Arch. Rational Mech. Anal. 115 (1991) 61–100.

Stability of Relative Equilibria. Part II: Application to Nonlinear Elasticity

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Communicated by P. HOLMES

Contents

Glossary: Summary of notation employed for elasticity

$Q = \mathrm{Emb}^+(\mathscr{B}, \mathbb{R}^3)$	*Configuration Space*, with elements denoted by $\varphi \in Q$.
TQ	*State Space*; points in the state space correspond to configurations and velocities and are denoted by $(\varphi, \dot{\varphi})$.

$P = T^*Q$	*Phase Space*; points in P correspond to configurations and momenta and are denoted by $z = (\varphi, p)$.
$(\delta\varphi, \delta p)$	Configuration-momentum variations in $T_\varphi Q \times T^*_\varphi P$.
SO(3)	*Special orthogonal group*; orthogonal 3×3 matrices with determinant 1.
$so(3)$	*Lie algebra* of SO(3); 3×3 skew symmetric matrices.
$\eta_Q(\varphi)$	*Infinitesimal generator*; $\eta_Q(\varphi) = \eta \times \varphi$.
$\langle\cdot,\cdot\rangle_g$	*Riemannian metric*; for elasticity the inner product $\langle \delta\varphi_1, \delta\varphi_2 \rangle_g = \int_{\mathscr{B}} \varrho_{\text{ref}}\, \delta\varphi_1 \cdot \delta\varphi_2\, dV.$
$\mathscr{I}(\varphi)$	*Locked inertia tensor*; defined as $\mathscr{I}(\varphi) = \int_{\mathscr{B}} (\lvert\varphi\rvert^2 \mathbf{1}_3 - \varphi \otimes \varphi)\, dV.$
$\mathsf{A}(\varphi)$	*First elasticity tensor*; defined as $\mathsf{A}(\varphi) = \dfrac{\partial^2 W}{\partial \boldsymbol{F}\, \partial \boldsymbol{F}}\Big\vert_{F=D\varphi}.$
$J: P \to so^*(3)$	*Angular momentum map*; $J(\varphi, p)\cdot\eta = \langle p, \eta_Q(\varphi)\rangle$.
$K: P \to \mathbb{R}$	*Kinetic* energy.
$V: Q \to \mathbb{R}$	*Potential* energy.
$H: P \to \mathbb{R}$	*Hamiltonian* function; $H = K + V$.
$H_\xi : P\times\mathbb{R}^3 \to \mathbb{R}$	*Energy-momentum* functional (*Routhian*); $H_\xi = K + V - (J - \mu_e)\cdot\xi.$
$\pounds_a b$	*Lie derivative* of b in direction a.
$\varrho_{\text{ref}} B(\varphi)$	Configuration dependent body force with potential $L: Q \to \mathbb{R}$.

§ 1. Introduction

The problem of the *dynamical stability* of mechanical systems has long been recognized as a fundamental problem of mechanics. An important class of such problems is concerned with the stability of steady motions. This type of problem arises naturally in the study of rotating systems and includes problems in celestial mechanics as well as classical problems in rigid-body mechanics concerned with spinning tops, gyrostats and so on. Over a century ago, in his Adams Prize-winning essay, ROUTH [1877] investigated the subject of *the criterion for dynamical stability*. In this now classic work he constructed a modified Lagrangian function, which subsequent authors have called the *Routhian*, by appending integrals of the motion and applying an 'energy criterion of stability'. This fundamental work is described in many standard references, such as WHITTAKER [1959], PARS [1965] or GOLDSTEIN [1981], and constitutes the point of departure of the modern theory of reduction of mechanical systems with symmetry; see, e.g., ARNOLD et al. [1988, Chapter 3], and MARSDEN & RATIU [1986] for recent surveys. Interestingly enough, ROUTH's work also played a significant role in stability analyses of classical linear control systems; e.g., the well-known Routh-Hurwitz criterion.

The modern point of view on stability of relative equilibria is initiated in the work of ARNOLD [1966a] where *explicit* conditions for formal stability are given

when the symmetry group coincides with the configuration space. This situation encompasses classical rigid-body dynamics and incompressible perfect fluids governed by Euler's equations. Rigorous nonlinear stability results for planar incompressible fluid flows are given in ARNOLD [1966b]. This method was subsequently formalized, extended and applied to additional examples, including fluids and plasmas, in HOLM, MARSDEN, RATIU & WEINSTEIN [1985] who coined the expression *Energy-Casimir* method. Further applications of ARNOLD's method include the works of KRISHNAPRASAD & MARSDEN [1987] on rigid bodies with a certain class of attached flexible appendages, and LEWIS [1989] on self-gravitating planar drops, among others. However, as noted in ARNOLD [1966a], in SIMO, POSBERGH & MARSDEN [1989] and in SIMO, LEWIS & MARSDEN [1990], hereafter referred to as Part I, the extension of the Energy-Casimir method to more general mechanical systems, even to cases in which the configuration space is isomorphic to the symmetry group, encounters a fundamental difficulty: Casimir functions and, in general, conserved quantities in the reduced (convective) representation, for many simple mechanical systems of interest are difficult to characterize or may indeed *not exist at all.* That this difficulty is not merely formal is illustrated by the following examples where the existence of Casimir is not known: three-dimensional elastodynamics, general three-dimensional (Cosserat) rods, plate and shell models, and three-dimensional incompressible fluid flow, (the only known Casimir function is the helicity). By contrast, these examples possess well-known conserved quantities in the *material* or canonical representation as defined by the corresponding momentum maps. For three-dimensional elasticity, rods, plates and shells these conserved quantities are the classical linear and angular momentum and for the isotropic case, Eshelby's energy-momentum tensor. For three-dimensional isentropic flows the conserved quantity in the canonical description is the circulation along closed loops, as a result of the classical Kelvin circulation theorem.

The basic difficulty alluded to above constitutes the main motivation for our approach to the stability of relative equilibria, which is formulated directly in the material representation and exploits in a crucial manner the energy-momentum mapping. This approach, referred to as the *energy-momentum* method, is introduced and applied to examples including rigid bodies with attached flexible appendages in SIMO, POSBERGH & MARSDEN [1989], and homogeneous elasticity in LEWIS & SIMO [1990]. Geometric aspects underlying the method are discussed in a general abstract setting in MARSDEN, SIMO, LEWIS & POSBERGH [1989]. An essential aspect in the application of the method is a result which states that a *block-diagonalization* of the second variation of the energy-momentum map can always be achieved by a suitable choice of coordinates which separates the rigid body modes associated with the action of the symmetry group from the internal vibrational modes. This block-diagonalization leads to particularly tractable stability conditions and is examined in detail in § 2.E of Part I.

For *simple mechanical system with symmetry* in the sense of SMALE [1970], the energy-momentum method can be reformulated in an alternative form, directly in terms of SMALE's *amended potential,* which achieves three crucial properties: (i) The stability results are optimal (sharp). (ii) Maximal reduction of dimension, as far as stability analysis is concerned, is achieved: The method

operates only in terms of configuration variations; in contrast with our original formulation of the energy-momentum method, variations in momenta play no role in the analysis. (iii) The enforcement of the conserved quantities is accomplished without explicit use of Lagrange multipliers. These properties, in particular, the maximal reduction of dimensions in the stability analysis, motivate the term *reduced energy-momentum method* coined in Part I of this work.

In this paper we consider the concrete application of the *reduced energy-momentum method* to an infinite-dimensional and highly non-trivial example: three-dimensional elasticity. Two main objectives motivate this work. First, we provide a detailed illustration of the abstract setting discussed in Part I of this paper in a concrete example which has a strong interest of its own. Second, we demonstrate that the application of the method to an infinite-dimensional example leads to explicit and readily tractable stability conditions, which can be implemented by means of numerical analysis techniques or modern symbolic computations. In particular, our analysis leads to the following results:

i. We provide a complete characterization of the possible relative equilibria of an anisotropic nonlinearly elastic body possessing a general form of stored energy function.

ii. We derive sufficient conditions for formal stability of the relative equilibria by exploiting in a crucial manner our block-diagonalization theorem for general simple mechanical systems with symmetry in the context of the reduced energy-momentum method.

iii. We provide a concrete mechanical interpretation for one set of stability conditions; namely, we show that stable stationary rotations are about the maximum axis of a certain locked inertia dyadic associated with the relative equilibrium. Furthermore, we give a constructive procedure for the remaining set of stability conditions in terms of a straightforward eigenvalue problem. This approach can be readily implemented in a numerical analysis context using a Galerkin finite element projection.

iv. We give a concrete mechanical interpretation of the block-diagonalization procedure and discuss in detail the structure of the symplectic two form in the context of elasticity. In particular, we show that the block diagonalization procedure also puts the linearized dynamics in normal form.

In contrast with the finite-dimensional case, in the present infinite-dimensional context our stability results are only formal. The reason for this formal nature of our results is the current status of existence theory in nonlinear elasticity, as summarized in CIARLET [1988, Chapter 7] or MARSDEN & HUGHES [1983, Chapter 6]. We remark, however, that the present analysis gives *conditional stability* results for the relative equilibria by appealing to arguments discussed in detail in BALL & MARSDEN [1984]. This conclusion rests on the following considerations. First, relative equilibria are characterized as minimizers of the amended potential V_{μ_e}. Second, the amended potential consists of the potential energy associated with the stored energy function, assumed to be polyconvex, plus a term which gives the potential energy of the loading associated with the centrifugal loading in stationary rotation. Accordingly, the results of BALL [1977], and BALL

& MARSDEN [1984] apply, provided that the loading potential is continuous as a mapping from $W^{1,p}(\mathscr{B})$ to $L^2(\mathscr{B})$. Such a requirement appears to hold in view of the structure of the locked inertia tensor in three-dimensional elasticity.

§ 2. Notation: Configurations and phase space for elasticity

In this section we summarize some basic notions of three-dimensional elasticity. For further details we refer to MARSDEN & HUGHES [1983] and CIARLET [1988].

§ 2.A. *Configuration and phase space*

We denote by $\mathscr{B} \subset \mathbb{R}^3$ the reference placement of an elastic body, and assume that $\mathscr{B}$ is open and bounded with smooth boundary $\partial\mathscr{B}$. We let

$$Q := \{\varphi : \mathscr{B} \to \mathbb{R}^3 \mid \det [D\varphi] > 0\} \tag{2.1}$$

be the *configuration manifold.* Typically, for elastodynamics, one assumes that $Q \subset H^s(\mathscr{B})$ with $s > \frac{3}{2} + 1$. The *velocity phase space*, i.e., the space of configuration-velocity fields, is the tangent bundle, defined as

$$TQ := \{V_\varphi = (\varphi, \delta\varphi) \mid \varphi \in Q \text{ and } \delta\varphi \in \chi(\mathscr{B}, \mathbb{R}^3)\}, \tag{2.2}$$

where $\chi(\mathscr{B}, \mathbb{R}^3) := \{\delta\varphi : \mathscr{B} \to \mathbb{R}^3\}$ is the space of smooth vector-valued functions on $\mathscr{B}$. The *canonical phase space* P is the space of configurations and momenta: the cotangent bundle $P = T^*Q$. We shall use the notation

$$T^*Q = \{z_\varphi = (\varphi, p) \mid \varphi \in Q \text{ and } p \in \text{Den}(\mathscr{B})\}, \tag{2.3}$$

where Den $(\mathscr{B})$ is the space of one-form densities on $\mathscr{B}$ (*i.e.*, one forms times the volume element). TQ and T^*Q are in duality via the L_2-pairing denoted by $\langle \cdot, \cdot \rangle$ in what follows. The canonical symplectic structure on P is induced by the *symplectic two-form*, $\Omega : TP \times TP \to \mathbb{R}$, defined by the standard expression

$$\Omega(z)(\delta z_1, \delta z_2) := \langle \delta p_2, \delta\varphi_1 \rangle - \langle \delta p_1, \delta p_2 \rangle, \tag{2.4}$$

for any $\delta z_\alpha = (\delta\varphi_\alpha, \delta p_\alpha) \in T_z P$, $(\alpha = 1, 2)$. The state space and the phase space are related through the *Legendre transformation* $\mathbf{FL} : TQ \to T^*Q$ defined by the standard formula

$$p := \varrho_{\text{ref}} \dot{\varphi}, \tag{2.5}$$

for $(\varphi, \dot{\varphi}) \in TQ$. Here $\dot{\varphi} = \partial\varphi/\partial t$ is the *material velocity* field, and $\varrho_{\text{ref}} : \mathscr{B} \to \mathbb{R}$ the *mass density* in the reference placement $\mathscr{B}$.

Classical three-dimensional elasticity is an infinite-dimensional Hamiltonian system with canonical phase space (P, Ω), and Hamiltonian $H : P \to \mathbb{R}$ equal to the sum of potential and kinetic energy, defined by the expression

$$H = K + V := \tfrac{1}{2} \int_{\mathscr{B}} |p|^2 \varrho_{\text{ref}}^{-1} \, dV + \int_{\mathscr{B}} W(X, D\varphi) \, dV. \tag{2.6}$$

Here $K: P \to \mathbb{R}$ is the kinetic energy, and $V: Q \to \mathbb{R}$ is the potential energy defined in terms of a *stored energy function* $W: \mathscr{B} \times GL^+(3, \mathbb{R}) \to \mathbb{R}$, which depends on the motion locally through the *deformation gradient* $\boldsymbol{F} := D\boldsymbol{\varphi}$. We assume that W is *frame-indifferent* in the sense that

$$W(\boldsymbol{X}, \boldsymbol{F}) = W(\boldsymbol{X}, \boldsymbol{AF}) \quad \text{for all} \quad \boldsymbol{A} \in \mathrm{SO}(3), \quad \boldsymbol{X} \in \mathscr{B} \quad \text{and} \quad \boldsymbol{F} \in GL^+(3, \mathbb{R}). \tag{2.7}$$

Equivalently, the stored energy function W is *left-invariant under the (left-) action of* SO(3). Given any $\boldsymbol{\eta} \in \mathbb{R}^3$ we denote by $\boldsymbol{\eta}_Q(\boldsymbol{\varphi}) \in T_{\boldsymbol{\varphi}}Q$ the *infinitesimal generator* of the SO(3)-action, defined by

$$\boldsymbol{\eta}_Q(\boldsymbol{\varphi}) := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \exp_{\mathrm{SO}(3)} [\varepsilon \hat{\boldsymbol{\eta}}]\boldsymbol{\varphi} = \hat{\boldsymbol{\eta}}\boldsymbol{\varphi} := \boldsymbol{\eta} \times \boldsymbol{\varphi}, \tag{2.8}$$

where $so(3)$ is the Lie algebra of SO(3). Recall that $so(3)$ is the linear space of skew-symmetric matrices whose Lie bracket is the ordinary matrix commutator denoted by $[\cdot, \cdot]$. The Lie algebra $(so(3), [\cdot, \cdot])$ is identified with $(\mathbb{R}^3, \times)$, where $\times$ denotes the ordinary cross product, in the standard fashion via the Lie algebra isomorphism $\hat{}: so(3) \to \mathbb{R}^3$ defined by (2.8). One has the standard relation

$$[\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}] = \widehat{\boldsymbol{\xi} \times \boldsymbol{\eta}}, \quad \text{for all} \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3. \tag{2.9}$$

With this notation at hand, the differentiation of the invariance condition (2.7) with respect to the group action and the use of (2.8) yields the relation

$$\hat{\boldsymbol{\eta}}\boldsymbol{F}: \partial_{\boldsymbol{F}} W(\boldsymbol{X}, \boldsymbol{F}) = \hat{\boldsymbol{\eta}}: \partial_{\boldsymbol{F}} W(\boldsymbol{X}, \boldsymbol{F})\, \boldsymbol{F}^T = \boldsymbol{0}, \quad \text{for all} \quad \boldsymbol{\eta} \in \mathbb{R}^3, \tag{2.10}$$

where the symbol " : " denotes the inner product of two rank-two tensors. This relation implies the classical symmetry condition on the Kirchhoff stress tensor:

$$\boldsymbol{\tau} := \partial_{\boldsymbol{F}} W(\boldsymbol{X}, \boldsymbol{F})\, \boldsymbol{F}^T = \boldsymbol{\tau}^T. \tag{2.11}$$

Of course, (2.11) is equivalent to the local form of balance of angular momentum.

§ 2.B. *Momentum maps*

The Hamiltonian $H: P \to \mathbb{R}$ defined by (2.6) is invariant under the left action of the orthogonal group in the sense that

$$H(\boldsymbol{\Lambda} \cdot z) = H(z), \quad \text{for all} \quad (z, \boldsymbol{\Lambda}) \in P \times \mathrm{SO}(3), \tag{2.12}$$

where the (symplectic) left action of SO(3) on P is defined by $\boldsymbol{\Lambda} \cdot z := (\boldsymbol{\Lambda}\boldsymbol{\varphi}, \boldsymbol{\Lambda}\boldsymbol{p})$ for $z = (\boldsymbol{\varphi}, \boldsymbol{p})$. The invariance property (2.12) follows from the assumption of frame-invariance on the stored energy function and the fact that rotations are isometries relative to the Euclidean dot product.

By the classical Noether theorem, associated with the invariance property (2.12) there is a *conserved quantity* called the *momentum map*, denoted by $\boldsymbol{J}: P \to so^*(3)$, and given by the abstract formula (see, e.g., ABRAHAM & MARSDEN [1978, p. 285])

$$\boldsymbol{J}(z) \cdot \boldsymbol{\eta} = \langle \boldsymbol{p}, \boldsymbol{\eta}_Q(\boldsymbol{\varphi}) \rangle \quad \text{for} \quad \hat{\boldsymbol{\eta}} \in so(3). \tag{2.13}$$

Here, $so^*(3)$ is the dual of the Lie algebra $so(3)$ identified with $\mathbb{R}^3$ via the Euclidean dot product and, in the present context, $\langle \cdot, \cdot \rangle$ is the L_2-pairing. Since $\eta_Q(\varphi) = \eta \times \varphi$, the abstract formula (2.13) gives

$$J(z) \cdot \eta = \int_{\mathscr{B}} \eta \times \varphi \cdot p \, dV = \eta \cdot \int_{\mathscr{B}} (\varphi \times p) \, dV, \tag{2.14}$$

for any $\eta \in \mathbb{R}^3$. Consequently, the associated momentum map as a function $J: P \to \mathbb{R}^3$ is given by the explicit expression

$$J(z) = \int_{\mathscr{B}} \varphi \times p \, dV, \tag{2.15}$$

which, as expected, is the classical definition of *total angular momentum* of the system.

From expression (2.6), it follows that the Hamiltonian function for elasticity is also invariant under the group of translations in phase space, i.e., under mappings $(\varphi, p) \mapsto (\varphi + c, p)$, for any $c \in \mathbb{R}^3$. The associated conserved quantity is the momentum map $\bar{J}: P \to \mathbb{R}^3$ defined via the general formula (2.13) by the expression

$$\bar{J}(z) = \int_{\mathscr{B}} p \, dV. \tag{2.16}$$

As expected, $\bar{J}$ defined by (2.16) gives the *total linear momentum* of the system. Note that the two momentum maps derived above correspond to the left action of the Euclidean group realized as the semidirect product $\mathbb{R}^3 \times SO(3)$.

According to the preceding discussion, we view classical three-dimensional elasticity as a particular instance of a Hamiltonian system with symmetry. The symmetry arises through the (symplectic) action of the Euclidean group on the canonical phase space, and gives rise to the conserved quantities (momentum maps) (2.15) and (2.16). Our next objective is to characterize explicitly particular solutions of Hamilton's equations known as *relative equilibria*, a terminology due to Poincaré, and examine their stability under finite perturbations of the initial conditions.

§ 3. The energy-momentum functional: First variation

By definition, *relative equilibria* are dynamic solutions of Hamilton's equations which are also group orbits. For SO(3) these are *uniformly rotating states*. It is a general fact that the relative equilibria of a Hamiltonian system with symmetry are critical points of the energy subject to the constraint of constant momentum map. This is the content of the general *Relative Equilibrium* Theorem in § 1.B of Part I. As discussed in § 2.B, for elasticity the symmetry group is the Euclidean group acting on the left and the momentum maps (the conserved quantities) are the total linear and angular momentum. According to this *constrained* variational characterization, the relative equilibria can be conveniently computed by the classical method of Lagrange multipliers as stationary points of the *energy*

momentum functional $H_{\mu_e,l_e}: P\times[\mathbb{R}^3\times\mathbb{R}^3]\to\mathbb{R}$ defined by

$$H_{\mu_e,l_e} := H - \xi\cdot(J-\mu_e) - u\cdot(\bar{J}-l_e). \tag{3.1}$$

Here, $(u, \xi)\in\mathbb{R}^3\times\mathbb{R}^3$ are the Lagrange multipliers, $(\mu_e, l_e)\in\mathbb{R}^3\times\mathbb{R}^3$ are the total angular momentum and linear momentum at the relative equilibrium $z_e\in P$, and $J(z)$ and $\bar{J}(z)$ are the total linear and angular momentum maps computed in § 2.B. Note that the identification $so(3)\approx\mathbb{R}^3$ is used in the preceding expression. The explicit computation of the stationary points of (3.1) (i.e., the relative equilibria) is considered in detail below.

§ 3.A. *The effective potential: First variation*

Our first step is to reformulate the energy-momentum map in an alternative format better suited for our subsequent stability analysis (see Proposition 2.1 and 2.2 of Part I for the general result). To this end, recall that the kinetic energy is associated with a Riemannian metric $\langle\cdot,\cdot\rangle_g$ on the state space TQ which, in the context of elasticity, is simply the L_2-inner product weighted by the density function; i.e.,

$$\langle\delta\varphi_1, \delta\varphi_2\rangle_g := \int_{\mathscr{B}} \varrho_{\text{ref}}\,\delta\varphi_1\cdot\delta\varphi_2\,dV, \tag{3.2}$$

for all $\delta\varphi_1$, $\delta\varphi_2$ in $T_\varphi Q$. Further, recall that $\xi_Q(\varphi) := \xi\times\varphi$ is the infinitesimal generator of the left action of $G = \mathrm{SO}(3)$ on Q.

Either by a direct computation, or from the general result in Proposition 2.1 of Part I, it follows that the energy momentum function (3.1) can be written

$$H_{\mu_e,l_e} = V_{\xi,u} + K_{\xi,u} + \xi\cdot\mu_e + u\cdot l_e. \tag{3.3}$$

Here, $V_{\xi,u}: Q\times[\mathbb{R}^3\times\mathbb{R}^3]\to\mathbb{R}$ is the *augmented potential function* defined in terms of the kinetic energy metric (3.2) by

$$V_{\xi,u} = V + L_{\xi,u},$$

where

$$L_{\xi,u}(\varphi) = -\tfrac{1}{2}\int_{\mathscr{B}} \varrho_{\text{ref}}\,|u + \xi\times\varphi|^2\,dV, \tag{3.4}$$

whereas $K_{\xi,u}: P\times[\mathbb{R}^3\times\mathbb{R}^3]\to\mathbb{R}$ is the *augmented kinetic energy*, given by the expression

$$K_{\xi,u}(z) = \tfrac{1}{2}\int_{\mathscr{B}} |p - \varrho_{\text{ref}}(u + \xi\times\varphi)|^2\,\varrho_{\text{ref}}^{-1}\,dV. \tag{3.5}$$

It will be explicitly shown below that each of the functions V_ξ and $K_{\xi,u}$ *independently* has a critical point at a relative equilibrium. This fact constitutes the main motivation for rephrasing the energy-momentum map as (3.3).

To compute the first variation of (3.3) we introduce some notation. Let $\varepsilon\mapsto\varphi_\varepsilon := (\varphi + \varepsilon\,\delta\varphi)\in Q$ be a curve in Q with $\varphi_\varepsilon|_{\varepsilon=0} = \varphi$, and $\left.\dfrac{d}{d\varepsilon}\right|_{\varepsilon=0}\varphi_\varepsilon = \delta\varphi$;

so that $\delta\varphi \in T_\varphi Q$. This curve induces a curve in $T^*Q = P$ obtained by setting

$$\varepsilon \mapsto z_\varepsilon := (\varphi_e, p + \varepsilon\, \delta p) \in P, \tag{3.6}$$

so that $(\delta\varphi, \delta p) \in T_z P$. We shall use the following notation for the directional derivative

$$DH(z) \cdot \delta z = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} H(z_\varepsilon). \tag{3.7}$$

We shall also denote by $(\delta H/\delta z)(z)$ the functional derivative, which is defined in the standard fashion as

$$DH(z) \cdot \delta z = \left\langle \delta z, \frac{\delta H(z)}{\delta z} \right\rangle, \tag{3.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing chosen, in the present context, as the L_2-pairing, and *not* the metric pairing (3.2).

With the preceding notation in hand, substitution of (3.6) into the potential energy part of the Hamiltonian (2.6), use of (3.7) and integration by parts yields

$$\begin{aligned} DV(\varphi) \cdot \delta\varphi &= \int_{\mathscr{B}} \partial_F W : \nabla(\delta\varphi)\, dV \\ &= -\int_{\mathscr{B}} \delta\varphi \cdot \operatorname{Div}(\partial_F W)\, dV + \int_{\partial\mathscr{B}} \delta\varphi \cdot [\partial_F W]\, N\, dV. \end{aligned} \tag{3.9}$$

After making use of a standard vector-product identity, we can write the first variation of the potential function $L_{\xi,u}$ defined by $(3.4)_2$ in the form

$$\begin{aligned} DL_{\xi,u}(\varphi) \cdot \delta\varphi &= -\int_{\mathscr{B}} \varrho_{\mathrm{ref}}(\xi \times \delta\varphi) \cdot (u + \xi \times \varphi)\, dV \\ &= \int_{\mathscr{B}} \delta\varphi \cdot \varrho_{\mathrm{ref}}[\xi \times (u + \xi \times \varphi)]\, dV. \end{aligned} \tag{3.10}$$

On the other hand, substitution of (3.6) into (3.5) and use of (3.7) yields the first variation of the augmented kinetic energy function as

$$DK_{\xi,u}(z) \cdot \delta z = \int_{\mathscr{B}} (\delta p - \varrho_{\mathrm{ref}}\, \xi \times \delta\varphi) \cdot [p - \varrho_{\mathrm{ref}}(u + \xi \times \varphi)]\, \varrho_{\mathrm{ref}}^{-1}\, dV. \tag{3.11}$$

The classical optimality conditions for $(z_e, \xi, u) \in P \times \mathbb{R}^3 \times \mathbb{R}^3$ to be a critical point of H_{μ_e, l_e} require that

$$\begin{gathered} \frac{\delta}{\delta z} H_{\mu_e, l_e}(z_e, \xi_e, u_e) = 0, \\ \frac{\delta}{\delta \xi} H_{\mu_e, l_e}(z_e, \xi_e, u_e) = 0, \quad \text{and} \quad \frac{\delta}{\delta u} H_{\mu_e, l_e}(z_e, \xi_e, u_e) = 0. \end{gathered} \tag{3.12}$$

The last two conditions simply tell us that $J(z_e) = \mu_e$ and $\bar{J}(z_e) = l_e$. From expressions (3.4), (3.5) and (3.3) it follows that the optimality condition $(3.12)_1$

holds if and only if

$$\frac{\delta V_{\xi_e,u_e}}{\delta \varphi}(\varphi_e) = \mathbf{0} \quad \text{and} \quad \frac{\delta K_{\xi_e,u_e}}{\delta z}(z_e) = \mathbf{0}, \tag{3.13}$$

which, in view of (3.9), (3.10) and (3.11), result in the following *relative equilibria conditions*:

$$\left.\begin{aligned} p_e &= \varrho_{\text{ref}}[\xi_e \times \varphi_e + u_e] \\ \text{Div}\,[\partial_F W_e] &= \xi_e \times \varrho_{\text{ref}}[\xi_e \times \varphi_e + u_e] \end{aligned}\right\} \text{ in } \mathscr{B}, \tag{3.14}$$

along with the stress-free boundary condition $[\partial_F W_e]\, N|_{\partial\mathscr{B}} = 0$. Here $N: \partial\mathscr{B} \to S^2$ denotes the unit-outward normal to $\partial\mathscr{B}$, where $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ is the unit sphere in $\mathbb{R}^3$.

Conditions (3.14), which characterize the relative equilibrium configuration $\varphi_e \in Q$ as critical points of the augmented potential V_{ξ_e,u_e} and characterize the momentum $p_e \in T^*_{\varphi_e}Q$ (defined by $(3.14)_1$)) as a critical point of $K_{\xi,u}$, are of course consistent with the general result in Propositions 2.1–2.2 of Part I.

An interesting question is concerned with the existence of solutions to the boundary value problem (3.14). In general, this is a problem to which the techniques of BALL [1977] may be applied. In this context, the problem reduces to the existence of minimizers for $V_{\xi,u}$ (or V_{μ_e,l_e}; see § 5.D for further details). Alternatively, if $\xi \in \mathbb{R}^3$ is small so that the relative equilibrium is near the reference (stress-free) state, the methods of CHILLINGWORTH, MARSDEN & WAN [1983] show that there exist slowly rotating smooth relative equilibria in a neighborhood of the reference state.

§ 3.B. *Interpretation of the relative equilibrium conditions*

Conditions (3.14) result in the following mechanical interpretation of the possible relative equilibria associated with the action of the Euclidean group (realized as $\mathbb{R}^3 \times \text{SO}(3)$) on the phase space P.

Theorem 3.1. *Let $z_e = (\varphi_e, p_e) \in P$ be a relative equilibrium. Define the total mass and the position of the center of mass by*

$$M := \int_{\mathscr{B}} \varrho_{\text{ref}}\, dV, \quad \varphi_e^0 := \frac{1}{M}\int_{\mathscr{B}} \varrho_{\text{ref}}\varphi_e\, dV. \tag{3.15}$$

Then the following results hold:

i. *The total linear momentum at equilibrium, $\bar{J}(z_e) = l_e$, is given by the classical formula*

$$l_e = M\dot{\varphi}_e^0, \quad \textit{with} \quad \xi_e \times l_e = \mathbf{0}. \tag{3.16}$$

ii. *The center of mass, with position vector φ_e^0, moves with constant velocity*

$$\dot{\varphi}_e^0 = u_e + \xi_e \times \varphi_e^0 \equiv \textit{constant}. \tag{3.17}$$

iii. *The angular velocity* $\xi_e \in \mathbb{R}^3$ *at equilibrium (for given angular momentum* μ_e*) satisfies the relation*

$$\mu_e = \mathscr{I}_e^0 \xi_e + \varphi_e^0 \times l_e, \tag{3.18}$$

where $\mathscr{I}_e^0 : \mathbb{R}^3 \to \mathbb{R}^3$ *is the locked inertia tensor relative to the center of mass of the relative equilibrium configuration* φ_e*, which is given by*

$$\mathscr{I}_e^0 := \mathscr{I}_e - M[|\varphi_e^0|^2 \mathbf{1}_3 - \varphi_e^0 \otimes \varphi_e^0].$$

where (3.19)

$$\mathscr{I}_e := \int_{\mathscr{B}} \varrho_{\text{ref}}[|\varphi_e|^2 \mathbf{1}_3 - \varphi_e \otimes \varphi_e]\, dV$$

and $\mathbf{1}_3$ *is the three-dimensional identity tensor.*

iv. *The total linear momentum and total angular momentum satisfy the condition*

$$u_e \times l_e + \xi_e \times \mu_e = \mathbf{0}. \tag{3.20}$$

v. *The vector* $\xi_e \in \mathbb{R}^3$*, which defines both the axis of stationary rotation and the angular velocity of the relative equilibrium satisfies the condition*

$$\xi_e \times \mathscr{I}_e^0 \xi_e = \mathbf{0} \quad \textit{if and only if} \quad \mathscr{I}_e^0 \xi_e = \lambda_e \xi_e. \tag{3.21}$$

Therefore, ξ_e *must be aligned with a principal direction of the locked inertia tensor* $\mathscr{I}_e^0$ *(relative to the center of mass).*

Proof. i. Integration of $(3.14)_2$ over $\mathscr{B}$ and use of the divergence theorem and the stress-free boundary condition yield

$$\xi_e \times \int_{\mathscr{B}} p_e\, dV = \mathbf{0}, \tag{3.22}$$

which in view of (2.16) and $(3.12)_2$ is equivalent to $(3.16)_2$. To prove $(3.16)_1$ we differentiate $(3.15)_2$ with respect to time and use the Legendre transformation to obtain

$$\int_{\mathscr{B}} \varrho_{\text{ref}} \dot{\varphi}_e\, dV = \int_{\mathscr{B}} p_e\, dV =: l_e = M\dot{\varphi}_e^0. \tag{3.23}$$

ii. To prove (3.17) we integrate $(3.14)_1$ over $\mathscr{B}$, and use (2.16) and $(3.12)_3$ along with (3.15) to obtain the result

$$l_e = M[u_e + \xi_e \times \varphi_e^0]. \tag{3.24}$$

Differentiation of this expression with respect to time and use of (3.16) yields

$$\frac{d}{dt} l_e = \xi_e \times M\dot{\varphi}_e^0 = \xi_e \times l_e = \mathbf{0}. \tag{3.25}$$

Hence, $l_e =$ constant, and (3.17) follows from (3.24) and $(3.16)_1$.

iii. By making use of (2.15), $(3.12)_2$ and $(3.14)_1$, we can compute the equilibrium value $\boldsymbol{\mu}_e = \boldsymbol{J}(z_e)$ as follows:

$$\begin{aligned}\boldsymbol{\mu}_e &= \int_{\mathscr{B}} \boldsymbol{\varphi}_e \times \boldsymbol{p}_e \, dV \\ &= \int_{\mathscr{B}} [\boldsymbol{\varphi}_e \times (\boldsymbol{\xi}_e \times \boldsymbol{\varphi}_e) + \boldsymbol{\varphi}_e \times \boldsymbol{u}_e] \, \varrho_{\text{ref}} \, dV \\ &= \left\{ \int_{\mathscr{B}} [|\boldsymbol{\varphi}_e|^2 \, \mathbf{1}_3 - \boldsymbol{\varphi}_e \otimes \boldsymbol{\varphi}_e] \, \varrho_{\text{ref}} \, dV \right\} \boldsymbol{\xi}_e + \left\{ \int_{\mathscr{B}} \varrho_{\text{ref}} \boldsymbol{\varphi}_e \, dV \right\} \times \boldsymbol{u}_e. \end{aligned} \tag{3.26}$$

The substitution of definitions (3.19) and $(3.15)_2$ into (3.26) results in the expression

$$\boldsymbol{\mu}_e = \mathscr{I}_e \boldsymbol{\xi}_e + M \boldsymbol{\varphi}_e^0 \times \boldsymbol{u}_e. \tag{3.27}$$

Now we use (3.24) to write the the second term in (3.27) as

$$\begin{aligned} M \boldsymbol{\varphi}_e^0 \times \boldsymbol{u}_e &= -M \boldsymbol{\varphi}_e^0 \times (\boldsymbol{\xi}_e \times \boldsymbol{\varphi}_e^0) + \boldsymbol{\varphi}_e^0 \times \boldsymbol{l}_e \\ &= -M[|\boldsymbol{\varphi}_e^0|^2 \, \mathbf{1}_3 - \boldsymbol{\varphi}_e^0 \otimes \boldsymbol{\varphi}_e^0] \, \boldsymbol{\xi}_e + \boldsymbol{\varphi}_e^0 \times \boldsymbol{l}_e. \end{aligned} \tag{3.28}$$

By combining (3.27) and (3.28), and using $(3.19)_1$ we obtain expression (3.18).

iv. To prove (3.20) we recall that the *left* SO(3)-*invariance condition* (2.11) on the stored energy function implies the relation

$$\sum_{A=1}^{3} \boldsymbol{\varphi}_{,A} \times \mathbf{T}^A = \mathbf{0}, \quad \text{where} \quad \mathbf{T}^A := [\partial_F W] \, \boldsymbol{E}^A. \tag{3.29}$$

Here $\{\boldsymbol{E}^A\}$ is the standard basis in $\mathbb{R}^3$ and $\mathbf{T}^A$ is the *nominal traction vector*. By the divergence theorem, the invariance requirement (3.29) along with the stress-free boundary condition results in the relation

$$\int_{\mathscr{B}} \boldsymbol{\varphi}_e \times \operatorname{Div} [\partial_F W_e] \, dV = \int_{\partial \mathscr{B}} \boldsymbol{\varphi}_e \times [\partial_F W_e \boldsymbol{N}] \, dV - \sum_{A=1}^{3} \int_{\mathscr{B}} \boldsymbol{\varphi}_{e,A} \times \mathbf{T}_e^A \, dV = \mathbf{0}. \tag{3.30}$$

The substitution of (3.14) into the left-hand side of (3.30) yields the equivalent expression

$$\int_{\mathscr{B}} \boldsymbol{\varphi}_e \times [\boldsymbol{\xi}_e \times \boldsymbol{p}_e] \, dV = \mathbf{0}. \tag{3.31}$$

By making use of Jacobi's identity, the equilibrium condition $(3.14)_1$, and (3.24) we find that

$$\begin{aligned} \mathbf{0} &= \boldsymbol{\xi}_e \times \left(\int_{\mathscr{B}} \boldsymbol{\varphi}_e \times \boldsymbol{p}_e \, dV \right) + \int_{\mathscr{B}} \boldsymbol{p}_e \times (\boldsymbol{\xi}_e \times \boldsymbol{\varphi}_e) \, dV \\ &= \boldsymbol{\xi}_e \times \boldsymbol{\mu}_e + \int_{\mathscr{B}} \varrho_{\text{ref}} (\boldsymbol{\xi}_e \times \boldsymbol{\varphi}_e + \boldsymbol{u}_e) \times (\boldsymbol{\xi}_e \times \boldsymbol{\varphi}_e) \, dV \\ &= \boldsymbol{\xi}_e \times \boldsymbol{\mu}_e + \boldsymbol{u}_e \times \int_{\mathscr{B}} \varrho_{\text{ref}} (\boldsymbol{\xi}_e \times \boldsymbol{\varphi}_e) \, dV \\ &= \boldsymbol{\xi}_e \times \boldsymbol{\mu}_e + \boldsymbol{u}_e \times \int_{\mathscr{B}} \boldsymbol{p}_e \, dV \\ &= \boldsymbol{\xi}_e \times \boldsymbol{\mu}_e + \boldsymbol{u}_e \times \boldsymbol{l}_e, \end{aligned} \tag{3.32}$$

which proves (3.20).

v. To prove (3.21) we use (3.18) along with (3.17), (3.16) and Jacobi's identity to obtain

$$\begin{aligned}\xi_e \times \mu_e &= \xi_e \times \mathscr{I}_e^0 \xi_e + \xi_e \times (\varphi_e^0 \times l_e) \\ &= \xi_e \times \mathscr{I}_e^0 \xi_e - \varphi_e^0 \times (l_e \times \xi_e) - l_e \times (\xi_e \times \varphi_e^0) \\ &= \xi_e \times \mathscr{I}_e^0 \xi_e - l_e \times (\dot{\varphi}_e^0 - u_e) \\ &= \xi_e \times \mathscr{I}_e^0 \xi_e - u_e \times l_e, \end{aligned} \tag{3.33}$$

which in view of (3.20) implies (3.21). □

Remarks 3.2. 1. Since the center of mass moves with constant velocity, and $l_e = \bar{J}(z_e)$ is constant, without loss of generality we may assume that

$$\varphi_e^0 = \mathbf{0} \quad \text{implies} \quad \dot{\varphi}_e^0 = \mathbf{0}. \tag{3.34}$$

This requirement amounts to selecting a reference frame with origin at the center of mass of the equilibrium configuration. Relative to this coordinate system one has

$$u_e = \mathbf{0}, \quad \text{and} \quad \mathscr{I}_e^0 = \mathscr{I}_e. \tag{3.35}$$

One then speaks of a *center-of-mass reduction.* In what follows we shall assume that this reduction is made and drop the subscript (i.e., $H_{\mu_e, l_e} = H_{\mu_e}$, etc.)

2. Condition (3.20) is the concrete version of the abstract condition $\mathrm{ad}^*_{\xi_e} \mu_e = \mathbf{0}$, (see Proposition 1.2.ii of Part I) since for $G = \mathrm{SO}(3)$ one has $\mathrm{ad}^*_{\xi_e} \mu_e = \xi_e \times \mu_e$.

We conclude this section by providing a mechanical interpretation of the effective potential V_{ξ_e} in the context of elasticity. As remarked above, we assume throughout that a center-of-mass reduction has been made so that (3.35) holds.

The relative equilibrium condition $(3.14)_2$, and the stress-free boundary condition, which constitute the optimality conditions for the effective potential V_{ξ_e}, lead to the following boundary value problem for the relative equilibrium configurations: Find the configurations $\varphi_e \in Q$ such that

$$\begin{aligned} \mathrm{Div}\,[\partial_F W(D\varphi_e)] + \varrho_{\mathrm{ref}} B(\varphi_e) &= \mathbf{0} \quad \text{in } \mathscr{B}, \\ [\partial_F W(D\varphi_e)]\, N &= \mathbf{0} \quad \text{on } \partial\mathscr{B}. \end{aligned} \tag{3.36}$$

Here, $\varrho_{\mathrm{ref}} B : Q \to \mathbb{R}^3$ is an equivalent *configuration-dependent* body force given by

$$\varrho_{\mathrm{ref}} B(\varphi_e) := \varrho_{\mathrm{ref}} |\xi_e|^2 \varphi_e^{\perp}, \tag{3.37}$$

where

$$\varphi_e^{\perp} := \varphi_e - (\varphi_e \cdot \xi_e) \frac{\xi_e}{|\xi_e|^2},$$

which has the physical interpretation illustrated in Figure 3.1. The body force $\varrho_{\mathrm{ref}} B(\varphi_e)$ gives the centrifugal force, acting on a configuration $\varphi_e \in Q$ in a relative equilibrium, corresponding to a stationary rotation about the axis $\xi_e/|\xi_e|$ with constant angular velocity $|\xi_e|$.

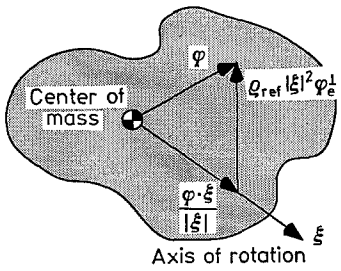

Fig. 3.1. Interpretation of the loading term $\varrho_{\text{ref}}\boldsymbol{B} = \varrho_{\text{ref}}\,|\,\xi_e\,|^2\,\varphi_e^{\perp}$ as a centrifugal force.

Note that the centrifugal force $\varrho_{\text{ref}}\boldsymbol{B}(\varphi)$ is a configuration-dependent *conservative* body force with potential function given by $L_{\xi_e}: Q \to \mathbb{R}$. Making use of a standard vector-product identity, we can write $(3.4)_2$ as (recall relation $(3.35)_2$)

$$L_{\xi_e}(\varphi) = -\tfrac{1}{2}\,\xi_e \cdot \mathscr{I}(\varphi)\,\xi_e, \tag{3.38}$$

where

$$\mathscr{I}(\varphi) := \int_{\mathscr{B}} \varrho_{\text{ref}}[\,|\varphi\,|^2\,\mathbf{1}_3 - \varphi \otimes \varphi]\,dV. \tag{3.39}$$

That $L_{\xi_e}(\varphi)$ furnishes the appropriate potential for the loading term $\varrho_{\text{ref}}\boldsymbol{B}(\varphi)$ in (3.36) is the result of (3.10).

§ 4. The pure traction boundary value problem for the relative equilibria

In this section we consider the structure of the boundary value problem for the relative equilibrium configurations. By examining the left-invariance properties of the *augmented potential* under the (left) action of the proper orthogonal group, we develop a crucial decomposition of the tangent space of variations, which plays a fundamental role in our subsequent stability analysis. We consider first the general case of the pure traction boundary value problem in elastostatics in which the configuration-dependent loading at an arbitrary configuration *does not possess full invariance* under the left action of SO(3). Subsequently, we specialize our results in § 4.C to the the boundary value problem for the relative equilibrium configurations.

§ 4.A. The pure traction boundary value problem in elasticity with partial SO(3)*-invariant, configuration-dependent loading*

Consider the following general Neumann boundary value problem: Find the configurations $\varphi_e \in Q$ such that

$$\begin{aligned} \text{Div}\,[\partial_F W(D\varphi_e)] + \varrho_{\text{ref}}\boldsymbol{B}(\varphi_e) &= \mathbf{0} \quad \text{in } \mathscr{B}, \\ [\partial_F W(D\varphi_e)]\,\boldsymbol{N} &= \mathbf{0} \quad \text{on } \partial\mathscr{B}. \end{aligned} \tag{4.1}$$

Here $\varrho_{\text{ref}}\boldsymbol{B}: Q \to \mathbb{R}^3$ now denotes a *configuration-dependent* body force which is assumed to be derived from a general potential $L: Q \to \mathbb{R}$ according to the relation

$$-DL(\boldsymbol{\varphi}_e) \cdot \boldsymbol{\delta\varphi} = \langle \boldsymbol{\delta\varphi}, \varrho_{\text{ref}}\boldsymbol{B}(\boldsymbol{\varphi}_e)\rangle \quad \text{for all } \boldsymbol{\delta\varphi} \in T_{\varphi_e}Q. \tag{4.2}$$

In view of the definition (3.8) of the functional derivative, this condition is equivalent to

$$\varrho_{\text{ref}}\boldsymbol{B}(\boldsymbol{\varphi}_e) = -\frac{\delta L}{\delta \boldsymbol{\varphi}}(\boldsymbol{\varphi}_e), \tag{4.3}$$

and thus boundary value problem (4.1) may be re-stated as

$$\frac{\delta}{\delta \boldsymbol{\varphi}}[V(\boldsymbol{\varphi}_e) + L(\boldsymbol{\varphi}_e)] = \mathbf{0}. \tag{4.1*}$$

The weak form of the boundary value problem (4.1) is given by

$$G(\boldsymbol{\varphi}_e, \boldsymbol{\delta\varphi}) := \int_{\mathscr{B}} [\partial_F W(D\boldsymbol{\varphi}_e)] : \nabla(\boldsymbol{\delta\varphi})\, dV - \int_{\mathscr{B}} \varrho_{\text{ref}}\boldsymbol{B} \cdot \boldsymbol{\delta\varphi}\, dV, \tag{4.4}$$

for all $\boldsymbol{\delta\varphi} \in T_{\varphi_e}Q$.

Next, we record the necessary conditions for the existence of solutions to boundary value problem (4.1).

i. Let $\boldsymbol{\delta\varphi} = \textit{constant}$ which, in view of (2.2), is in $T_{\varphi_e}Q$. Since $\nabla(\boldsymbol{\delta\varphi}) = \mathbf{0}$, we conclude that

$$\int_{\mathscr{B}} \varrho_{\text{ref}}\boldsymbol{B}(\boldsymbol{\varphi}_e)\, dV = -\int_{\mathscr{B}} \frac{\delta L}{\delta \boldsymbol{\varphi}}(\boldsymbol{\varphi}_e)\, dV = \mathbf{0}. \tag{4.5}$$

ii. Let $\boldsymbol{\delta\varphi} = \boldsymbol{\zeta}_Q(\boldsymbol{\varphi}_e) = \boldsymbol{\zeta} \times \boldsymbol{\varphi}_e$ be a superposed infinitesimal rigid-body motion, which, in view of (2.2), is also in $T_{\varphi_e}Q$. It follows that

$$\nabla \boldsymbol{\zeta}_Q(\boldsymbol{\varphi}_e) = \hat{\boldsymbol{\zeta}} D\boldsymbol{\varphi}_e. \tag{4.6}$$

Since $\tau_e = \partial_F W(D\boldsymbol{\varphi}_e)\,[D\boldsymbol{\varphi}_e]^T$ is symmetric by (2.11), we have

$$\begin{aligned} G(\boldsymbol{\varphi}_e, \boldsymbol{\zeta}_Q(\boldsymbol{\varphi}_e)) &= \int_{\mathscr{B}} [\partial_F W(D\boldsymbol{\varphi}_e)\, D\boldsymbol{\varphi}_e^T] : \hat{\boldsymbol{\zeta}}\, dV - \int_{\mathscr{B}} \varrho_{\text{ref}}\boldsymbol{B}(\boldsymbol{\varphi}_e) \cdot \boldsymbol{\zeta} \times \boldsymbol{\varphi}_e\, dV \\ &= -\boldsymbol{\zeta} \cdot \int_{\mathscr{B}} \boldsymbol{\varphi}_e \times \varrho_{\text{ref}}\boldsymbol{B}(\boldsymbol{\varphi}_e)\, dV = 0, \end{aligned} \tag{4.7}$$

which must hold for all $\boldsymbol{\zeta} \in \mathbb{R}^3$; hence

$$\int_{\mathscr{B}} \boldsymbol{\varphi}_e \times \varrho_{\text{ref}}\boldsymbol{B}(\boldsymbol{\varphi}_e)\, dV = \mathbf{0}. \tag{4.8}$$

Conditions (4.5) and (4.8) are the statement of *force balance* and *moment balance for the body force at equilibrium.*

An equivalent formulation of condition (4.8) in terms of the potential L is

$$\begin{aligned} 0 &= -\int_{\mathscr{B}} \varrho_{\text{ref}} \boldsymbol{B}(\boldsymbol{\varphi}_e) \cdot \boldsymbol{\zeta}_Q(\boldsymbol{\varphi}_e)\, dV \\ &= \left\langle \frac{\delta L}{\delta \boldsymbol{\varphi}_e}(\boldsymbol{\varphi}_e),\, \boldsymbol{\zeta}_Q(\boldsymbol{\varphi}_e) \right\rangle \\ &= DL(\boldsymbol{\varphi}_e) \cdot \boldsymbol{\zeta}_Q(\boldsymbol{\varphi}_e), \end{aligned} \tag{4.9}$$

which, upon using the definition of Lie derivative (denoted by £), leads to

$$(\pounds_{\zeta_Q} L)(\boldsymbol{\varphi}_e) = 0, \quad \text{for all } \hat{\boldsymbol{\xi}} \in so(3). \tag{4.10}$$

The preceding conditions are therefore equivalent to the requirement that the loading potential $L: Q \to \mathbb{R}$ be *infinitesimally left-invariant* under the action of the Euclidean group at an *equilibrium solution* $\boldsymbol{\varphi}_e \in Q$. Assuming that translational invariance holds, we are led to the requirement that

$$L(\boldsymbol{\varphi}_e) \text{ must possess full infinitesimal SO(3)-invariance at } \boldsymbol{\varphi}_e \in Q. \tag{4.11}$$

The crucial observation to be made is that while L must possess full SO(3)-invariance *at an equilibrium solution* $\boldsymbol{\varphi}_e \in Q$ of the boundary value problem (4.1), in general, $L: Q \to \mathbb{R}$ *need not be invariant for all* $\boldsymbol{\varphi} \in Q$; Equivalently, the loading term $\varrho_{\text{ref}} \boldsymbol{B}(\boldsymbol{\varphi}_e)$ need not be fully SO(3)-invariant, even at the equilibrium point $\boldsymbol{\varphi}_e$. As we shall see below, this is the case for loading which results from centrifugal forces in a stationary rotation of the body.

Our objective in this section is to provide a precise characterization of the space of admissible variations for boundary value problem (4.1), and to introduce a split of this space that plays a crucial role in our subsequent analysis. This characterization is intimately related to the invariance properties of the loading term $\varrho_{\text{ref}} \boldsymbol{B}(\boldsymbol{\varphi}_e)$. To motivate our discussion, we start out by considering the case of a potential function $L: Q \to \mathbb{R}$ which possesses full SO(3)-invariance at *any* $\boldsymbol{\varphi} \in Q$. The more important case of interest in which L is only *partially* SO(3)-*invariant* (away from equilibrium) will be considered subsequently.

4.A.1. The case of full SO(3)-invariant loading. Assume that $L: Q \to \mathbb{R}$ is SO(3)-invariant at *any* $\boldsymbol{\varphi} \in Q$, not necessarily an equilibrium solution of boundary value problem (4.1). This assumption means that

$$L(\boldsymbol{\Lambda}\boldsymbol{\varphi}) = L(\boldsymbol{\varphi}), \quad \text{for all } \boldsymbol{\varphi} \in Q \text{ and } \boldsymbol{\Lambda} \in \text{SO(3)}. \tag{4.12}$$

Differentiating (4.12) we find that it implies objectivity of $\varrho_{\text{ref}} \boldsymbol{B}(\boldsymbol{\varphi})$ at $\boldsymbol{\varphi}_e \in Q$, i.e.,

$$\boldsymbol{\Lambda}^T \varrho_{\text{ref}} \boldsymbol{B}(\boldsymbol{\Lambda}\boldsymbol{\varphi}_e) = \varrho_{\text{ref}} \boldsymbol{B}(\boldsymbol{\varphi}_e), \quad \text{for all } \boldsymbol{\Lambda} \in \text{SO(3)}. \tag{4.13}$$

By taking $\boldsymbol{\Lambda} = \exp[\varepsilon \hat{\boldsymbol{\eta}}]$ for any $\hat{\boldsymbol{\eta}} \in so(3)$ and differentiating (4.13) at $\varepsilon = 0$ we obtain its infinitesimal counterpart, i.e.,

$$(\pounds_{\eta_Q}[\varrho_{\text{ref}} \boldsymbol{B}])(\boldsymbol{\varphi}_e) := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \exp[-\varepsilon \hat{\boldsymbol{\eta}}]\, \varrho_{\text{ref}} \boldsymbol{B}(\exp[\varepsilon \hat{\boldsymbol{\eta}}]\, \boldsymbol{\varphi}_e) = \mathbf{0}. \tag{4.14}$$

With the preceding observations in mind, we examine the characterization of those trivial solutions associated with a given solution to the boundary value problem (4.1). Let $\varphi_e \in \mathcal{Q}$ be a solution of (4.1). Consider an SO(3)-*orbit* of configurations:

$$\varphi_e^+ := \Lambda\varphi_e, \quad \text{for } \Lambda \in \mathrm{SO}(3). \tag{4.15}$$

One writes $\varphi_e^+ \in [\mathrm{SO}(3)\cdot\varphi_e]$. Since the stored energy function is, by assumption, objective we have $\partial_F W(\Lambda D\varphi) = \Lambda\, \partial_F W(D\varphi)$. Therefore

$$\begin{aligned} G(\varphi_e^+, \delta\varphi) &= \int_{\mathscr{B}} [\Lambda\, \partial_F W(D\varphi_e)] : \nabla\, (\delta\varphi)\, dV - \int_{\mathscr{B}} \varrho_{\mathrm{ref}} \boldsymbol{B}(\Lambda\varphi_e)\cdot\delta\varphi\, dV \\ &= \int_{\mathscr{B}} \varrho_{\mathrm{ref}}[\Lambda \boldsymbol{B}(\varphi_e) - \boldsymbol{B}(\Lambda\varphi_e)]\cdot\delta\varphi\, dV, \end{aligned} \tag{4.16}$$

which vanishes for all $\delta\varphi \in T_{\varphi_e}\mathcal{Q}$ if and only if the condition of objectivity (4.13) holds.

The preceding (standard) argument shows that any configuration given by (4.15) in the orbit $\mathrm{SO}(3)\cdot\varphi_e$ of a solution $\varphi_e \in \mathcal{Q}$ is also a solution of boundary value problem (4.1). Trivial solutions associated with a given solution to (4.1) are therefore eliminated by restricting the admissible configurations for problem (4.1) to the *orbit space*:

$$\mathscr{C} := \mathcal{Q}/\mathrm{SO}(3). \tag{4.17}$$

That is, configurations $\varphi_e \in \mathcal{Q}$ modulo rigid rotations of the form $\varphi_e^+ = \Lambda\varphi_e$, for $\Lambda \in \mathrm{SO}(3)$, are identified by the choice of $\mathscr{C}$ in (4.17). The *space of admissible variations*, denoted by $\mathscr{V}$, is then simply the tangent space to the quotient manifold $\mathscr{C}$; i.e.,

$$\mathscr{V} := T_{\varphi_e}\mathcal{Q}/[so(3)\cdot\varphi_e], \tag{4.18}$$

where

$$[so(3)\cdot\varphi_e] := \{\boldsymbol{\eta}_{\mathcal{Q}}(\varphi_e) = \boldsymbol{\eta}\times\varphi_e \in T_{\varphi_e}\mathcal{Q} \mid \hat{\boldsymbol{\eta}} \in so(3)\}. \tag{4.19}$$

An explicit realization of $\mathscr{V}$ is obtained through the identification

$$\mathscr{V} \cong \{\delta\varphi \in T_{\varphi_e}\mathcal{Q} \mid \langle \delta\varphi, \boldsymbol{\eta}_{\mathcal{Q}}(\varphi_e)\rangle_g = 0,\ \boldsymbol{\eta} \in \mathbb{R}^3\}, \tag{4.20}$$

where $\langle\cdot,\cdot\rangle_g$ is the Riemannian metric defined by (3.2).

4.A.2. The case of partially SO(3)-invariant loading: Decomposition of the space of variations. Now consider the case of interest for which the loading term $\varrho_{\mathrm{ref}}\boldsymbol{B}(\varphi_e)$ is only partially SO(3)-invariant. In view of condition (4.10) this assumption means the existence of non-zero elements $\hat{\boldsymbol{\eta}} \in so(3)$ for which $(\pounds_{\eta_{\mathcal{Q}}}\varrho_{\mathrm{ref}}\boldsymbol{B})(\varphi_e) \neq 0$. Consequently,

$$\tilde{\mathscr{G}} := \{\hat{\boldsymbol{\xi}} \in so(3) \mid (\pounds_{\xi_{\mathcal{Q}}}[\varrho_{\mathrm{ref}}\boldsymbol{B}])(\varphi_e) = 0\} \subset so(3) \tag{4.21}$$

is no longer the full Lie algebra $so(3)$. Using standard properties of the Lie derivative one can easily show that $\tilde{\mathscr{G}}$ is, in fact, a Lie subalgebra of $so(3)$, since $[\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\nu}}] \in \tilde{\mathscr{G}}$ for $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\nu}} \in \tilde{\mathscr{G}}$. Let $\tilde{G}$ be the associated symmetry group; i.e.,

$$\tilde{G} = \{\Lambda \in \mathrm{SO}(3) \mid \Lambda^T\varrho_{\mathrm{ref}}\boldsymbol{B}(\Lambda\varphi_e) = \varrho_{\mathrm{ref}}\boldsymbol{B}(\varphi_e)\}. \tag{4.22}$$

As in the case of a fully SO(3)-invariant loading, the configuration manifold is now taken as

$$\mathscr{C} := Q/G. \tag{4.23}$$

The tangent space of admissible variations, now given by $T_{\varphi_e}Q/[\tilde{\mathscr{G}} \cdot \varphi_e]$, is then isomorphic to the constrained subspace

$$\mathscr{V} := \{\delta\varphi \in T_{\varphi_e}Q \mid \langle \delta\varphi,\ \zeta_Q(\varphi_e)\rangle_g = 0,\ \text{for all}\ \hat{\zeta} \in \tilde{\mathscr{G}}\}. \tag{4.24}$$

Next, we introduce a splitting of $\mathscr{V}$ of the form

$$\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT} \tag{4.25}$$

by means of the following construction:

i. *The tangent space of rigid body variations* $\mathscr{V}_{RIG}$. We define a subspace $\tilde{\mathscr{G}}^{\perp} \subset so(3)$ by the orthogonality condition

$$\tilde{\mathscr{G}}^{\perp} = \{\hat{\eta} \in so(3) \mid \langle \eta_Q(\varphi_e),\ \zeta_Q(\varphi_e)\rangle_g = 0,\ \text{for all}\ \hat{\zeta} \in \tilde{\mathscr{G}}\}. \tag{4.26}$$

Note that from the definition of infinitesimal generator and (3.2), we have

$$\langle \eta_Q(\varphi_e),\ \zeta_Q(\varphi_e)\rangle_g = \eta \cdot \mathscr{I}_e \zeta, \tag{4.27}$$

where $\mathscr{I}_e$ is the locked inertia tensor at the relative equilibrium configuration φ_e defined by (3.19). (Recall that $\mathscr{I}_e = \mathscr{I}_e^0$, since a center-of-mass reduction is assumed throughout). Since $\mathscr{I}_e$ is positive-definite, (4.27) defines an inner product on $\mathscr{G}$. Moreover, since $\tilde{\mathscr{G}} \oplus \tilde{\mathscr{G}}^{\perp} = so(3)$, we have

$$(\pounds_{\eta_Q}[\varrho_{\text{ref}}\boldsymbol{B}])\,(\varphi_e) \neq \mathbf{0}, \quad \text{for all}\ \hat{\eta} \in \tilde{\mathscr{G}}^{\perp}; \tag{4.28}$$

i.e., $\tilde{\mathscr{G}}^{\perp} \subset so(3)$ generates those rigid body variations for which the loading is not SO(3)-invariant. We set

$$\mathscr{V}_{RIG} := \{\eta_Q(\varphi_e) \in T_{\varphi_e}Q \mid \hat{\eta} \in \tilde{\mathscr{G}}^{\perp}\}. \tag{4.29}$$

From (4.24) and (4.26) it follows that $\mathscr{V}_{RIG} \subset \mathscr{V}$. To motivate the defining condition of the subspace $\mathscr{V}_{INT}$ given below, we first observe that the following result holds.

Lemma 4.1. *For any* $\hat{\eta} \in \tilde{\mathscr{G}}^{\perp}$ *and* $\hat{\zeta} \in \tilde{\mathscr{G}}$ *one has*

$$\left\langle \left(\pounds_{\eta_Q} \frac{\delta L}{\delta \varphi}\right)(\varphi_e),\ \zeta_Q(\varphi_e) \right\rangle_g = -\langle (\pounds_{\eta_Q}\varrho_{\text{ref}}\boldsymbol{B})\,(\varphi_e),\ \zeta_Q(\varphi_e)\rangle_g = 0. \tag{4.30}$$

Proof. From relation (4.3) and the definition of the Lie derivative we have

$$\begin{aligned}\left(\pounds_{\eta_Q} \frac{\delta L}{\delta \varphi}\right)(\varphi_e) &= -(\pounds_{\eta_Q}\varrho_{\text{ref}}\boldsymbol{B})\,(\varphi_e) \\ &= -\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \exp\,[-\varepsilon\hat{\eta}]\,\varrho_{\text{ref}}\boldsymbol{B}(\exp\,[\varepsilon\hat{\eta}]\,\varphi_e) \\ &= \hat{\eta}\varrho_{\text{ref}}\boldsymbol{B}(\varphi_e) - \varrho_{\text{ref}}\,\nabla\boldsymbol{B}(\varphi_e)\,\hat{\eta}\varphi_e.\end{aligned} \tag{4.31}$$

Since $\nabla \boldsymbol{B}(\boldsymbol{\varphi}_e) = -\nabla^2 L(\boldsymbol{\varphi}_e)$, it follows that

$$\left\langle \left(\pounds_{\eta_Q} \frac{\delta L}{\delta \boldsymbol{\varphi}} \right)(\boldsymbol{\varphi}_e),\, \zeta_Q(\boldsymbol{\varphi}_e) \right\rangle = \int_{\mathscr{B}} [\zeta_Q(\boldsymbol{\varphi}_e) \cdot \nabla^2 L(\boldsymbol{\varphi}_e)\, \eta_Q(\boldsymbol{\varphi}_e) + \hat{\boldsymbol{\eta}} \varrho_{\mathrm{ref}} \boldsymbol{B}(\boldsymbol{\varphi}_e) \cdot \zeta \times \boldsymbol{\varphi}_e]\, dV. \tag{4.32}$$

Using Jacobi's identity on the second term of (4.32) we obtain

$$\begin{aligned} \left\langle \left(\pounds_{\eta_Q} \frac{\delta L}{\delta \boldsymbol{\varphi}} \right)(\boldsymbol{\varphi}_e),\, \zeta_Q(\boldsymbol{\varphi}_e) \right\rangle &= -\int_{\mathscr{B}} \eta_Q(\boldsymbol{\varphi}_e) \cdot \varrho_{\mathrm{ref}} [\nabla \boldsymbol{B}(\boldsymbol{\varphi}_e)\, \hat{\zeta} \boldsymbol{\varphi}_e - \hat{\zeta} \boldsymbol{B}(\boldsymbol{\varphi}_e)]\, dV \\ &\quad - \int_{\mathscr{B}} \varrho_{\mathrm{ref}} \boldsymbol{B}(\boldsymbol{\varphi}_e) \cdot [\widehat{\boldsymbol{\eta} \times \zeta}]\, \boldsymbol{\varphi}_e\, dV \\ &= -\int_{\mathscr{B}} \eta_Q(\boldsymbol{\varphi}_e) \cdot (\pounds_{\zeta_Q} \varrho_{\mathrm{ref}} \boldsymbol{B})(\boldsymbol{\varphi}_e)\, dV \\ &\quad - \int_{\mathscr{B}} \varrho_{\mathrm{ref}} \boldsymbol{B}(\boldsymbol{\varphi}_e) \cdot [\hat{\eta}, \hat{\zeta}]_Q(\boldsymbol{\varphi}_e)\, dV. \end{aligned} \tag{4.33}$$

By definition of $\tilde{\mathscr{G}}$, $(\pounds_{\zeta_Q} \varrho_{\mathrm{ref}} \boldsymbol{B})(\boldsymbol{\varphi}_e) = \mathbf{0}$ for $\hat{\zeta} \in \tilde{\mathscr{G}}$. On the other hand, by the invariance condition (4.10), we have

$$-\int_{\mathscr{B}} \varrho_{\mathrm{ref}} \boldsymbol{B}(\boldsymbol{\varphi}_e) \cdot [\hat{\eta}, \hat{\zeta}]_Q(\boldsymbol{\varphi}_e)\, dV = DL(\boldsymbol{\varphi}_e) \cdot [\hat{\eta}, \hat{\zeta}]_Q(\boldsymbol{\varphi}_e) = 0, \tag{4.34}$$

so that (4.33) vanishes, and the result holds. □

ii. *The tangent-space of internal deformations* $\mathscr{V}_{INT}$. Loosely speaking, the result in the lemma above says that the loading term $\dfrac{\delta L}{\delta \boldsymbol{\varphi}}(\boldsymbol{\varphi}_e)$, which is not SO(3)-invariant as a result of condition (4.28), "looks invariant" when tested by variations $\zeta_Q(\boldsymbol{\varphi}_e)$ generated by $\tilde{\mathscr{G}}$. We define the space $\mathscr{V}_{INT} \subset \mathscr{V}$ precisely by this condition, but now enforced on $\tilde{\mathscr{G}}^{\perp}$, i.e.,

$$\mathscr{V}_{INT} := \left\{ \delta \boldsymbol{\varphi} \in \mathscr{V} \;\middle|\; \left\langle \left(\pounds_{\eta_Q} \frac{\delta L}{\delta \boldsymbol{\varphi}} \right)(\boldsymbol{\varphi}_e), \delta \boldsymbol{\varphi} \right\rangle = 0, \quad \text{for all } \hat{\eta} \in \tilde{\mathscr{G}}^{\perp} \right\}. \tag{4.35}$$

From Lemma 4.1 we conclude that $\mathscr{V}_{RIG} \cap \mathscr{V}_{INT} = \{\mathbf{0}\}$. Furthermore

$$\dim [\mathscr{V}_{RIG}] = \dim [\tilde{\mathscr{G}}^{\perp}] = \dim [so(3)] - \dim [\tilde{\mathscr{G}}]; \tag{4.36}$$

since the number of constraints in (4.35) equals $\dim [\tilde{\mathscr{G}}^{\perp}]$ it follows that $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$ as required.

We show below that the preceding construction leads to the following crucial result: *The second variation of* $V + L$ *block-diagonalizes on* $\mathscr{V}_{RIG} \times \mathscr{V}_{INT}$. Observe that by assumption, $V + L$ has a critical point at $\boldsymbol{\varphi}_e \in Q$ and, therefore, the second variation $D^2[V + L](\boldsymbol{\varphi}_e)$ is *symmetric* and makes intrinsic sense.

Remarks 4.2. 1. The space $\mathscr{V}_{INT}$ may be viewed as a constrained subspace of $T_{\varphi_e} Q$ obtained by enforcing the orthogonality condition $\langle \delta \boldsymbol{\varphi}, \zeta_Q(\boldsymbol{\varphi}_e) \rangle_g = 0$

along with the requirement in (4.35). The result in Lemma 4.1 then shows that these two conditions are consistent.

2. Alternatively, one could define $\mathscr{V}_{INT}$ by the requirement that $\langle \boldsymbol{\eta}_Q(\varphi), \delta\varphi \rangle_g = 0$ for $\hat{\boldsymbol{\eta}} \in \tilde{\mathscr{G}}^{\perp}$. However, as shown in § 6.D, such a definition *does not lead to a block-diagonal structure of the second variation of* $[V + L]$. This latter definition, on the other hand, results in a diagonal structure of the symplectic two-form. The structure of the symplectic two-form is examined in Section § 5.C.

3. The construction given above leads to the following decomposition of the tangent space $T_{\varphi_e}Q$:

$$T_{\varphi_e}Q = \underbrace{\underbrace{[\tilde{\mathscr{G}} \cdot \varphi_e]}_{\text{SO(3)-invariant rigid body variations}} \oplus \mathscr{V}_{RIG}}_{\text{all rigid body variations}} \oplus \underbrace{\mathscr{V}_{INT}}_{\text{infinitesimal deformations}}. \tag{4.37}$$

4. In general, the decomposition (4.25) is valid if the locked inertia tensor is nonsingular or, more generally, if the Arnold form is non-degenerate; see Part I, § 2 for a detailed discussion.

§ 4.B. *The second variation: Block-diagonalization in elasticity*

Here, we examine the structure of the second variation at a relative equilibrium $\varphi_e \in Q$. We show in the concrete setting of elasticity that the decomposition (4.37) leads to a block-diagonalization of the second variation of the effective potential V_{ξ}.

4.B.1. The second tangent of the internal energy. We start our analysis of the second variation by recalling the standard expression for the second tangent associated with the internal energy, namely,

$$D^2V(\varphi)(\delta\varphi_1, \delta\varphi_2) = \int_{\mathscr{B}} \nabla(\delta\varphi_1) : \mathbf{A}(\varphi) : \nabla(\delta\varphi_2)\, dV, \tag{4.38}$$

for all $\delta\varphi_1, \delta\varphi_2 \in T_{\varphi}Q$, where $\mathbf{A}(\varphi)$ is the first elasticity tensor at $\varphi \in Q$ given by

$$\mathbf{A}(\varphi) := \left.\frac{\partial^2 W}{\partial \boldsymbol{F}\, \partial \boldsymbol{F}}\right|_{F = D\varphi}. \tag{4.39}$$

Observe that, except for *dead loading*, in general the first variation of the internal energy term, $V: Q \to \mathbb{R}$, does not have a critical point and, therefore, the second variation does not make intrinsic sense; see ABRAHAM, MARSDEN & RATIU [1988, p. 113]. We shall, therefore, refer to (4.38) as the *second tangent* to $V(\varphi)$ at $\varphi \in Q$.

Next, we recall that the frame-indifference condition (2.7) implies that $W(\boldsymbol{X}, \boldsymbol{F}) = \bar{W}(\boldsymbol{X}, \boldsymbol{C})$ where $\boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy-Green tensor. We use the notation $\boldsymbol{C}(\varphi) := D\varphi^T D\varphi$. From definition (4.39) of the elasticity tensor

$\mathbf{A}(\varphi)$, we obtain the relation

$$\nabla(\delta\varphi_1) : \mathbf{A}(\varphi) : \nabla(\delta\varphi_2) = \nabla(\delta\varphi_1) : [\nabla(\delta\varphi_2)\, S] + \tfrac{1}{4}\,[D\mathbf{C}(\varphi)\cdot\delta\varphi_1] : \mathbf{C}(\varphi) : [D\mathbf{C}(\varphi)\cdot\delta\varphi_2]. \tag{4.40}$$

Here $S := F^{-1}\,\partial_F W = 2\partial_C \overline{W}$ is the symmetric (second) Piola-Kirchhoff tensor, and

$$\mathbf{C}(\varphi) := 4\,\frac{\partial^2 W}{\partial \boldsymbol{C}\,\partial \boldsymbol{C}}\bigg|_{C=[D\varphi]^T D\varphi} \tag{4.41}$$

is the second elasticity tensor. Furthermore, we have

$$D\mathbf{C}(\varphi)\cdot\delta\varphi := D\varphi^T\,\nabla(\delta\varphi) + [\nabla(\delta\varphi)]^T\,D\varphi. \tag{4.42}$$

Formulae (4.39)–(4.41) are standard; see, e.g., MARSDEN & HUGHES [1983, Chapter 2].

At a relative equilibrium $\varphi_e \in Q$, the second tangent of V, as a bilinear form evaluated on $(\eta_Q(\varphi_e), \delta\varphi) \in \mathscr{V}_{RIG}\times\mathscr{V}$, is given by the following

Proposition 4.3. *At an equilibrium configuration $\varphi_e \in Q$, the second tangent of V is*

$$D^2V(\varphi_e)\,(\delta\varphi, \eta_Q(\varphi_e)) = \int_{\mathscr{B}} \delta\varphi\cdot\hat{\eta}\varrho_{\mathrm{ref}}\boldsymbol{B}(\varphi_e)\,dV, \tag{4.43}$$

for any $\delta\varphi \in \mathscr{V}$ and $\eta_Q(\psi_e) \in \mathscr{V}_{RIG}$.

Proof. First observe that since $\eta_Q(\varphi) = \eta\times\varphi$ we have $\nabla(\eta_Q(\varphi)) = \hat{\eta}F$, where $F = D\varphi$. Relation (4.42) then yields

$$D\mathbf{C}(\varphi)\cdot\eta_Q(\varphi) = F^T\hat{\eta}F + F^T\hat{\eta}^TF = \mathbf{0}, \tag{4.44}$$

since $\hat{\eta} + \hat{\eta}^T = \mathbf{0}$. Hence, $\ker\,[D\mathbf{C}(\varphi_e)] = [so(3)\cdot\varphi_e]$. Consequently, if $\eta_Q(\varphi_e) \in \mathscr{V}_{RIG}$, then by (4.38), (4.40) and (4.44) we have

$$D^2V(\varphi_e)\,(\delta\varphi, \eta_Q(\varphi)) = \int_{\mathscr{B}} \nabla(\delta\varphi) : [\nabla(\eta_Q(\varphi_e))\,S_e]\,dV. \tag{4.45}$$

Using the divergence theorem, the stress-free boundary condition and the fact that $\nabla(\eta_Q(\varphi_e)) = \hat{\eta}F_e$, we reduce (4.45) as follows:

$$\begin{aligned} D^2V(\varphi_e)\,(\delta\varphi, \eta_Q(\varphi_e)) &= \int_{\mathscr{B}} \nabla(\delta\varphi) : [\hat{\eta}F_eS_e]\,dV \\ &= -\int_{\mathscr{B}} \delta\varphi\cdot\mathrm{Div}\,[\hat{\eta}\,(\partial_F W_e)]\,dV \\ &\quad + \int_{\partial\mathscr{B}} \delta\varphi\cdot\eta\times[\partial_F W_e N]\,dA \\ &= -\int_{\mathscr{B}} \delta\varphi\cdot\hat{\eta}\,\mathrm{Div}\,[\partial_F W_e]\,dV \\ &= \int_{\mathscr{B}} \delta\varphi\cdot\hat{\eta}\varrho_{\mathrm{ref}}\boldsymbol{B}(\varphi_e)\,dV, \end{aligned} \tag{4.46}$$

which completes the proof. □

4.B.2. The second tangent of the loading potential. Now consider the potential term $L: Q \to \mathbb{R}$. We have

Proposition 4.4. *At a relative equilibrium* $\varphi_e \in Q$:

$$D^2L(\varphi_e)\,(\delta\varphi,\, \eta_Q(\varphi)) = -\int_{\mathscr{B}} \delta\varphi \cdot \varrho_{\text{ref}} \nabla \boldsymbol{B}(\varphi_e)\, \eta_Q(\varphi_e)\, dV, \tag{4.47}$$

for all $\eta_Q(\varphi_e) \in \mathscr{V}_{RIG}$ *and* $\delta\varphi \in \mathscr{V}$.

Proof. Recalling that $\dfrac{\delta L}{\delta \varphi} = -\varrho_{\text{ref}}\boldsymbol{B}$ we have

$$DL(\varphi)\cdot\delta\varphi = -\int_{\mathscr{B}} \varrho_{\text{ref}}\boldsymbol{B}(\varphi)\cdot\delta\varphi\, dV, \tag{4.48}$$

so that

$$D[DL(\varphi_e)\cdot\delta\varphi]\cdot\eta_Q(\varphi_e) = -\int_{\mathscr{B}} \delta\varphi\cdot\varrho_{\text{ref}}\nabla\boldsymbol{B}(\varphi_e)\,\eta_Q(\varphi_e)\,dV, \tag{4.49}$$

which proves (4.47). □

As alluded to above, *neither* $D^2V(\varphi_e)$ *nor* $D^2L(\varphi_e)$ make intrinsic sense (independent of the coordinate chart) since these functions do not possess a critical point at $\varphi_e \in Q$. However, $V + L$ does have a critical point at $\varphi_e \in Q$ and its second variation defines a symmetric bilinear form. Furthermore, this bilinear form possesses the following crucial property:

Theorem 4.5. (Configuration Block-Diagonalization Theorem). *Let* $\mathscr{V}_{RIG}$ *and* $\mathscr{V}_{INT}$ *be defined by* (4.29) *and* (4.35), *respectively. Then*

$$D^2[V+L]\,(\varphi_e)\cdot(\eta_Q(\varphi_e),\,\delta\varphi) = 0, \tag{4.50}$$

for $\delta\varphi \in \mathscr{V}_{INT}$ *and* $\eta_Q(\varphi_e) \in \mathscr{V}_{RIG}$.

Proof. Combining the results in Propositions 4.3 and 4.4, and making use of the definition of the Lie derivative we obtain

$$\begin{aligned} D^2[V+L](\varphi_e)\cdot(\eta_Q(\varphi_e),\,\delta\varphi) &= -\int_{\mathscr{B}} \delta\varphi\cdot\varrho_{\text{ref}}[-\hat{\eta}\boldsymbol{B}(\varphi_e) + \nabla\boldsymbol{B}(\varphi_e)\cdot\delta\varphi]\,dV \\ &= -\int_{\mathscr{B}} \delta\varphi\cdot\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \exp[-\varepsilon\hat{\eta}]\,\varrho_{\text{ref}}\boldsymbol{B}(\exp[\varepsilon\hat{\eta}]\varphi_e)\,dV \\ &= -\int_{\mathscr{B}} \delta\varphi\cdot(\pounds_{\eta_Q}\varrho_{\text{ref}}\boldsymbol{B}])\,(\varphi_e)\,dV \\ &= \left\langle \left(\pounds_{\eta_Q}\frac{\delta L}{\delta\varphi}\right)(\varphi_e),\,\delta\varphi\right\rangle. \end{aligned} \tag{4.51}$$

Thus, if $\varrho_{\text{ref}}\boldsymbol{B} = -\dfrac{\delta L}{\delta\varphi}$ *is* SO(3)*-invariant*, the second variation of $V+L$ *vanishes identically* when evaluated at $(\eta_Q(\varphi_e),\,\delta\varphi) \in \mathscr{V}_{RIG}\times\mathscr{V}$. On the other hand, if $\varrho_{\text{ref}}\boldsymbol{B}$ *only possesses partial* SO(3)*-invariance*, result (4.50) follows from (4.51) and the defining condition of $\mathscr{V}_{INT}$ in (4.35). □

§ 4.C. *Application: Centrifugal body force in a stationary rotation*

We now apply the results of the preceding sections to the boundary value problem for the relative equilibrium configurations $\varphi_e \in Q$ formulated in § 3.B, [see (3.36)–(3.37)]. Here the potential function for the external loading is $L_\xi : Q \times \mathbb{R}^3 \to \mathbb{R}$ defined by $(3.4)_2$. We have

i. The Lie algebra $\tilde{\mathcal{G}} \subset \mathrm{SO}(3)$ associated with the symmetry group of L_ξ at a configuration $\varphi \in Q$ is $\tilde{\mathcal{G}} \equiv \mathcal{G}_{\xi_e}$, defined by

$$\mathcal{G}_{\xi_e} = \{\zeta \in so(3) \mid \zeta \times \xi_e = 0\} = \{\hat{\zeta} \in so(3) \mid [\hat{\zeta}, \hat{\xi}_e] = 0\}. \tag{4.52}$$

This result follows from the computation below which uses the fact that SO(3) acts by isometries:

$$\begin{aligned}(\pounds_{\zeta_Q} L_{\xi_e})(\varphi) &= -\langle \pounds_{\zeta_Q(\varphi)} \xi_{eQ}(\varphi),\ \xi_{eQ}(\varphi)\rangle_g \\ &= -\langle [\zeta,\ \xi_e]_Q(\varphi),\ \xi_{eQ}(\varphi)\rangle_g \\ &= 0, \quad \text{for } [\hat{\xi}_e,\ \hat{\zeta}] = 0.\end{aligned} \tag{4.53}$$

It follows that $\dim [\mathcal{G}_{\xi_e}] = 1$. Clearly $G_{\xi_e} \equiv \tilde{G} \subset \mathrm{SO}(3)$ is the group of rotations about ξ_e.

ii. *At a relative equilibrium* $z_e = (\varphi_e, p_e) \in P$, with total angular momentum $\mu_e = J(z_e)$, $\mathcal{G}_{\xi_e}$ *coincides with* $\mathcal{G}_{\mu_e}$, the Lie subalgebra invariant under the coadjoint action, i.e.,

$$\mathcal{G}_{\mu_e} := \{\hat{\xi} \in so(3) \mid \mathrm{ad}^*_{\xi}(\mu_e) = \xi \times \mu_e = 0\}. \tag{4.54}$$

Furthermore, the subspace $\mathcal{G}_{\xi_e}^{\perp}$ *coincides with the orthogonal complement* $\mathcal{G}_{\mu_e}^{\perp}$ in *the standard Euclidean inner product*, i.e.,

$$\mathcal{G}_{\mu_e}^{\perp} := \{\hat{\eta} \in so(3) \mid \eta \cdot \xi_e = 0\}. \tag{4.55}$$

This conclusion follows from the equilibrium condition $\mathcal{I}_e \xi_e = \lambda \xi_e$, expression (4.27) and definition (4.26).

iii. *At a relative equilibrium configuration* $\varphi_e \in Q$, *the loading is equilibrated* in the sense that conditions (4.5) and (4.8) hold. Equivalently, by (4.10), $L_{\xi_e}(\varphi_e)$ is infinitesimally left-invariant under the full group SO(3). The abstract proof of this fact is contained in Part I.

With these results in hand, we consider the explicit characterizations of the spaces $\mathcal{V}$, $\mathcal{V}_{RIG}$ and $\mathcal{V}_{INT}$ for boundary value problem (3.36). We have

Proposition 4.6. *A relative equilibrium configuration* $\varphi_e \in Q$ *is a solution of the weak form* (4.4) *of the boundary value problem* (3.36) *with tangent space of admissible variations given by*

$$\mathcal{V} = \left\{\delta\varphi \in T_{\varphi_e} Q \mid \xi_e \cdot \int_{\mathcal{B}} \varrho_{\mathrm{ref}} \varphi_e \times \delta\varphi \, dV = 0\right\}. \tag{4.56}$$

Moreover, $\mathscr{V}$ *splits as* $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$, *where*

$$\mathscr{V}_{RIG} = \{\boldsymbol{\eta}_Q(\boldsymbol{\varphi}_e) = \boldsymbol{\eta}\times\boldsymbol{\varphi}_e \in \mathscr{V} \mid \hat{\boldsymbol{\eta}} \in \mathscr{G}_{\mu_e}^{\perp} \text{ i.e., } \boldsymbol{\eta}\cdot\boldsymbol{\xi}_e = 0\}, \tag{4.57}$$

and

$$\mathscr{V}_{INT} = \{\delta\boldsymbol{\varphi} \in \mathscr{V} \mid \boldsymbol{\eta}\cdot \text{ident}_{\xi_e}(\delta\boldsymbol{\varphi}) = 0, \textit{ for all } \hat{\boldsymbol{\eta}} \in \mathscr{G}_{\mu_e}^{\perp}\}, \tag{4.58}$$

where

$$\text{ident}_{\xi_e}(\delta\boldsymbol{\varphi}) := \int_{\mathscr{B}} \varrho_{\text{ref}}[2(\boldsymbol{\xi}_e\times\boldsymbol{\varphi}_e)\times\delta\boldsymbol{\varphi} - \boldsymbol{\xi}_e\times(\boldsymbol{\varphi}_e\times\delta\boldsymbol{\varphi})]\, dV. \tag{4.59}$$

Note that the condition in (4.58) *is equivalent to requiring that* $\text{ident}_{\xi_e}(\delta\boldsymbol{\varphi}) \in \mathscr{G}_{\mu_e}$ *for all* $\delta\boldsymbol{\varphi} \in \mathscr{V}_{INT}$.

Proof. The characterizations in (4.56) and (4.57) follow from the definitions (4.24), (4.29) and (4.26) along with (4.54).

To prove (4.59) we compute the defining condition in (4.35). Using the definition of Lie derivative and the fact that $\varrho_{\text{ref}}\boldsymbol{B}(\boldsymbol{\varphi}_e) = -\varrho_{\text{ref}}\boldsymbol{\xi}_e\times(\boldsymbol{\xi}_e\times\boldsymbol{\varphi}_e)$, we have

$$\begin{aligned}\left(\pounds_{\eta_Q}\frac{\delta L_{\xi_e}}{\delta\boldsymbol{\varphi}}\right)(\boldsymbol{\varphi}_e) &= -\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \exp[-\varepsilon\hat{\boldsymbol{\eta}}]\,\varrho_{\text{ref}}\boldsymbol{B}(\exp[\varepsilon\hat{\boldsymbol{\eta}}]\boldsymbol{\varphi}_e)\\ &= \varrho_{\text{ref}}\{\boldsymbol{\eta}\times\boldsymbol{B}(\boldsymbol{\varphi}_e) - \nabla\boldsymbol{B}(\boldsymbol{\eta}\times\boldsymbol{\varphi}_e)\}\\ &= -\varrho_{\text{ref}}\{\boldsymbol{\eta}\times[\boldsymbol{\xi}_e\times(\boldsymbol{\xi}_e\times\boldsymbol{\varphi}_e)] - \boldsymbol{\xi}_e\times[\boldsymbol{\xi}_e\times(\boldsymbol{\eta}\times\boldsymbol{\varphi}_e)]\}.\end{aligned} \tag{4.60}$$

Making use of standard vector-product identities, we reduce (4.60) to

$$\left(\pounds_{\eta_Q}\frac{\delta L_{\xi_e}}{\delta\boldsymbol{\varphi}}\right)(\boldsymbol{\varphi}_e) = 2(\boldsymbol{\xi}_e\times\boldsymbol{\eta})\times\boldsymbol{p}_e - [(\boldsymbol{\xi}_e\times\boldsymbol{\eta})\times\boldsymbol{\xi}_e]\times\varrho_{\text{ref}}\boldsymbol{\varphi}_e, \tag{4.61}$$

where $\boldsymbol{p}_e := \varrho_{\text{ref}}\boldsymbol{\xi}_e\times\boldsymbol{\varphi}_e$. Therefore, the condition in (4.35) yields

$$\begin{aligned}\left\langle \delta\boldsymbol{\varphi}, \left(\pounds_{\eta_Q}\frac{\delta L}{\delta\boldsymbol{\varphi}}\right)(\boldsymbol{\varphi}_e)\right\rangle &= (\boldsymbol{\xi}_e\times\boldsymbol{\eta})\cdot\int_{\mathscr{B}}[2\boldsymbol{p}_e\times\delta\boldsymbol{\varphi} - \boldsymbol{\xi}_e\times\varrho_{\text{ref}}(\boldsymbol{\varphi}_e\times\delta\boldsymbol{\varphi})]\, dV\\ &= (\boldsymbol{\xi}_e\times\boldsymbol{\eta})\cdot\text{ident}_{\xi_e}(\delta\boldsymbol{\varphi}) = 0.\end{aligned} \tag{4.62}$$

Since $\widehat{\boldsymbol{\xi}_e\times\boldsymbol{\eta}} = \mathbf{0}$ lies in $\mathscr{G}_{\mu_e}^{\perp}$ for any $\hat{\boldsymbol{\eta}} \in \mathscr{G}_{\mu_e}^{\perp}$, the result is proved. □

Remarks 4.7. 1. The expression (4.57) agrees with definition (2.26) of Part I for $\text{ident}_{\xi_e} : \mathscr{V} \to \mathscr{G}^*$. In fact, from (3.19) we have

$$\begin{aligned}\text{ident}_{\xi_e}(\delta\boldsymbol{\varphi}) &:= -D[\mathscr{I}(\boldsymbol{\varphi}_e)\cdot\delta\boldsymbol{\varphi}]\,\boldsymbol{\xi}_e\\ &= -D\left[\int_{\mathscr{B}}\boldsymbol{\varphi}_e\times\varrho_{\text{ref}}(\boldsymbol{\xi}_e\times\boldsymbol{\varphi}_e)\, dV\right]\cdot\delta\boldsymbol{\varphi}\\ &= -\int_{\mathscr{B}}[\delta\boldsymbol{\varphi}\times\varrho_{\text{ref}}(\boldsymbol{\xi}_e\times\boldsymbol{\varphi}_e) + \boldsymbol{\varphi}_e\times\varrho_{\text{ref}}(\boldsymbol{\xi}_e\times\delta\boldsymbol{\varphi})]\, dV.\end{aligned} \tag{4.63}$$

Using Jacobi's identity and rearranging terms we recover (4.57).

2. Expressions (4.57) and (4.58) for $\mathscr{V}_{RIG}$ and $\mathscr{V}_{INT}$ respectively agree, therefore, with the general definitions in Part I, namely, equations (2.34) and (2.35).

3. There is also a block-diagonalization result on variations in phase space; see Part I and § 6 below.

§ 5. Stability analysis by the reduced energy-momentum method

In this section we examine the stability of the relative equilibria characterized by Theorem 3.1 using the *reduced* Energy-Momentum method discussed in the general setting of Hamiltonian systems with symmetry in § 2 of Part I. The method exploits a reformulation of the energy-momentum map H_{μ_e}, which has two remarkable properties:

i. Enforcement of conservation of total angular momentum is built at the outset into the Hamiltonian, which now coincides with the energy-momentum map, thereby bypassing the need for Lagrange multipliers.

ii. The second variation test for formal stability of the relative equilibria is formulated solely in terms of the configuration variables. Equivalently, the method operates on the configuration space Q and not on the full phase space $P = T^*Q$ and leads, therefore, to a *substantial reduction* of the original problem.

As discussed below, the crucial idea is to introduce a change of variables in the original Hamiltonian via a *shifting* operator which projects the phase space P onto the level set $J^{-1}(\mathbf{0})$. In terms of these shifted variables, the restriction to the level set $J^{-1}(\mathbf{0})$ of the original Hamiltonian can be easily enforced at the outset, and leads to a particularly convenient expression involving an *amended potential* introduced by SMALE [1970a, b] but having its origins in early work of RIEMANN [1860] and ROUTH [1877]. In this setting, the critical points of the kinetic energy correspond to *zero values of the momenta*, and the critical points of the amended potential define the relative equilibrium configurations. Furthermore, the test for definiteness of the second variation reduces to a test for definiteness of the amended potential at the relative equilibrium configuration.

§ 5.A. *The reduced Hamiltonian and Smale's amended potential*

The reparametrization of the energy momentum function possessing properties (i) and (ii) summarized above is constructed according to the following steps:

Step 1. *The shifting map.* Introduce a map $\Sigma: P \to P$ which projects the phase space P onto the level set $J^{-1}(\mathbf{0}) \subset P$ by means of the formula

$$\begin{aligned}\Sigma(z) &:= (\varphi, p - \mathbf{FL}((\mathscr{I}^{-1}(\varphi)\, J(z))_Q\,(\varphi))) \\ &= (\varphi, p - \varrho_{\text{ref}} \mathscr{I}^{-1}(\varphi)\, J(z) \times \varphi),\end{aligned} \tag{5.1}$$

where $J(z)$ is the total angular momentum. Note that $\mathscr{I}^{-1}(\varphi)\, J(z)$ gives the *angular velocity* associated with the angular momentum $J(z)$. That the map defined by (5.1) does in fact project P onto $J^{-1}(\mathbf{0})$ can be verified by the following calculation (see Proposition 2.2 of Part I for the general proof):

$$\begin{aligned} J(\Sigma(z)) &= \int_{\mathscr{B}} \varphi \times [p - \varrho_{\text{ref}}(\mathscr{I}^{-1}(\varphi)\, J(z)) \times \varphi]\, dV \\ &= J(z) - \left[\int_{\mathscr{B}} \varrho_{\text{ref}}[|\varphi|^2\, \mathbf{1}_3 - \varphi \otimes \varphi]\, dV\right] (\mathscr{I}^{-1}(\varphi)\, J(z)) \\ &= J(z) - \mathscr{I}(\varphi)\, \mathscr{I}^{-1}(\varphi)\, J(z) = \mathbf{0}. \end{aligned} \tag{5.2}$$

The shift term $\varrho_{\text{ref}} \mathscr{I}^{-1}(\varphi)\, J(z) \times \varphi$ can therefore be interpreted as defining the momenta of an 'equivalent' rigid body with shape defined by $\varphi \in Q$ and total angular momentum $J(z)$. One speaks of the '*locked system*' at configuration φ.

Step 2. *Reformulation of the energy-momentum map.* In terms of the shifting map $\Sigma: P \to P$, the original Hamiltonian $H = V + K$, where V and K are the potential and kinetic energies respectively, can be expressed as

$$H(z) = V_{J(z)}(\varphi) + K(\Sigma(z)), \tag{5.3a}$$

where

$$V_{J(z)}(\varphi) := V(\varphi) + \tfrac{1}{2}\, J(z)\, \mathscr{I}^{-1}(\varphi)\, J(z). \tag{5.3b}$$

This result can be verified by a direct calculation using definition (5.1) (see Proposition 2.2 of Part I). Restricted to the level set $J^{-1}(\mu_e) \subset P$, the energy-momentum function and the Hamiltonian coincide. From (5.3) we therefore conclude that

$$H_{\mu_e}(z)|_{J^{-1}(\mu_e)} = H(z)|_{J^{-1}(\mu_e)} = V_{\mu_e}(\varphi) + K(\Sigma(z))|_{J^{-1}(\mu_e)}, \tag{5.4}$$

where $V_{\mu_e}(\varphi): Q \to \mathbb{R}$ is Smale's amended potential defined by (5.3b) with $J(z) = \mu_e$.

Step 3. *Change of variables.* Finally, we consider the Hamiltonian $H|_{J^{-1}(\mu_e)}$ defined by (5.4) as a function of the *shifted variables*

$$\tilde{z} = (\varphi, \tilde{p}) := (\varphi, p - \varrho_{\text{ref}}(\mathscr{I}^{-1}(\varphi)\, \mu_e) \times \varphi_e) \in J^{-1}(\mathbf{0}). \tag{5.5}$$

This change of variables leads to a Hamiltonian function $h_{\mu_e}: J^{-1}(\mathbf{0}) \to \mathbb{R}$, referred to as the *reduced Hamiltonian* in what follows, and given from (5.4) and (5.5) as

$$\begin{aligned} h_{\mu_e}(\tilde{z}) &= V_{\mu_e}(\varphi) + K(\tilde{p}_e), \\ V_{\mu_e}(\varphi) &:= V(\varphi) + \tfrac{1}{2}\, \mu_e \cdot \mathscr{I}^{-1}(\varphi)\, \mu_e. \end{aligned} \tag{5.6}$$

The preceding construction results in a (reduced) Hamiltonian function in which the restriction to the level set (e.g., the constraint of constant angular momentum) is replaced by the requirement that $\tilde{z}$ be in the level set $J^{-1}(\mathbf{0})$. This proves particularly convenient in calculations. In particular, in terms of these shifted vari-

abies, the relative equilibrium conditions take the following form. From (5.6), the first variation of h_{μ_e} is given by

$$Dh_{\mu_e}(\tilde{z}) \cdot \delta\tilde{z} = DV_{\mu_e}(\varphi) \cdot \delta\varphi + \int_{\mathscr{B}} \varrho_{\text{ref}}^{-1} \tilde{p} \cdot \widetilde{\delta p}\, dV. \tag{5.7}$$

Making use of the well known expression for the derivative of the inverse of a tensor, we can express the first term in (5.7) as

$$DV_{\mu_e}(\varphi) \cdot \delta\varphi = DV(\varphi) \cdot \delta\varphi - \tfrac{1}{2}\, \mathscr{I}^{-1}(\varphi)\, \mu_e \cdot [D\mathscr{I}(\varphi) \cdot \delta\varphi]\, \mathscr{I}^{-1}(\varphi)\, \mu_e. \tag{5.8}$$

In view of (3.4), (3.9) and (3.10) (recall that $u = 0$ since a reduction to the center of mass is assumed at the outset), it follows from (5.7) and (5.8) that the critical points of h_{μ_e} are characterized by the conditions

$$\frac{\delta V_{\mu_e}}{\delta\varphi}(\varphi_e) = \frac{\delta V_{\xi_e}}{\delta\varphi}(\varphi_e) \equiv 0 \quad \text{and} \quad \tilde{p}_e = 0, \tag{5.9}$$

where $\xi_e := \mathscr{I}^{-1}(\varphi_e)\, \mu_e$. Conditions (5.9) are equivalent to conditions (3.13) and therefore equivalent to the relative equilibrium conditions (3.14) since, by the change of variables (5.5), we have

$$\tilde{p}_e = 0 \quad \text{if and only if} \quad p_e = \varrho_{\text{ref}} \xi_e \times \varphi_e \tag{5.10}$$

which is the relative equilibrium condition $(3.14)_1$. In summary, the relative equilibrium conditions (5.9) associated with the reduced Hamiltonian h_{μ_e} are identical to those associated with the energy momentum map H_{μ_e} derived in § 1. For the general statement of this result, essentially due to SMALE [1970a, b], see Part I, § 2. To simplify our notation in what follows we shall write ξ in place of $\xi_e := \mathscr{I}^{-1}(\varphi_e)\, \mu_d$.

§ 5.B. *The second variation of the reduced Hamiltonian: The reduced test for formal stability*

We start our analysis of the second variation by characterizing the space of admissible variations at a relative equilibrium $\tilde{z}_e \in J^{-1}(0)$, denoted by $\mathscr{S}_0 \subset T_{\tilde{z}_e} P$ in what follows. First, any $\delta\tilde{z} \in \mathscr{S}_0$ must lie in $\ker [J^{-1}(0)]$; equivalently,

$$DJ(\tilde{z}) \cdot \delta\tilde{z} = \int_{\mathscr{B}} [\delta\varphi \times \tilde{p}_e + \varphi_e \times \widetilde{\delta p}]\, dV = \int_{\mathscr{B}} \varphi_e \times \widetilde{\delta p}\, dV = 0, \tag{5.11}$$

since $\tilde{p}_e = 0$. By taking the dot product of (5.11) with any $\zeta \in \mathbb{R}^3$ we obtain

$$\langle \widetilde{\delta p},\, \zeta_Q(\varphi_e) \rangle = \int_{\mathscr{B}} \zeta_Q(\varphi_e) \cdot \widetilde{\delta p}\, dV = 0 \tag{5.12}$$

for all $\hat{\zeta} \in so(3)$. Hence, admissible variations $\widetilde{\delta p}$ are *L_2-orthogonal* to infinitesimal rigid body variations of φ_e. Equivalently, $\widetilde{\delta p}$ must lie in the annihilator (relative to the L_2-pairing) of the tangent space $so(3) \cdot \varphi_e$ to the orbit $\mathrm{SO}(3) \cdot \varphi_e$,

denoted in what follows by $[so(3)\cdot\varphi_e]^A$. Second, admissible variations $(\delta\varphi, \widetilde{\delta p}) \in \mathscr{S}_0$ must be taken modulo directions along which the second variation of h_{μ_e} vanishes identically. These directions are precisely superposed infinitesimal rotations with axis $\xi \in \mathbb{R}^3$. The preceding two conditions lead to the following concrete realization of the space $\mathscr{S}_0$:

$$\mathscr{S}_0 := \{(\delta\varphi, \widetilde{\delta p}) \mid \delta\varphi \in \mathscr{V} \text{ and } \langle \widetilde{\delta p}, \zeta_Q(\varphi_e)\rangle = 0 \text{ for all } \zeta \in \mathbb{R}^3\}. \quad (5.13a)$$

Equivalently, $\mathscr{S}_0$ is given by

$$\mathscr{S}_0 = \mathscr{V} \oplus [so(3)\cdot\varphi_e]^A. \quad (5.13b)$$

Next, we compute the second variation of the reduced Hamiltonian h_{μ_e}. Differentiation of expression (5.7) for the first variation gives

$$D^2 h_{\mu_e}(\tilde{z}_e)(\delta\tilde{z}_1, \delta\tilde{z}_2) = D^2 V_{\mu_e}(\varphi_e)(\delta\varphi_1, \delta\varphi_2) + \int_{\mathscr{B}} \varrho_{\text{ref}}^{-1}\, \widetilde{\delta p}_1 \cdot \widetilde{\delta p}_2\, dV. \quad (5.14)$$

The second variation of the amended potential V_{μ_e} is readily obtained from (5.8) as follows. Using the chain rule, the relative equilibrium conditions, and the fact that $\text{ident}_\xi(\delta\varphi) := -D[\mathscr{I}(\varphi_e)\,\xi]\cdot\delta\varphi$, we obtain

$$D^2 V_{\mu_e}(\varphi_e)(\delta\varphi_1, \delta\varphi_2) = D^2 V(\varphi_e)(\delta\varphi_1, \delta\varphi_2) - \xi\cdot[D^2\mathscr{I}(\varphi_e)\cdot(\delta\varphi_1, \delta\varphi_2)]\,\xi + \text{ident}_\xi(\delta\varphi_1)\cdot\mathscr{I}^{-1}(\varphi_e)\,\text{ident}_\xi(\delta\varphi_2). \quad (5.15)$$

In view of $(3.4)_1$ and $(3.10)_1$, the first two terms in (5.15) give the second variation of $V_\xi(\varphi_e)$, which can be recast in the compact form:

$$D^2 V_\xi(\varphi_e)(\delta\varphi_1, \delta\varphi_2) = D^2 V(\varphi_e)(\delta\varphi_1, \delta\varphi_2) - \int_{\mathscr{B}} \varrho_{\text{ref}}\, \xi\times\delta\varphi_1 \cdot \xi\times\delta\varphi_2\, dV. \quad (5.16)$$

The second variation $D^2 V_{\mu_e}$ then takes the final form

$$D^2 V_{\mu_e}(\varphi_e)(\delta\varphi_1, \delta\varphi_2) = D^2 V_\xi(\varphi_e)(\delta\varphi_1, \delta\varphi_2) + \text{ident}_\xi(\delta\varphi_1)\cdot\mathscr{I}^{-1}(\varphi_e)\,\text{ident}_\xi(\delta\varphi_2). \quad (5.17)$$

In view of (5.17), it is apparent that expression (5.14) does not involve terms coupling $\delta\varphi$ and $\widetilde{\delta p}$. Furthermore, the term involving $\widetilde{\delta p}$ is the second variation of the kinetic energy, which is clearly positive-definite. Therefore, it follows that definiteness of the reduced Hamiltonian restricted to the constrained subspace $\mathscr{S}_0$ holds if and only if definiteness of V_{μ_e} restricted to $\mathscr{V}$ holds (see Part I, § 2 for a general proof). To summarize:

$$D^2 V_{\mu_e}(\varphi_e)|_{\mathscr{V}\times\mathscr{V}} > 0 \Rightarrow z_e = (\varphi_e, p_e) \in P \text{ is formally stable.} \quad (5.18)$$

Thus, as pointed out above, the reduced test (5.18) for formal stability involves only configurations and configuration variations. Furthermore, the constraint that the total linear and angular momentum be conserved is automatically enforced.

§ 5.C. *Implementation of the reduced stability test: Block-diagonalization of the amended potential*

The implementation of the reduced test (5.18) for formal stability of a relative equilibrium relies crucially on the fact that $D^2V_{\mu_e}(\varphi_e)$ block-diagonalizes on $\mathscr{V}_{RIG}\times\mathscr{V}_{INT}$. A concrete proof of this result in the context of elasticity is given below, and rests on the following identities.

Lemma 5.1. *Let* $\hat{\eta}\in so(3)$ and $\delta\varphi\in\mathscr{V}$. *Then, the following identities hold*:

i.

$$D^2V_\xi(\varphi_e)\,(\eta_Q,\delta\varphi)=\xi\times\eta\cdot\mathrm{ident}_\xi\,(\delta\varphi). \tag{5.19}$$

ii.

$$\mathrm{ident}_\xi(\eta_Q(\varphi_e))=-\eta\times\mu_e-\mathscr{I}(\xi\times\eta). \tag{5.20}$$

Proof. To prove (5.19) we make use of the general result (4.50). From expression (3.37) for the body force in a relative equilibrium, the definition of Lie derivative, and repeated use of Jacobi's identity we obtain

$$\begin{aligned}-(\pounds_{\eta_Q}\varrho_{\mathrm{ref}}^{\varphi}B)\,(\varphi_e)&=\varrho_{\mathrm{ref}}\{-\eta\times[\xi\times(\xi\times\varphi_e)]+\xi\times[\xi\times(\eta\times\varphi_e)]\}\\&=\varrho_{\mathrm{ref}}\{(\xi\times\varphi_e)\times(\eta\times\xi)+\xi\times[(\xi\times\eta)\times\varphi_e]\}\\&=\varrho_{\mathrm{ref}}\{2(\xi\times\eta)\times(\xi\times\varphi_e)-[(\xi\times\eta)\times\xi]\times\varphi_e\}.\end{aligned} \tag{5.21}$$

Substitution of this expression into (4.51) and comparison with (4.59) in Lemma 4.6 yields (5.19).

Making use of (4.59) and repeated use of Jacobi's identity, we obtain

$$\begin{aligned}\mathrm{ident}_\xi\,(\eta_Q(\varphi_e))&=\int_{\mathscr{B}}[2p_e\times(\eta\times\varphi_e)-\varrho_{\mathrm{ref}}(\xi\times(\varphi_e\times(\eta\times\varphi_e)))\}\,dV\\&=\int_{\mathscr{B}}\{[-\eta\times(\varphi_e\times p_e)-\varphi_e\times(p_e\times\eta)]\\&\quad-\varrho_{\mathrm{ref}}[(\eta\times\varphi_e)\times(\xi\times\varphi_e)+\xi\times(\varphi_e\times(\eta\times\varphi_e))]\}\,dV\\&=\int_{\mathscr{B}}\{-\eta\times(\varphi_e\times p_e)+\varrho_{\mathrm{ref}}\varphi_e\times[(\xi\times\varphi_e)\times\eta]\\&\quad+\varrho_{\mathrm{ref}}\varphi_e\times[\xi\times(\eta\times\varphi_e)]\}\,dV\\&=\int_{\mathscr{B}}\{-\eta\times(\varphi_e\times p_e)-\varrho_{\mathrm{ref}}\varphi_e\times[(\xi\times\eta)\times\varphi_e]\}\,dV\\&=-\eta\times\mu_e-\mathscr{I}(\xi\times\eta),\end{aligned} \tag{5.22}$$

which proves (5.20). □

Note that relation (5.20) follows directly from the abstract result in Proposition 2.3 of Part I [eq. (2.38)] merely by observing that $\mathrm{ad}^*_\eta\,\mu_e=\eta\times\mu_e$ for $G=\mathrm{SO}(3)$. The preceding lemma immediately yields the following block diagonalization result.

Proposition 5.2. (Block-diagonalization for elasticity). *The second variation of* V_{μ_e} *at* $\varphi_e \in Q$ *equals zero on* $\mathscr{V}_{RIG} \times \mathscr{V}_{INT}$, *i.e.*,

$$D^2 V_{\mu_e}(\varphi_e)\,(\eta_Q(\varphi_e), \delta\varphi) = 0, \quad \textit{for } (\hat{\eta}, \delta\varphi) \in \mathscr{G}_{\mu_e}^{\perp} \times \mathscr{V}_{INT}. \tag{5.23}$$

Proof. That $D^2 V_{\xi}(\varphi_e)\,(\eta_Q(\varphi_e), \delta\varphi) = 0$ follows from (4.59), (5.19) and the fact that $\widehat{\eta \times \xi} \in \mathscr{G}_{\mu_e}^{\perp}$ for $\hat{\eta} \in \mathscr{G}_{\mu_e}^{\perp}$, a result also in agreement with Theorem 4.5. On the other hand, by (5.20) we have

$$\text{ident}_{\xi}(\delta\varphi) \cdot \mathscr{I}^{-1}(\varphi_e)\,\text{ident}_{\xi}(\delta\varphi) = \text{ident}_{\xi}(\delta\varphi) \cdot [\mathscr{I}^{-1}(\eta \times \mu_e) + \eta \times \xi]. \tag{5.24}$$

Next, we observe that $\mathscr{I}^{-1}(\eta \times \mu_e)$ is in $\mathscr{G}_{\mu_e}^{\perp}$, since

$$\begin{aligned} \xi \cdot \mathscr{I}^{-1}(\eta \times \mu_e) &= \mathscr{I}^{-1}\xi \cdot \eta \times \mu_e \\ &= \mu_e \cdot \eta \times \mu_e = 0. \end{aligned} \tag{5.25}$$

Thus, (5.24) vanishes by the characterization of $\mathscr{V}_{INT}$ in (4.59), and so (5.23) follows. □

As a result of the preceding block-diagonalization theorem, the reduced test (5.18) for formal stability of a relative equilibrium is equivalent to the following two *uncoupled* conditions

$$D^2 V_{\mu_e}(\varphi_e)|_{\mathscr{V}_{RIG} \times \mathscr{V}_{RIG}} > 0 \quad \text{and} \quad D^2 V_{\mu_e}(\varphi_e)|_{\mathscr{V}_{INT} \times \mathscr{V}_{INT}} > 0. \tag{5.26}$$

These conditions for formal (orbital) stability agree with the abstract results in § 2 and § 3 of Part I, and lead to the following explicit results.

5.C.1. Stability conditions associated with $\mathscr{V}_{RIG}$. Combining expressions (5.17), (5.19) and (5.20), and using the equilibrium condition $\mu_e = \mathscr{I}(\varphi_e)\,\xi = \lambda_e \xi$, we obtain

$$\begin{aligned} D^2 V_{\mu_e}(\varphi_e)\,(\eta_Q(\varphi_e), \nu_Q(\varphi)) &= -\mathscr{I}^{-1}(\varphi_e)\,\eta \times \mu_e \cdot \text{ident}_{\xi}(\nu_Q(\varphi_e)) \\ &= \eta \times \mu_e \cdot [\mathscr{I}^{-1}(\varphi_e)\,[\nu \times \mu_e] + \xi \times \nu] \\ &= \eta \times \mu_e \cdot [\mathscr{I}^{-1}(\varphi_e) - \lambda_e^{-1}\,\mathbf{1}]\,\nu \times \mu_e. \end{aligned} \tag{5.27}$$

Since $\widehat{\eta \times \mu_e} \in \mathscr{G}_{\mu_e}^{\perp}$ for $\hat{\eta} \in \mathscr{G}_{\mu_e}^{\perp}$, we conclude that (5.27) cannot vanish, provided that $\dim[\text{span}(\xi)] = 1$. Under this condition, it follows that (5.27) *is definite if and only if* $\xi \in \mathbb{R}^3$ *is an axis of either maximum or minimum inertia of the locked inertia tensor* $\mathscr{I}(\varphi_e)$. We must choose the *maximum* value of λ_e since the kinetic energy term in the second variation of the reduced Hamiltonian is always positive-definite.

5.C.2. Stability conditions associated with $\mathscr{V}_{INT}$. The second stability requirement $(5.26)_2$ involves a test for positive-definiteness on the *constrained* subspace $\mathscr{V}_{INT}$ defined by (4.58). We show below that condition $(5.26)_2$ can be recast in terms of an eigenvalue problem formulated in the *entire* (unrestricted) tangent

space of variations $T_{\varphi_e}Q$, which is amenable to a *straightforward implementation* in concrete applications.

First, we recall that the block-diagonalization theorem implies that $\mathscr{V}_{RIG}$ and $\mathscr{V}_{INT}$ are in fact *orthogonal* with respect to the second variation $D^2V_{\mu_e}$ at $\varphi_e \in Q$, i.e.,

$$\delta\varphi \in \mathscr{V}_{INT} \quad \text{if and only if} \quad D^2V_{\mu_e}(\varphi_e)\,(\delta\varphi, \eta_Q(\varphi_e)) = 0 \quad \text{for all} \quad \hat{\eta} \in \mathscr{G}_{\mu_e}^{\perp}. \tag{5.28}$$

Also recall from Lemma 5.1 that $D^2V_{\mu_e}(\varphi_e)\,(\delta\varphi, \xi_Q(\varphi_e)) = 0$ for any $\delta\varphi \in T_{\varphi_e}Q$.

Next, we select a particularly convenient basis for *all* rigid body variations in $so(3)\cdot\varphi_e$, namely, the principal directions of the locked inertia dyadic $\mathscr{I}$. Accordingly, let $\xi^{(A)}$, $A = 1, 2$, be such that

$$\mathscr{I}\xi^{(A)} = \lambda^{(A)}\xi^{(A)}, \quad \lambda^{(A)} > 0, \tag{5.29}$$

where, we also assume that no added symmetries are present to that the eigenvalues $\lambda^{(A)}$ are distinct. Set

$$\psi_1 = \xi^{(1)} \times \varphi_e, \quad \psi_2 = \xi^{(2)} \times \varphi_e, \quad \psi_3 = \xi \times \varphi_e, \tag{5.30}$$

so $\{\psi_1, \psi_2, \psi_3\}$ is an orthogonal basis which spans $[so(3)\cdot\varphi_e]$. Let $B_e : T_{\varphi_e}Q \times T_{\varphi_e}Q \to \mathbb{R}$ be the bilinear form induced by $D^2V_{\mu_e}$ at $\varphi_e \in Q$, i.e.,

$$B_e(\cdot, \cdot) := D^2V_{\mu_e}(\varphi_e)\,(\cdot, \cdot). \tag{5.31}$$

Note that $B_e(\psi_3, \psi_3) = 0$. Our implementation of conditions $(5.26)_2$ is formulated in terms of the modified bilinear form $\tilde{B}_e : T_{\varphi_e}Q \times T_{\varphi_e}Q \to \mathbb{R}$ defined as

$$\tilde{B}_e(\cdot, \cdot) = B_e(\cdot, \cdot) - \sum_{A=1}^{2} \frac{B_e(\psi_A, \cdot) \otimes B_e(\psi_A, \cdot)}{B_e(\psi_A, \psi_A)} \tag{5.32}$$

where $B_e(\psi_A, \cdot) : T_{\varphi_e}Q \to T_{\varphi_e}Q$ is the linear operator associated with the bilinear form, B_e and $\psi_A \in [so(3)\cdot\varphi_e]$. The modified bilinear form $\tilde{B}_e$ is the unique extension of the operator B_e from $\mathscr{V}_{INT} \times \mathscr{V}_{INT}$ to $T_{\varphi_e} \times T_{\varphi_e}$ with the following convenient property.

Lemma 5.3. *The bilinear form $\tilde{B}_e$ has the canonical form*

$$\tilde{B}_e := \left[\begin{array}{ccc|c} 0 & & & \\ & 0 & & O \\ & & 0 & \\ \hline & O & & \tilde{B}_e|_{\mathscr{V}_{INT}\times\mathscr{V}_{INT}} \end{array}\right] \begin{array}{l} \Big\} so(3)\cdot\varphi_e \\ \Big\} \mathscr{V}_{INT} \end{array} \tag{5.33}$$

$$\underbrace{\qquad\qquad}_{so(3)\cdot\varphi_e} \quad \underbrace{\qquad\qquad\qquad}_{\mathscr{V}_{INT}}$$

where

$$\tilde{B}_e|_{\mathscr{V}_{INT}\times\mathscr{V}_{INT}} = B_e|_{\mathscr{V}_{INT}\times\mathscr{V}_{INT}}. \tag{5.34}$$

Proof. That $\tilde{B}_e(\delta\varphi_1, \psi_A) = 0$ for *any* $\delta\varphi \in T_{\varphi_e}Q$ follows directly from (5.32) since

$$\tilde{B}_e(\delta\varphi, \psi_A) = B_e(\delta\varphi, \psi_A) - B_e(\delta\varphi, \psi_A) = 0; \quad \text{for all } \delta\varphi \in T_{\varphi_e}Q. \tag{5.35}$$

Consequently, $\dim [\ker \tilde{B}_e] \geqq 3$, and the diagonal structure in (5.33) holds. Furthermore, again from (5.32) along with the orthogonality condition (5.28), we have

$$\begin{aligned} \tilde{B}_e(\delta\varphi, \psi) &= B_e(\delta\varphi, \psi) - \sum_{A=1}^{2} \frac{B_e(\psi_A, \delta\varphi)\, B_e(\psi_A, \psi)}{B_e(\psi_A, \psi_A)} \\ &= B_e(\delta\varphi, \psi) \quad \text{for all } \psi, \delta\varphi \in \mathscr{V}_{INT}, \end{aligned} \tag{5.36}$$

so that (5.33) holds. □

We show below that the preceding result reduces the test for orbital stability on the subspace $\mathscr{V}_{INT}$ to the solution of a standard eigenvalue problem. In particular, if a standard Galerkin finite-element projection is introduced; see, e.g., CIARLET [1978], the test for stability merely reduces to the computation of the lower part of the spectrum of a symmetric matrix. From a numerical analysis standpoint, this task is straightforward and can be easily accomplished by using well-known algorithms for the standard symmetric eigenvalue problem; see, e.g., GOLUB & VAN LOAN [1989] for a recent overview.

The unrestricted eigenvalue problem. The stability test on $\mathscr{V}_{INT} \times \mathscr{V}_{INT}$ reduces to the following unconstrained test: Find the *four* lowest eigenpairs $(\lambda, \psi) \in \mathbb{R} \times T_{\varphi_e}Q$ such that

$$\tilde{B}_e(\delta\varphi, \psi) = \lambda \langle \delta\varphi, \psi \rangle_{L_2(\mathscr{B})}. \tag{5.37}$$

Lemma 5.3 ensures that $\{\psi_1, \psi_2, \psi_3\}$ are eigenvectors of $\tilde{B}_e$ associated with the zero eigenvalue $\lambda = 0$. The stability conditions $(5.26)_2$ then reduces to testing whether the next eigenvalue of problem (5.37) is *positive*. If the condition

$$\tilde{\lambda}_e = \min_{\langle \delta\varphi, \psi_A \rangle_{L_2(\mathscr{B})} = 0} \frac{\tilde{B}_e(\delta\varphi, \delta\varphi)}{\langle \delta\varphi, \delta\varphi \rangle_{L_2(\mathscr{B})}} > 0, \tag{5.38}$$

holds, then we conclude formal stability on $\mathscr{V}_{INT} \times \mathscr{V}_{INT}$.

§ 5.D. *Polyconvexity and conditional stability*

In this section we discuss some of the severe technical difficulties involved in a rigorous rather than merely formal stability analysis of relative requilibria.

Definition 5.4. *A relative equilibrium* $z_e \in P$ *is* conditionally stable *relative to a* G_{μ_e}*-invariant metric* d *if for all initial data* $z \in P$ *near to* z_e *(in the metric* d*), the solution of the initial value problem with initial condition* z *remains near to the* G_{μ_e}*-orbit of* z_e *for as long as it is defined in a given function space.*

We assume that a function space, typically a Sobolev space, has been chosen for the existence and uniqueness theory. Furthermore, we assume that in this function space conservation of energy has been established (or at least that the energy is a non-increasing function). For example, one possible choice of function spaces is that in HUGHES, KATO & MARSDEN [1977]. It should be noted, however, that the metric d usually involves a topology that is weaker than the topology for which current existence and uniqueness theory is known. This is, however, the best one can hope for, given the state of the art in existence theory. One would expect that with the addition of dissipation one can say more since, with our definition of formal stability, such an addition will move the spectrum of the linearized equations into the left half-plane and thus be helpful to the existence theory. For example, in the case of rods, one should be able to use this to prove global existence of smooth solutions near a stable solution.

There is a method for establishing conditional stability based on Theorem 4.9 of BALL & MARSDEN [1984]. This result is given for equilibrium (rather than relative equilibrium) solutions of nonlinear elasticity. It states that if the equilibrium satisfies certain technical conditions [(H1)–(H7) of that paper with $p > 3$], which include, for example, constitutive relations discussed in BALL [1977] and CIARLET & GEYMONAT [1982], and if the equilibrium is a strict local minimum of the stored energy function in a metric d induced by the $W^{1,1}$-topology and finiteness of the energy, (see BALL & MARSDEN [1984], p. 270) then the equilibrium lies in a potential well with respect to the metric d. In particular, if these conditions hold, then one has conditional stability. The metric d can be replaced by a metric ϱ which is related to the $W^{1,p}$-norm (loc. cit. p. 274).

One of the important assumptions that examples in BALL & MARSDEN [1984] show is not easy to omit, is the assumption that the equilibrium is a strict local minimizer. Certainly formal stability is necessary for this, but the examples show that it is not sufficient. This is an unfortunate obstacle to a satisfactory theory. The example constructed in BALL & MARSDEN [1984] is actually a homogeneous equilibrium (in fact, the identity in a natural state) for which the second variation of the energy is positive-definite in the $W^{1,1}$-topology, yet it is not a local minimum in the $W^{1,r}$-topology intersected with the C^0-topology for any $r < 1 + (3/p)$. This is not a contradiction because the energy function is not differentiable on these spaces. One cannot, on the other hand, use a stronger topology in which calculus does guarantee a local minimum since it is known that there is no potential well in these cases. Of course at a global minimum of the energy, the minimum condition is automatically satisfied, and so one has conditional stability. In summary, the quoted result of BALL & MARSDEN may be useful for establishing conditional stability, but one is left with a fairly nontrivial hypothesis to check (that the equilibrium is a local minimizer). The examples show that it is not obvious how to make use of the positivity of the second variation to establish this condition.

There are several ways out of this apparent dilemma which warrant further investigation in the future. First, one could take advantage of dissipation; the formal stability results will then guarantee that the spectrum moves into the left half-plane, and one can attempt to use this fact to get an improved asymptotic stability result. Second, one can truncate the system by a finite-element model,

for example, and argue that this is stable and that the high-frequency modes cut out by this process are not of interest anyway. Third, one can employ special arguments for particular materials, such as the convexity arguments used in fluid mechanics (see HOLM et al. [1985]) to show by methods other than calculus methods that one has a local minimizer. Then again one has conditional stability. A successful application of this last approach to the stability of planar rotating liquid drops is contained in the work of LEWIS [1989]. For elastic rods with stored energy function of the Saint Venant-Kirchhoff type, see SIMO, POSBERGH & MARSDEN [1990].

In our context, all of the above arguments must, of course, be modified to take into account the fact that we are dealing with a relative equilibrium and not a true equilibrium. Notice that the modification required in the transition from the stored energy function to the amended potential involves terms that are algebraic in the deformation, and so do not affect the technical potential-well arguments. Another point that requires further attention is the sense in which any of the relative equilibria of LEWIS & SIMO [1990] are also equilibria of the full three-dimensional elasticity problem. By contrast, the situation is far more clear in classical hydrodynamics; see, e.g., CHANDRASEKHAR [1977].

§ 6. Block-diagonalization and symplectic structure

The decomposition $\mathscr{V} = \mathscr{V}_{INT} \oplus \mathscr{V}_{RIG}$ at the configuration level defined above induces in a natural manner decompositions at the phase-space level relative to which both the Hamiltonian and the reduced Hamiltonian block-diagonalize. The simplest proof of these results uses the following two-step construction:

i. First, on the level set $\boldsymbol{J}^{-1}(\mathbf{0})$ of zero momentum, define the split $\mathscr{S}_0 = \mathscr{S}_{0INT} \oplus \mathscr{S}_{0RIG}$ of the tangent space $\mathscr{S}_0$ given by (5.13), so as to achieve a block-diagonal structure of the reduced Hamiltonian h_{μ_e}. As shown below, the space $\mathscr{S}_{0RIG}$ consists of rigid configuration variations (in $\mathscr{V}_{RIG}$) with *zero momentum*. This choice defines the entire construction.

ii. Second, map forward the result in i to the tangent to the level set $\boldsymbol{J}^{-1}(\boldsymbol{\mu}_e)$ using the tangent to the (inverse) shifting map *restricted* to level set. This restriction, denoted by $\Sigma_{\mu_e}^{-1} : \boldsymbol{J}^{-1}(\boldsymbol{\mu}_e) \to \boldsymbol{J}^{-1}(\mathbf{0})$, is defined from (5.1) as

$$\Sigma_{\mu_e}^{-1}(\boldsymbol{\varphi}, \tilde{\boldsymbol{p}}) = (\boldsymbol{\varphi}, \tilde{\boldsymbol{p}} + \boldsymbol{p}_{\mu_e}); \quad \text{where} \quad \boldsymbol{p}_{\mu_e} := \varrho_{\text{ref}}([\mathscr{J}(\boldsymbol{\varphi})]^{-1}\, \boldsymbol{\mu}_e) \times \boldsymbol{\varphi}. \tag{6.1}$$

We show below that this construction does in fact yields a split of the tangent to the level set $\boldsymbol{J}^{-1}(\boldsymbol{\mu}_e)$ relative to which the second variation of the original Hamiltonian block-diagonalizes.

§ 6.A. *Block-diagonalization on the zero-momentum level set*

We define the space $\mathscr{S}_{0RIG} \subset \mathscr{S}_0$ of *rigid variations* simply by appending *zero-momentum variations* to the space $\mathscr{V}_{RIG}$ of rigid configuration variations. With this definition and view of the characterization (5.13) of the space $\mathscr{S}_0$, the

only possible choice is to define the space $\mathscr{S}_{0INT} \subset \mathscr{S}_0$ merely by appending all possible momentum variations $\widetilde{\delta p} \in [so(3) \cdot \varphi_e]^A$ to the space $\mathscr{V}_{INT}$. Accordingly, we set

$$\mathscr{S}_{0RIG} := \mathscr{V}_{RIG} \oplus \{\mathbf{0}\}, \quad \text{and} \quad \mathscr{S}_{0INT} := \mathscr{V}_{INT} \oplus [so(3) \cdot \varphi_e]^A. \tag{6.2}$$

It follows from $(6.2)_1$ that $\dim \mathscr{S}_{0RIG} = \dim \mathscr{V}_{RIG} = 2$. Moreover, again from (5.13) and the fact that $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$ we have

$$\mathscr{S}_0 = \mathscr{S}_{0RIG} \oplus \mathscr{S}_{0INT}, \tag{6.3}$$

as required. Finally, inspection of expression (5.14) for the second variation of the reduced Hamiltonian along with (5.13) and definition (6.2) of the split, yields the following block-diagonal structure

$$D^2 h_{\mu_e}|_{\mathscr{S}_0 \times \mathscr{S}_0} = \begin{array}{c} \begin{matrix} \mathscr{V}_{RIG} & \mathscr{V}_{INT} & [so(3) \cdot \varphi_e]^A \end{matrix} \\ \begin{bmatrix} D^2 V_{\mu_e}|_{\mathscr{V}_{RIG} \times \mathscr{V}_{RIG}} & O & O \\ O & D^2 V_{\mu_e}|_{\mathscr{V}_{INT} \times \mathscr{V}_{INT}} & O \\ O & O & \langle \varrho_{\text{ref}} \cdot, \cdot \rangle \end{bmatrix} \end{array}. \tag{6.4}$$

Each column of the matrix of (6.4) lies in the space shown above it. Observe that the definition of $\mathscr{S}_{0RIG}$ in $(6.2)_1$ determines the entire construction leading to the block diagonalization result (6.4). The mechanical motivation for $(6.2)_1$ is clear: If $\mathscr{S}_{0RIG}$ is to model rigid-body variations about the equilibrium $\tilde{z}_e = (\varphi_e, \mathbf{0})$, then $\widetilde{\delta p} = \mathbf{0}$ is the only possible rigid body momentum variation consistent with the constraint of zero total angular momentum.

§ 6.B. *Block-diagonalization on the μ_e-momentum level set*

Let $\mathscr{S} \subset T_{z_e} J^{-1}(\mu_e)$ be the tangent space of admissible configuration-momentum variations associated with the relative equilibrium point $z_e = (\varphi_e, p_e)$. Variations in $\mathscr{S}$ satisfy the linearized constant angular momentum condition modulo infinitesimal rotations about the axis μ_e (which must be parallel to ξ). Formally, we thus have:

$$\mathscr{S} = \ker [DJ(z_e)]/[so(3)_{\mu_e} \cdot \varphi_e]. \tag{6.5}$$

In the developments that follow, however, it is more convenient to characterize $\mathscr{S}$ simply as the image of $\mathscr{S}_0$ under the linearization of the inverse of the *restricted* shifting map $\Sigma_{\mu_e}^{-1} : J^{-1}(\mathbf{0}) \to J^{-1}(\mu_e)$ defined in (6.1). It is clear from (6.1) that this map is one-to-one and onto. The split $\mathscr{S} = \mathscr{S}_{RIG} \oplus \mathscr{S}_{INT}$ is then defined merely as the image of the split of $\mathscr{S}_0$ according to the diagram:

$$\mathscr{S}_0 = \mathscr{S}_{0RIG} \oplus \mathscr{S}_{0RIG} \xrightarrow{D\Sigma_{\mu_e}^{-1}(\tilde{z}_e)} \mathscr{S} = \mathscr{S}_{RIG} \oplus \mathscr{S}_{RIG}. \tag{6.6}$$

If we set $(\delta\varphi, \delta p) = D\Sigma_{\mu_e}^{-1}(\tilde{z}_e) \cdot (\delta\varphi, \widetilde{\delta p})$, a direct computation from (6.1) that uses the relative equilibrium conditions $(3.14)_1$ and the relation $\text{ident}_\xi(\delta\varphi) =$

$-D[\mathscr{I}(\varphi_e)\cdot\delta\varphi]\,\xi$ (see (4.63)) yields the result

$$\delta p = \widetilde{\delta p} + \varrho_{\text{ref}}\xi\times\delta\varphi + \varrho_{\text{ref}}\mathscr{I}_e^{-1}[\text{ident}_{\xi}(\delta\varphi)]. \tag{6.7}$$

This expression leads to an explicit characterization of the variations in $\mathscr{S}_{RIG}$, denoted by $\Delta z = (\Delta\varphi, \Delta p)$, which is useful in computations. In fact, since variations in $\mathscr{S}_{0RIG}$ are of the form $\eta\times\varphi_e$ with $\eta\cdot\xi = 0$, from (6.6), (6.7) and (5.20) along with Jacobi's identity, we conclude that

$$\Delta\varphi = \eta\times\varphi_e, \quad \text{and} \quad \Delta p = \varrho_{\text{ref}}\mathscr{I}_e^{-1}[\mu_e\times\eta] + \eta\times p_e. \tag{6.8}$$

The block-diagonalization result (6.4) for the reduced Hamiltonian h_{μ_e}, along with definition (6.6) of the decomposition $\mathscr{S} = \mathscr{S}_{RIG}\oplus\mathscr{S}_{INT}$, determine an analogous block-diagonalization result for the restriction $H|_{J^{-1}(\mu_e)}$ of the original Hamiltonian to the μ_e-momentum level set. The proof of this result rests on the identity

$$H(\Sigma_{\mu_e}^{-1}(\tilde{z})) = h_{\mu_e}(\tilde{z}), \quad \forall\, z\in J^{-1}(0), \tag{6.9}$$

which follows at once from (5.3) and (6.1). Differentiation of (6.9), and use of the equilibrium condition $Dh_{\mu_e}(\tilde{z}_e)\cdot\widetilde{\delta z} = 0$ along with the chain rule yields

$$D^2H(\Sigma_{\mu_e}^{-1}(\tilde{z}_e))\cdot(D\Sigma_{\mu_e}^{-1}\cdot\widetilde{\delta z}, D\Sigma_{\mu_e}^{-1}\cdot\widetilde{\Delta z}) = h_{\mu_e}(\tilde{z}_e)\cdot(\widetilde{\delta z}, \widetilde{\Delta z}), \tag{6.10}$$

for all $\widetilde{\delta z}, \widetilde{\Delta z}\in\mathscr{S}_0$. In particular, if we choose $\widetilde{\delta z}\in\mathscr{S}_{0INT}$ and $\widetilde{\Delta z}\in\mathscr{S}_{0RIG}$, it follows from (6.6) that

$$\delta z := D\Sigma_{\mu_e}^{-1}\cdot\widetilde{\delta z}\in\mathscr{S}_{INT} \quad \text{and} \quad \Delta z := D\Sigma_{\mu_e}^{-1}\cdot\widetilde{\Delta z}\in\mathscr{S}_{RIG}. \tag{6.11}$$

Since $D^2h_{\mu_e}(\tilde{z}_e)\cdot(\widetilde{\delta z}, \widetilde{\Delta z}) = 0$, it follows from (6.10) that $D^2H(z_e)\cdot(\delta z, \Delta z) = 0$. This result can also be checked by a direct computation using formula (6.8), as in SIMO, POSBERGH & MARSDEN [1990].

§ 6.C. *Block-diagonalization and the symplectic two-form*

We conclude this section with a few remarks on the structure taken by the symplectic two-form when the splitting $\mathscr{S} = \mathscr{S}_{RIG}\oplus\mathscr{S}_{INT}$ is introduced. Although this structure is not relevant to the stability analysis of relative equilibria, it does play a crucial role in the formulation of the linearized dynamics and in the study of the possible bifurcations at the relative equilibrium.

Let $\Omega: TP\times TP\to\mathbb{R}$ be the *canonical symplectic two-form* defined by (2.4). If the first entry is restricted to $\mathscr{S}_{RIG}$, then $\Omega(z_e)|_{\mathscr{S}_{RIG}\times\mathscr{S}}$ becomes

$$\Omega(z_e)(\Delta z, \delta z) = -\int_{\mathscr{B}}\varrho_{\text{ref}}(\zeta\times\varphi_e)\cdot\delta\varphi\, dV =: -\langle\zeta_Q(\varphi_e), \delta\varphi\rangle_g \tag{6.12}$$

for any $\delta z = (\delta\varphi, \delta p)$ in $\mathscr{S}$ and any Δz in $\mathscr{S}_{RIG}$ as given by (6.8), where we have set $\zeta := \mathscr{I}(\mu_e\times\eta)$. This result can be checked by a direct calculation as

follows. Substitute (6.8) into (2.4), and identify the explicit expression for $DJ(z_e)$ to obtain

$$\begin{aligned}\Omega(z_e)(\Delta z, \delta\varphi) &= \int_{\mathscr{B}} [(\eta\times\varphi_e)\cdot\delta p - (\varrho_{\text{ref}}\zeta\times\varphi_e + \eta\times p_e)\cdot\delta\varphi]\, dV \\ &= \eta\cdot\int_{\mathscr{B}} [\varphi_e\times\delta p + \delta\varphi\times p_e)]\, dV - \zeta\cdot\int_{\mathscr{B}} \varrho_{\text{ref}}\varphi_e\times\delta p\, dV \\ &= \eta\cdot[DJ(z_e)\cdot\delta z] - \zeta\cdot\int_{\mathscr{B}} \varrho_{\text{ref}}\varphi_e\times\delta\varphi\, dV. \qquad (6.13)\end{aligned}$$

Equation (6.12) then follows merely by noting that δz restricted to $\mathscr{S}$ satisfies the condition $DJ(z_e)\cdot\delta z = 0$. It should be noted that the symplectic two-form restricted to $\mathscr{S} = \mathscr{S}_{RIG} \oplus \mathscr{S}_{INT}$ *does not* block-diagonalize since the coupling term in (6.12) does not generally vanish. Accordingly, we have the following structure:

$$\Omega(z_e)(\Delta z, \delta z) = \begin{array}{c} \quad\mathscr{S}_{RIG} \qquad\qquad \mathscr{S}_{INT} \\ \left[\begin{array}{cc} \left[\begin{array}{c}\text{rigid}\\ \text{component}\end{array}\right] & \left[\begin{array}{c}\text{rigid-internal}\\ \text{coupling}\end{array}\right] \\ -\left[\begin{array}{c}\text{rigid-internal}\\ \text{coupling}\end{array}\right] & \left[\begin{array}{c}\text{internal}\\ \text{component}\end{array}\right] \end{array}\right] \end{array}. \qquad (6.14)$$

For elasticity this is a particular instance of a general result pointed out in Remark 2.8 of Part I. The preceding observations suggest at least two possible options to define the split $\mathscr{S} = \mathscr{S}_{RIG} \oplus \mathscr{S}_{INT}$:

i. Define $\mathscr{S}_{INT}$ according to the diagram in (6.6). Then, the second variation becomes block-diagonal restricted to $\mathscr{S}_{INT}\times\mathscr{S}_{RIG}$ but, in view of (6.14), the symplectic form is not block-diagonal.

ii. Define $\mathscr{V}_{INT}$ by the orthogonality condition

$$\langle\eta_Q(\varphi_e), \delta\varphi\rangle_g = 0, \quad \text{for } \delta\varphi\in\mathscr{V}_{INT} \text{ and } \hat{\eta}\in\mathscr{G}_{\mu_e}^{\perp}. \qquad (6.15)$$

Then, the second variation restricted to $\mathscr{S}_{INT}\times\mathscr{S}_{RIG}$ is not block-diagonal, but the symplectic form now block-diagonalizes on $\mathscr{S}_{INT}\times\mathscr{S}_{RIG}$.

Clearly, option i above is the most convenient one from the point of view of a stability analysis of relative equilibria.

§ 7. Summary and concluding remarks

The Relative Equilibrium Theorem characterizes relative equilibria as extremals of the energy subject to the constraint of constant angular momentum. In § 3 this constrained variational principle is transformed into an *unconstrained* variational principle, which involves only the configurations, by using Smale's effective potential. The associated momenta play no role, and can be computed explicitly from the relative equilibrium configurations. For elasticity, we have shown that the effective potential is merely the total stored energy in the body augmented by the potential of the centrifugal loading. Formally, the associated

Euler-Lagrange equations lead to a Neumann boundary value problem in elastostatics with configuration-dependent body force defined by the centrifugal potential, as discussed in detail in § 4. Although a fairly complete qualitative characterization of the relative equilibria is possible (see Theorem 3.1), explicit solutions in the general case may only be obtainable numerically.

Our main contribution lies in the explicit characterization of the formal stability conditions for relative equilibria in nonlinear elasticity, along with a framework for future development of bifurcation analysis. First, the *constrained* test for formal stability dictated by the energy-momentum map is reduced to an essentially *unconstrained* test formulated in terms of the *amended* potential V_{μ_e}, which involves only configuration variations. Second, the implementation of the final stability test is remarkably simplified by introducing the decomposition $\mathscr{V} = \mathscr{V}_{RIG} \oplus \mathscr{V}_{INT}$ relative to which the second variation of the reduced Hamiltonian becomes block-diagonal, i.e.,

$$D^2 H_{\mu_e}|_{\mathscr{S} \times \mathscr{S}} = \underbrace{\begin{bmatrix} D^2 V_{\mu_e}|_{\mathscr{V}_{RIG} \times \mathscr{V}_{RIG}} & O & O \\ O & D^2 V_{\mu_e}|_{\mathscr{V}_{INT} \times \mathscr{V}_{INT}} & O \\ O & O & \langle \varrho_{\text{ref}} \cdot, \cdot \rangle \end{bmatrix}}_{\text{configuration variations}} \underbrace{\qquad}_{\text{momentum variations}}.$$

The conditions for formal stability can be read off directly from this expression. In fact, as shown in § 5.C.1, the conditions associated with the $\mathscr{V}_{RIG}$-block are explicit and generalize the classical rigid body conditions. On the other hand, as shown in § 5.C.2, the stability conditions associated with the $\mathscr{V}_{INT}$-block reduce to the computation of the lowest eigenvalue of a standard self-adjoint eigenvalue problem. This computation can be easily implemented numerically.

In addition to these stability results, for purposes of bifurcation theory, it is remarkable that the *same* choice of coordinates leading to our block-diagonalization result also brings the symplectic form, and hence the linearized equations of motion, into normal form. See LEWIS, MARSDEN, RATIU & SIMO [1990] for more information on this point.

Acknowledgments. We are indebted to D. LEWIS for her involvement in this work and her many contributions to the subject of this paper. We thank P. HOLMES and P. S. KRISHNAPRASAD for their comments and suggestions of the subject of this paper. The research of SIMO was partially supported by AFOSR contracts 2-DJA-544 and 2-DJA-771. The research of MARSDEN was partially supported by NSF Grant DMS-8922704 and DOC contract DE-FGO-88ER-25064.

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(Received September 1, 1990)