the causality condition. If it turns out that, because of the laws of physics, some of *these* numbers do in fact depend on each other, then we call these numbers redundant.

that the number of independent probabilities obtainable from these measurements is

$$(N_1-1)\,g(N_1)+(N_2-1)\,g(N_2)+(N_1-1)(N_2-1)\,g(N_1)\,g(N_2) \quad (15)$$

On the other hand, according to our assumptions, the number of independent numbers needed to specify the state of Z_1+Z_2 is

$$(N_1N_2-1)\,g(N_1N_2) \quad (16)$$

Therefore, if the measurements are to give a complete and nonredundant characterization of the state, we must have

$$\begin{aligned}(N_1-1)\,g(N_1)+(N_2-1)\,g(N_2)+(N_1-1)(N_2-1)\,g(N_1)\,g(N_2)\\ =(N_1N_2-1)\,g(N_1N_2)\end{aligned} \quad (17)$$

In any world in which our assumptions are satisfied, the positive-integer-valued function $g(N)$ must satisfy Eq. (17) for all positive integers N_1 and N_2. We can think of Eq. (17) as follows: it is the condition that must be satisfied in order that *those measurements which are just sufficient for determining the states of the subsystems are, when performed jointly, also just sufficient for determining the state of the whole system.* In quantum mechanics, $g(N)=N+1$, which does satisfy Eq. (17). Another solution is $g(N)=1$, which corresponds to a world where on each system there is only one complete measurement that can be performed. Such a world could be called "classical," although it differs from the world of classical physics in that the outcomes are discrete. Any function $g(N)$ of the form $g(N)=\sum_{k=0}^{K}N^k$ is also a solution of Eq. (17). It would be interesting to know if there are any other solutions. In any case, the fact that the $g(N)$ for quantum mechanics does satisfy Eq. (17) made it possible for us to treat composite systems as we did.

In conclusion, we have here the beginning of a formulation of quantum mechanics expressed purely in terms of quantities which are in principle directly observable. The way in which states are represented is quite different from that of the usual formulation, and the rule for computing transition probabilities also looks quite different. But the physical content is the same. The question, "why complex amplitudes?," could therefore be rephrased to read, "why does the transition rule involve the particular combination of probabilities given by $\{p,p'\}$?" The restatement may or may not be easier to answer than the original question. It certainly gives us a new perspective on the problem.

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