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s-PRIME ELEMENTS IN MULTIPLICATIVE LATTICES

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Abstract

Let $\mathcal{L}$ be a C-lattice which is strong join principally generated. In this paper, we consider prime elements of $\mathcal{L}$ for which every semiprimary element is primary. We show, for example, that a compact nonmaximal prime p with this property is principal. We also show that if every prime $p \leq m$ has this property, then $\mathcal{L}_m$ is either a one dimensional domain or a primary lattice. It follows that if every prime p satisfies the property, and if there are only a finite number of minimal primes in $\mathcal{L}$, then $\mathcal{L}$ is the finite direct product of one-dimensional domains and primary lattices.

By a ring, we mean a commutative, associative ring with identity. By a *multiplicative lattice*, we main a complete but not necessarily modular lattice L on which there is defined a commutative, associative, completely join distributive product. We denote the least element of L by 0, the greatest element of L by 1, and we assume the 1 is a compact multiplicative identity.

We denote the set of all compact elements of L by L_*. By a *C-lattice*, we mean a multiplicative lattice L which is generated by a multiplicatively closed subset C of compact elements. In a C-lattice, the set L_* is multiplicatively closed. The ideal lattice $L(R)$ of any ring R provides an example of a C-lattice.

An element $p < 1$ of a multiplicative lattice L is *prime* if for $a, b \in L_*$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $q < 1$ of L is said to be *primary* if, for $a, b \in L_*$, $ab \leq q$ implies $a \leq q$ or $b \leq \sqrt{q} = \vee\{x \mid x^n \leq q \text{ for some } n\}$. An element $q \in L$ is p-primary if q is primary, p is prime and $\sqrt{q} = p$. These definitions agree with those given by R.P. Dilworth [4] if L satisfies the ascending chain condition, and in general for the ideal lattice $L(R)$ of a ring.

If $k \in L$, the interval $[k, 1]$ is denoted L/k. The lattive L/k is again a multiplicative lattice with $a \circ b = ab \vee k$. The element $a \vee k$ of L/k is frequently denoted a/k. Under this convention, $a/k \circ b/k = ab/k$. Is is well known that for a ring R and an ideal $I \in L(R)$, the lattices $L(R/I)$ and $[I, R] = L(R)/I$ are isomorphic.

Like the ideal lattice of a ring, any C-lattice can be localized at a multiplicatively closed set. If S is a multiplicatively closed subset of L_* in a C-lattice L,

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then for $a \in L$, $a_S = \vee\{x \in L_* \mid xs \leq a$ for some $s \in S\}$ and $L_S = \{x_S \mid x \in L\}$. L_S is again a multiplicative lattice under the same order as L with the product $a_S \circ b_S = (a_S b_S)_S$, where the right hand side is evaluated in L. The meet operation is the same for L_S as for L, and the join operation for L is given by $a \vee b = (a \vee b)_S$, where again the right-hand side is evaluated in L. For $a, b \in L$, $a : b$ is the greatest element c satisfying $cb \leq a$ (or, equivalently, $a : b = \vee\{x \in L \mid xb \leq a\}$). If b is compact, then $a_S : b_S = (a : b)_S$. The right-hand side is evaluated in L. The left-hand side can be evaluated either in L or in L_S. If $p \in L$ is prime and $S = \{x \in L_* \mid x \leq p\}$, then L_S is denoted L_p. As one would hope, the localization theory is such that $a \leq b$ iff $a \leq b_m$ for all maximal elements m. In particular, $a = b$ if and only if $a_m = b_m$ for all maximal elements m. This definition of localization is somewhat more general than that given by Dilworth in [4]. Localization at a prime agrees with Dilworth's definition if L satisfies ACC. If R is a ring, S is a multiplicatively closed subset of R, and $\overline{S} = \{(s) \mid s \in S\}$, then $L(R_S) \simeq L(R)_{\overline{S}}$.

The relation of the prime and primary elements of L to those in L_S and L/k is the same as it is for ideals of rings. In particular, any maximal element m is prime, and any power m^n of a maximal element m is m-primary. In both L/k and L_S, the products are frequently denoted by juxtaposition.

We say an element e of a multiplicative lattice L is *meet principal* if it satisfies the indentity (i) $a \wedge be = ((a : e) \wedge b)e$. Dually, e is said to be *join principal* if it satisfies the identity (ii) $a \vee (b : e) = (ae \vee b) : e$. An element $e \in L$ satisfying the weaker identity (i)' $a \wedge e = (a : e)e$ obtained from (i) setting $b = 1$ is said to be *weak meet principal* and an element e satisfying the identity (ii)' $a \vee (0 : e) = ae : e$ obtained from (ii) by setting $b = 0$ is called *weak join principal*. Elements which are both (weak) meet principal and (weak) join principal are called *(weak) principal*. These identities were introduced by R.P. Dilworth in [4]. In a C-lattice, weak principal elements are compact ([2], Theorem 1.3). Principal ideals in the ideal lattice $L(R)$ of a commutative ring R with 1 are examples of principal ideals. The compact elements of $L(R)$ are the finitely generated ideals. Hence, for any commutative ring R with 1, $L(R)$ is a principally generated C-lattice. Principal elements have been studied extensively by D.D. Anderson, P.J. McCarthy, the current authors, and others. See, for example, [2], [7], [8].

If e is compact in L, then e/k is compact in L/k and e_S is compact in L_S. If e is join principal in L, e/k is join principal in L/k. If e is join principal and compact in L, then e_S is join principal in L_S.

An element $e \in L$ is said to be a *strong join principal element* if it is compact and join principal. L is said to be *quasi local* if it has a unique maximal element.

An element $a \in L$ is said to be *semiprimary* if $\sqrt{a}$ is a prime element. If $\sqrt{a} = p$ is prime, then a is said to be *p-semiprimary*. R.W. Gilmer and J.L. Mott have studied extensively commutative rings in which semiprimary ideals are primary (see [5], [6] and [9]). For various characterizations of multiplicative lattices in which semiprimary are primary, the reader is referred to [1]. The reader is referred to [2] for general background and terminology.

Throughout, L denotes a multiplicative lattice with $1 \in L_*$, and $\mathcal{L}$ denotes a C-lattice which is generated by strong join-principal elements.

The following basic facts are all known. We record them for the reader's convenience.

PROPERTIES OF LOCALIZATION. *Assume L is a C-lattice and S is a multiplicatively closed subset of L_*. Then the following properties hold.*

(0.1) $(L_S)_* = (L_*)_S = \{a_S \mid a \in L_*\}$.

(0.2) If p is maximal in the complement of S (i.e., maximal with respect to the property that $s \in S$ implies $s \not\leq p$) then p is prime.

(0.3) For $a \in L$, $\sqrt{a}$ is the intersection of the prime elements of L over a.

(0.4) If $a \in L$, then $\sqrt{a_S} = (\sqrt{a})_S = (\sqrt{a_S})_S$.

(0.5) If $p \in L$ is a prime minimal over a then a_p is primary.

(0.6) If $p \in L$ is a prime minimal over a then $\sqrt{a_p} = p$.

(0.7) If q is p-primary and p does not meet S (in the sense that $s \in S$ implies $s \not\leq p$) then $q_S = q$, $p_S = p$, and q_S is p_S-primary in L_S.

(0.8) If q_S is p_S-primary in L_S, then q_S is p_S-primary in L.

PROOF. (0.1) is known but the statement does not appear in the literature. The proof is routine. The proofs of the remaining statements parallel the proofs of their ring theoretic analogues. The proof of this version of (0.2) is similar to [2], Theorem 2.2. Also, (0.3) is similar to [2], Theorem 2.4. The proof of (0.4) is routine (cf. [3], Proposition 3.11). Here, $\sqrt{a_S}$ can be computed either in L or in L_S, the other two terms in L. The proof of (0.5) is also routine. Then (0.6) follows from (0.4), (0.5) and the definition of p-primary. (0.7) follows from (0.1), as does (0.8). ■

We introduce the following definitions.

DEFINITON 1. A prime element $p \in L$ is said to be a weak s-prime if every p-semiprimary element is p-primary.

DEFINITION 2. A prime element p of L is said to be an s-prime if every prime element $q \leq p$ is a weak s-prime.

Note that maximal elements are weak s-primes (cf. [10], p. 153). Complemented maximal elements and prime elements which are both maximal and minimal are examples of s-primes. A minimal prime is an s-prime if and only if is a weak s-prime. Every prime is a weak s-prime element iff every maximal prime is an s-prime iff every semiprimary element is primary.

We first consider weak s-primes.

THEOREM 1. *Let p be a nonmaximal prime of $\mathcal{L}$. Then the following statements are equivalent:*

(i) p is a weak s-prime element.

(ii) p is the only p-semiprimary element.

PROOF. (ii)⇒(i). Clear. (i)⇒(ii). Suppose q is p-semiprimary. As p is nonmaximal and 1 is compact, $p < m$ for some maximal element m of L. Choose any join principal element $b \leq p$ such that $b \not\leq q$. Then $(q : b) \leq p < m$ as q is p-primary. Then $mb \leq q \vee mb$ $q \vee mb$ is p-primary, and $p < m$, so $b \leq q \vee mb$. But b is join principal, so $1 = b : b \leq (q \vee mb) : b = (q : b) \vee m = m$, a contradiction. Therefore $q = p$. ■

We use the following result which is proved in [1], Lemma 1.1.

LEMMA 1. *Let L be a multiplicative lattice with 1 compact. Suppose $d = \vee_{i=1}^{n} a_i$, where the a_i's are join-principal elements. If $d \leq b \vee cd$, then $c \vee (b : d) = 1$. Hence if $d = cd$, then $c \vee (0 : d) = 1$.*

LEMMA 2. *Let L be a C-lattice. If e is compact and idempotent, then e is principal.*

PROOF. An element $e \in L$ is principal if and only if e is compact and e_m is principal in L_m for every maximal element m of L. As e is compact, we can assume L is quasi local with maximal element m. By Lemma 1, e idempotent implies $e \vee (0 : e) = 1$. As L is quasi local, it follows that $e = 1$ or $e = 0$. In either case, e is principal. ■

THEOREM 2. *Let p be a nonmaximal prime element of $\mathcal{L}$. Then the following statements are equivalent.*

(i) p is a compact idempotent element.

(ii) p is a compact weak s-prime element.

(iii) p is a compact element and $p = pm$ for all maximal elements $m > p$.

PROOF. (i)$\Rightarrow$(ii). Suppose (i) holds. As p is compact and idempotent, $\sqrt{a} = p$ implies $a = p$. (ii)$\Rightarrow$(i). $\sqrt{p^2} = p$, so p^2 is p-semiprimary. By Theorem 1, $p^2 = p$. (i)$\Rightarrow$(iii). If p is idempotent then $p^2 = p$, so for any maximal element $m \geq p$, also $p = mp$. (iii)$\Rightarrow$(i). Suppose (iii) holds. By Lemma 1, $m \vee (0 : p) = 1$ for every maximal element $m > p$. It follows that $p \vee (0 : p)$ is not contained in any maximal element, so $p \vee (0 : p) = 1$. But then $p = p^2$. ■

LEMMA 3. *Let p be a prime element of $\mathcal{L}$. If p is a weak s-prime, then for every maximal element $m \geq p$, p_m is a weak s-prime element of $\mathcal{L}$. Conversely, if p_m is a weak s-prime element in $\mathcal{L}_m$ for every maximal element $m \geq p$, then p is a weak s-prime element of $\mathcal{L}$.*

PROOF. If $\sqrt{a_m} = p_m$ in $\mathcal{L}_m$, then $\sqrt{a_m} = p$ in $\mathcal{L}$ (0.4). Then p is a weak s-prime in $\mathcal{L}$ implies a_m is p-primary in $\mathcal{L}$, which implies a_m is p-primary in $\mathcal{L}_m$ (0.7). Now, assume that $p = p_m$ is a weak s-prime in $\mathcal{L}_m$ for every maximal element $m \geq p$. Assume $a \in L$ satisfies $\sqrt{a} = p$. If p is a maximal prime, then a is p-primary. If p is not maximal, and if $m > p$ is maximal, then $\sqrt{a_m} = p_m$ in $\mathcal{L}_m$, so (Theorem 1) $a_m = p_m$. Hence, $a_m = p_m$ for every maximal element $m > a$, For any maximal element $m \not> a$, $a_m = 1 = p_m$. Hence $a_m = p_m$ for all maximal elements

m, and so $a = p$. ■

THEOREM 3. *Let p be a nonmaximal prime element of $\mathcal{L}$. Then the following statements are equivalent.*

(i) p is weak s-prime and locally compact.

(ii) p is idempotent and locally compact.

(iii) $x = xp$ for all compact elements $x \leq p$.

(iv) p is weak meet principal and $p = pm$ for all maximal elements $m > p$.

(v) p is idempotent and weak meet principal.

PROOF. (i)⇔(ii). This follows from Theorem 2 and Lemma 3. (ii)⇒(iii). Assume p satisfies (ii) and let $m > p$ be a maximal element of $\mathcal{L}$. Fix some compact element $x \leq p$. Then in $\mathcal{L}_m$, p_m is idempotent and compact, and so (Lemma 2) satisfies $p_m \vee (0_m : p_m) = 1_m$. Then $x_m \leq p_m$, so $p_m x_m = x_m$. If m is maximal and $m \not> p$, then $p_m = 1_m$, so again $p_m x_m = x_m$. In $\mathcal{L}$, this yields $(px)_m = x_m$ for every maximal element m, and hence $px = x$. (iii)⇒(iv). Assume (iii). By hypothesis, $\mathcal{L}$ is compactly generated, so p is weak meet principal ([2], Proposition 1.1). Let $x \leq p$ be a join-principal element. Then $x = xp$, so $p \vee (0 : x) = 1$ (Lemma 1). If $m > p$ is a maximal element, then also $m \vee (0 : x) = 1$, so $x = mx$. As p is the join of join-principal elements, $p = mp$. (iv)⇒(v). Assume (iv). Let $m > p$ be a maximal element. If $a_m \leq p_m$ in $\mathcal{L}_m$, then $a_m \leq p_m = p$ in $\mathcal{L}$, so $a_m = pc$ for some $c \in \mathcal{L}$. It follows that $a_m = p_m c_m$ in $\mathcal{L}_m$, and hence that p_m is weak meet principal in $\mathcal{L}_m$. Pass to $\mathcal{L}_m$. As $\mathcal{L}_m$ is also a strong join principally generated C-lattice, it follows by [2], Theorem 1.2, that p_m is (completely) join irreducible, and hence weak principal. As $p_m = p_m m_m$, it follows that $1_m = m_m \vee (0_m : p_m)$. Hence, $p_m = 0_m$. But then $(p^2)_m = p_m$. Also, if m is maximal and $m \not> p$, then $p_m = 1_m$, so again $(p^2)_m = p_m$. Hence, p is idempotent, and so (iv)⇒(v). (v)⇒(ii). Assume p is weak meet principal and idempotent. Then, as in the proof of (iv)⇒(v), p is locally (completely) join irreducible, and hence locally compact. ■

We now consider s-primes.

LEMMA 4. *Let p be an s-prime of $\mathcal{L}$. Then p is either minimal or maximal.*

PROOF. Let p be an s-prime with $p_1 < p$ for some prime element p_1 of $\mathcal{L}$. Choose any strong join principal $d \leq p$ such that $d \not\leq p_1$. Let p_2 be any prime minimal over $d^2 \vee p_1$ such that $p_2 \leq p$. Note that $(d^2 \vee p_1)_{p_2}$ is p_2-semiprimary. If p_2 is nonmaximal, by Theorem 1, $p_2 = (d^2 \vee p_1)_{p_2}$. So $d \leq (d^2 \vee p_1)_{p_2}$, and therefore $dy \leq d^2 \vee p_1$ for some $y \not\leq p_2$. Then $y \leq (d^2 \vee p_1) : d = (p_1 : d) \vee d \leq p_1 \vee d \leq p_2$. But this contradicts the choice of $y \not\leq p_2$. Therefore $p_2 \leq p$ is maximal, and hence $p = p_2$ is maximal. ■

COROLLARY 1. *If $\mathcal{L}$ is a domain, then nonzero s-prime elements are maximal.*

LEMMA 5. *Let p be a nonminimal prime which is an s-prime element of $\mathcal{L}$*

and let Q_p be the collection of all p-primary elements. Then $\wedge Q_p$ is prime and the only prime element properly contained in p.

PROOF. Let p_1 be any prime element such that $p_1 < p$. If q is p-primary, then $p_1 q$ is p_1-semiprimary, so (Theorem 1) $p_1 q = p_1$, hence $p_1 \leq \wedge Q_p$. Let $d \leq \wedge Q_p$ be any strong join-principal element such that $d \nleq p_1$. Then p is a minimal prime over $(d^2 \vee p_1)$ (by Lemma 4), so $(d^2 \vee p_1)_p$ is p-primary. Then $d \nleq (d^2 \vee p_1)_p$. Otherwise, as in the proof of Lemma 4, there is an element $y \nleq p$ with $dy \leq d^2 \vee p_1$, and then $y \leq p$, a contradiction. But this contradicts the choice of $d \leq \wedge Q_p$. Therefore $\wedge Q_p$ is the only prime element properly contained in p. ■

THEOREM 4. *Let m be a prime element of $\mathcal{L}$. Then m is an s-prime element if and only if m satisfies any one of the following conditions.*

(i) m is both maximal and minimal.

(ii) m is a minimal prime and a weak s-prime.

(iii) m is maximal but not a minimal prime, there exists a unique prime $p < m$, and p is a weak s-prime.

PROOF. Suppose m is an s-prime. By Lemma 4, m satisfies either (i) or (ii). Suppose m is maximal but nor minimal. Let p be a prime element such that $p < m$. Then (Lemma 5) p is the only prime contained in m and p is a weak s-prime. The condition that a maximal element m is an s-prime is equivalent to the condition that every prime $p < m$ is a weak s-prime. Hence, each of (i)–(iii) implies m is an s-prime. ■

L is said to be a *domain* if the zero element is prime. L is a *primary lattice* if it contains exactly one prime element m (say). In the latter case, it is clear that L is quasi local and that every element $q < 1$ is primary for the maximal element.

THEOREM 5. *Suppose m is a maximal element of $\mathcal{L}$. Then m is an s-prime if and only if $\mathcal{L}$ is either a one-dimensional domain or a primary lattice.*

PROOF. We can assume L is quasi local with maximal element m. ($\Rightarrow$). Assume that m is an s-prime. If m is minimal, the result is immediate. Assume that m is not minimal. Then (iii) of Theorem 4 holds. Let p be the unique minimal prime satisfying $p < m$. Then $p = \sqrt{0}$, so 0 is p-semiprimary, and therefore (Theorem 1) $p = 0$. ($\Rightarrow$). It is clear that every prime in a primary lattice or one-dimensional domain is an s-prime. ■

COROLLARY 2. *The following statements are equivalent for $\mathcal{L}$.*

(i) Every maximal element is an s-prime.

(ii) For each element m, $\mathcal{L}_m$ is either a one-dimensional domain or a primary lattice.

By Corollary 2, if every maximal element is an s-prime, then every nonmaximal prime is a locally compact s-prime. This and Theorem 3 yield the following.

COROLLARY 3. *The following statements are equivalent for $\mathcal{L}$.*
(i) Every maximal prime is an s-prime.
(ii) Every nonmaximal element is idempotent and locally compact.

By the kernel of an element $a \in L$, we mean the meet of all primary elements of the minimal primes of L/a. We denote the kernel of a by a^*. It is easy to see that $a^* = \wedge\{a_p \mid p \text{ is a prime minimal over } a\}$. The following result can be obtained from a related result in [1]. However, we give an independent proof.

LEMMA 6. *Every maximal element of $\mathcal{L}$ is an s-prime if and only if every element is equal to its kernel.*

PROOF. ($\Rightarrow$). Assume every nonmaximal prime is a weak s-prime. Fix $a \in \mathcal{L}$. Let Q be the set of primes minimal over a, and let M be the set of maximal primes containing a. For $q \in Q$, set $M_q = \{m \in M \mid m \geq q\}$. Note that if $q \in Q$ and q is maximal, then for $m \in M$, $q = m$ and hence $a_m = a_q$. On the other hand, if $q \in Q$ is not maximal, then (Theorem 5) for $m \in M_q$, $q_m = 0_m \leq a_q \leq q = q_m$. Hence, in either case, $a_q = a_m$ for all $m \in M_q$. Then $a = \wedge_{m\in M} a_m = \wedge_{q\in Q} \wedge_{m\in M_q} a_m = \wedge_{q\in Q} a_q = a^*$. ($\Leftarrow$). Assume every element is its own kernel. Let p be a nonmaximal prime and assume $\sqrt{a} = p$. Then p is the unique prime minimal over a, so $a = a_p$. As a_p is primary in $\mathcal{L}_p$, it is also primary in $\mathcal{L}$. Hence a is p-primary. It follows that every nonmaximal element is a weak s-prime, and hence that every maximal element is an s-prime. ■

THEOREM 6. *Assume $\mathcal{L}$ is modular. Then the following are equivalent.*
(i) $\mathcal{L}$ contains only finitely many minimal primes and every nonmaximal prime is idempotent and weak meet principal.
(ii) $\mathcal{L}$ is a finite direct product of primary C-lattices and one-dimensional C-lattice domains, and each factor of the direct product is strong join-principally generated.

PROOF. (i)$\Rightarrow$(ii). Let $p_1, p_2, \ldots, p_k, p_{k+1}, \ldots p_n$ be the distinct minimal prime elements. By Theorem 3, nonmaximal primes are weak s-primes, and so maximal primes are s-primes. Assumes that for $1 \leq i \leq k$, p_i is nonmaximal and for $k+1 \leq j \leq n$, p_j is maximal. By Lemma 5, p_i's are pairwise comaximal. By Lemma 6 and Theorem 1, $0 = p_1 \wedge \cdots \wedge p_k \wedge q_{k+1} \wedge \cdots \wedge q_n$, where each q_j is p_j-primary for $k+1 \leq j \leq n$. Again since the p_i's are pairwise comaximal, it follws that the p_i's and q_i's are pairswise comaximal and so $\mathcal{L} \simeq \mathcal{L}/p_1 \times \cdots \times \mathcal{L}/p_k \times \mathcal{L}/q_{k+1} \times \cdots \times \mathcal{L}/q_n$. Note that by Theorem 5, each $\mathcal{L}/p_i$, $(1 \leq i \leq k)$ is a one-dimensional domain and each $\mathcal{L}/q_j$ $(k+1 \leq j \leq n)$ is a primary lattice. (ii)$\Rightarrow$(i) is straightforward. ■

COROLLARY 4. *Let $\mathcal{L}$ be an r-lattice. Then the following statements are equivalent:*

(i) $\mathcal{L}$ contains only finitely many minimal primes and every nonmaximal prime is idempotent and weak meet principal.

(ii) $\mathcal{L}$ is a finite direct product of primary r-lattice and one-dimensional r-lattice domains.

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