AN AXIALLY SYMMETRIC CONTACT PROBLEM FOR A HALF-SPACE WITH AN ELASTICALLY REINFORCED CYLINDRICAL CAVITY

P. Ya. Malits

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We consider the pressure of a plate on a half-space with a round cylindrical cavity. The surface of the cavity is reinforced by elastic elements that are modeled by very general operators. The problem is reduced to a Fredholm integral equation of second kind. A detailed study is made of the case of reinforcement described by the Winkler law. An approximate solution is obtained in the form of the asymptotics with respect to the radii of the plate and the cavity.

One table. Bibliography: 3 titles.

We study the pressure of a round convex plate on an elastic homogeneous and isotropic half-space $z \geq 0$ with the elastically reinforced cavity $r = a$, $0 \leq z < \infty$. A plate with the equation $z = f(r)$ is pressed on with axial force P. We assume that friction is absent in the region of contact. The displacements u_r and u_z of the surface of the cavity are certain operators on the stresses applied to this surface. For a large class of reinforcement models the connection of the displacements and strains $\tau(z) = \tau_{rz}(a, z)$, $\sigma(z) = \sigma_r(a, z)$ has the form

$$\begin{aligned} 2\mu a^{-1}\bar{u}_z^s(a,\xi) &= -l_{11}(\xi)\bar{\sigma}^c(\xi) + l_{12}(\xi)\bar{\tau}^s(\xi) + w^*(\xi), \\ 2\mu a^{-1}\bar{u}_r^c(a,\xi) &= l_{21}(\xi)\bar{\sigma}^c(\xi) - l_{22}(\xi)\bar{\tau}^s(\xi) + u^*(\xi), \end{aligned} \tag{1}$$

in the space of Fourier cosine- and sine-transforms, where the elements of the functional matrix $l_{ij}(\xi)$ are determined by the model of reinforcement, $u^*(\zeta)$ and $w^*(\zeta)$ are known functions connected with the boundary conditions in noncontact reinforcement surfaces and with the mass forces.

As a preliminary step, assuming for the time being that the displacement of the entire surface $z = 0$ is known and neglecting friction, i.e., under the boundary conditions

$$u_z = w(r), \quad a \leq r < \infty, \quad \tau_{rz} = 0, \quad a < r < \infty, \tag{2}$$

we shall find the tension-deformed state of the medium under the boundary conditions on the cavity given by the relations (1).

We represent the functions occurring in the Pakovich-Neiber representation [1] as expansions

$$\begin{aligned} \psi(r,z) &= \frac{2}{\pi}\int_0^\infty A_1(\xi)K_1(\xi r)\cos\xi z\, d\xi, \\ \varphi_1(r,z) &= \frac{2}{\pi}\int_0^\infty A_2(\xi)K_0(\xi r)\cos\xi z\, d\xi + \int_0^\infty \alpha(\xi)e^{-\xi z}\frac{\chi_{1,0}(\xi,r)}{|H_1^{(1)}(\xi a)|^2}\, d\xi, \\ \varphi_2(r,z) &= \int_0^\infty \xi\beta(\xi)e^{-\xi z}\frac{\chi_{1,0}(\xi,r)}{|H_1^{(1)}(\xi a)|^2}\, d\xi, \\ \chi_{1,0}(\xi,r) &= Y_0(\xi r)J_1(\xi a) - Y_1(\xi a)J_0(\xi a), \end{aligned} \tag{3}$$

where $A_j(\xi)$, $\alpha(\xi)$, and $\beta(\xi)$ are functions to be determined.

Substituting the representations (3) into the condition (2) and then applying Weber transformations leads to a system of equations from which we find

$$\alpha(\xi) = -\frac{1-2\nu}{1-\nu}\mu\bar{w}(\xi), \quad \beta(\xi) = \frac{\mu}{1-\nu}\bar{w}(\xi),$$

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where ν is the Poisson coefficient.

To determine $A_1(\zeta)$ and $A_2(\zeta)$ we substitute the representations (3) into conditions (1). As a result we obtain a system of linear algebraic equations

$$\begin{cases} A_1(\xi)d_{11}(\xi) + \xi A_2(\xi)d_{21}(\xi) = w^*(\xi) - \bar{w}_2^s(\xi) + l_{11}(\xi)\bar{\gamma}^c(\xi) = V(\xi), \\ A_1(\xi)d_{21}(\xi) + \xi A_2(\xi)d_{22}(\xi) = u^*(\xi) - l_{21}(\xi)\bar{\gamma}^c(\xi) = U(\xi). \end{cases} \tag{4}$$

Here

$$\begin{aligned} d_{11}(\xi) &= \xi a K_1(\xi a) - l_{11}(\xi)m_1(\xi) - l_{12}(\xi)m_2(\xi)\xi a, \\ d_{12}(\xi) &= K_0(\xi a) - l_{11}(\xi)m_3(\xi) + l_{12}(\xi)K_1(\xi a)\xi a, \\ d_{21}(\xi) &= \xi a K_0(\xi a) + 4(1-\nu)K_1(\xi a) + l_{21}(\xi)m_1(\xi) - l_{22}(\xi)m_2(\xi)\xi a, \\ d_{22}(\xi) &= K_1(\xi a) + l_{21}(\xi)m_3(\xi) - l_{22}(\xi)K_1(\xi a)\xi a, \\ m_1(\xi) &= \big(4(1-\nu) + \xi^2 a^2\big)K_1(\xi a) + (3-2\nu)\xi a K_0(\xi a), \\ m_2(\xi) &= 2(1-\nu)K_1(\xi a) + \xi a K_0(\xi a), \quad m_3(\xi) = K_1(\xi a) + \xi a K_0(\xi a), \\ \bar{\gamma}^c(\xi) &= \frac{\xi\mu}{(1-\nu)K_1(\xi a)} \int_a^\infty t w(t) T(\xi,t)\,dt, \\ \bar{w}_2^s(\xi) &= \frac{\mu}{(1-\nu)aK_1(\xi a)} \int_a^\infty t w(t)\big(2(1-\nu) + T(\xi,t)\big)\,dt, \\ T(\xi,t) &= \frac{1}{K_1(\xi a)}\big(\xi a K_2(\xi a)K_0(\xi t) - \xi t K_1(\xi a)K_1(\xi t)\big). \end{aligned}$$

We now take up the solution of the problem of the pressure of the plate on the base when the boundary conditions differ from (1) and (2) in that for $z=0$

$$u_z = c - f(r) = w_0(r), \ \text{for } a \le r \le b, \quad \sigma_z = 0, \ \text{for } r > b, \quad \tau_{rz} = 0, \ \text{for } a \le r \le \infty. \tag{5}$$

The first and third of the relations just written are satisfied if we set $w(r) = w_0(r)$ in the expressions for $\alpha(\xi)$, $\beta(\xi)$, $A_1(\xi)$, and $A_2(\xi)$ for $a \le r \le b$ and leave $w(r)$ to be determined on the interval $r > b$.

We now write out the expression for the normal tensions $\sigma_z(r,0)$

$$\sigma_z(r,0) = \frac{\mu}{1-\nu}\int_0^\infty \xi \frac{\bar{g}_1(\xi)\bar{g}_2(\xi)}{|H_1^{(1)}(\xi a)|^2}\,d\xi + \frac{2}{\pi}\int_0^\infty \xi[K_0(\xi r)(\xi A_2(\xi) - 2\nu A_1(\xi)) + \xi r K_1(\xi r)A_1(\xi)]\,d\xi.$$

Here $\bar{g}_1(\xi)$ and $\bar{g}_2(\xi)$ are defined in [2]. If we express $w(r)$ in terms of the auxiliary function $\omega(t)$ using the results of [2], we obtain

$$\frac{\pi(1-\nu)}{2\mu}\sigma_z(r,0) = -\frac{\omega(b) + \psi(b)}{\sqrt{b^2 - r^2}}\theta(b-r) - \int_r^\infty \left\{ \begin{matrix} \frac{d}{dt}[(\mathrm{I} - K)\omega - K^*(t) + R\omega], & t > b \\ \frac{d}{dt}[R\omega - K\omega - K^*(t) - \psi(t)], & t < b \end{matrix} \right\} \frac{dt}{\sqrt{t^2 - r^2}}, \tag{6}$$

where $K\omega$, $K^*(t)$, and $\psi(t)$ are determined by the formulas in [2] and

$$R\omega = \frac{1-\nu}{\mu}\int_0^\infty [\xi A_2(\xi) + (1 - 2\nu + \xi t)A_1(\xi)]e^{-\xi t}\,d\xi, \quad \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

To represent the operator $R\omega$ in explicit form, we express $A_1(\xi)$ and $A_2(\xi)$ in terms of $\omega(t)$

$$A_i(\xi) = \frac{\mu\xi^{1-i}}{1-\nu}\Big[-\int_b^\infty \omega(t)R_i(t,\xi)\,dt + w_0(a)R_i^*(\xi) + \int_a^b tR_i(t,\xi)\,dt \int_a^t \frac{w_0'(s)\,ds}{\sqrt{t^2 - s^2}} + A_i^*(\xi)\Big], \quad i = 1,2.$$

Here

$$A_i^*(\xi) = \frac{(-1)^i}{\Delta(\xi)}[u^*(\xi)d_{1m}(\xi) - w^*(\xi)d_{2m}(\xi)], \quad m = 0.5[3-(-1)^i],$$

$$\xi a\Delta(\xi)R_i(\xi t) = D(\xi,t)F_i(\xi) + 2(1-\nu)K_1^{-1}(\xi a)e^{-\xi t}d_{2m}(\xi),$$

$$\xi^2 a\Delta(\xi)R_i^*(\xi) = -D^*(\xi)F_i(\xi) + 2(1-\nu)K_1^{-1}(\xi a)[e^{-\xi b} - \xi aK_1(\xi a)]d_{2m}(\xi),$$

$$D(\xi,t) = [\xi aK_2(\xi a) - (1+\xi t)K_1(\xi a)]K_1^{-1}(\xi a)e^{-\xi t},$$

$$D^*(\xi) = [\xi aK_0(\xi a) - \xi bK_1(\xi a)]K_1^{-1}(\xi a)e^{-\xi b},$$

$$F_1(\xi) = \xi al_{11}(\xi) - l_{21}(\xi) + \xi al_{22}(\xi) - \xi a\lambda(\xi) - 1,$$

$$F_2(\xi) = \xi aK_0(\xi a)K_1^{-1}(\xi a)[\xi al_{11}(\xi) - (3-2\nu)l_{21}(\xi) + \xi al_{22}(\xi) - \lambda(\xi)\xi^2a^2 - 1] + 2(1-\nu)[2\xi al_{11}(\xi) - 2l_{21}(\xi) + \xi al_{22}(\xi) - \xi^2a^2\lambda(\xi) - 2],$$

$$\lambda(\xi) = l_{11}(\xi)l_{22}(\xi) - l_{12}(\xi)l_{21}(\xi),$$

and $\Delta(\xi)$ is the determinant of the system (4). We can now write out the expressions for $R\omega$

$$R\omega = -\mathrm{I}\omega + L^*(t), \tag{7}$$

$$L\omega = \int_b^\infty \omega(x)\Big[\int_0^\infty (R_2(x,\xi) + (1-2\nu+\xi t)R_1(x,\xi))e^{-\xi t}\,d\xi\Big]\,dx,$$

$$L^*(t) = w_0(a)\int_0^\infty \big[R_2^*(\xi) + (1-2\nu+\xi t)R_1^*(\xi)\big]e^{-\xi t}\,d\xi + \int_a^b x\int_0^\infty \big[R_2(x,\xi) + (1-2\nu+\xi t)R_1(x,\xi)\big]e^{-\xi t}\,d\xi\,dx\int_a^x \frac{w_0'(s)\,ds}{\sqrt{x^2-s^2}} + \int_0^\infty \big[A_2^*(\xi) + (1-2\nu+\xi t)A_1^*(\xi)\big]e^{-\xi t}\,d\xi.$$

We require that the second of conditions (5) hold. Formula (6), as in [2], makes it possible to obtain an integral equation of second kind on the half-line

$$\omega(t) = (K+L)\omega + K^*(t) - L^*(t), \quad t > b, \tag{8}$$

and an expression for the contact tension

$$\frac{\pi(1-\nu)}{2\mu}\sigma_z(r,0) = -\frac{\omega(b)+\psi(b)}{\sqrt{b^2-r^2}} + \int_r^b \frac{d[(K+L)\omega + K^*(t) - L^*(t) + \psi(t)]}{\sqrt{t^2-r^2}}\,dt.$$

Using the asymptotic expansions of the Laplace integrals [3] one can show that $K^*(t)-L^*(t) \in L_1(b,\infty)$ for physically realizable problems and the kernel of the operator L is square-integrable on $(b,\infty)\times(b,\infty)$. Consequently the expression (8) is a Fredholm integral equation of second kind.

We now consider the case when the reinforcement of the cavity is modeled by a Winkler medium. We shall neglect the friction between the reinforcement and the half-space. Within the limits of this model we have $u_z(a,z) = k\sigma_z(a,z)$, $\tau_{rz}(a,z) = 0$, and we arrive at the conditions (1) for $l_{11}(\xi) = l_{22}(\xi) = 0$, $u^*(\xi) = w^*(\xi) = 0$, $l_{12}(\xi) = \infty$, $l_{21}(\xi) = 2\mu a^{-1}k > 0$.

After a passage to the limit as $l_{12} \to \infty$ the kernel of Eq. (8) becomes symmetric. The corresponding integral operator turns out to be completely continuous on $L_1(b,\infty)$. Since the free term belongs to $L_1(b,\infty)$, it follows that $\omega(t) \in L_1(b,\infty)$ also.

The connection between the initial displacement of the plate and the impressive force is given by the relation

$$\frac{1-\nu}{4\mu}P = bc - \int_a^b \frac{f(r)\,dr}{\sqrt{b^2-r^2}} - \int_b^\infty \omega(t)\,dt, \tag{9}$$

and by the preceding remark this last integral converges.

If $\beta = b/a \gg 1$, then, as can easily be shown by an asymptotic estimate of the free term and the kernel of Eq. (8), we have $\omega(t) \sim O(1/\beta^2)$. Then up to quantities of order β^{-4} one can neglect the terms containing $\omega(t)$ in the expressions for $\sigma_r(z,a)$ and $\sigma_\theta(a,z)$. As a result the radial and tangential tensions can be expressed by certain integrals which in turn can be replaced by their asymptotic expansions.

In the case of a flat plate we obtain

$$\sigma_r(a,r) = -\frac{2\mu c}{\pi(1-\nu)(1+l_{21})a} X\left(\beta, \frac{z}{a}\right) + O\left(\left|\frac{\ln^2\left(\beta + i\frac{z}{a}\right)}{\left(\beta + i\frac{z}{a}\right)^4}\right|\right),$$

$$\sigma_\theta(a,z) = -\frac{2\mu c}{\pi(1-\nu)a}\left[2\nu Y\left(\beta, \frac{z}{a}\right) - \frac{l_{21}}{1+l_{21}} X\left(\beta, \frac{z}{a}\right)\right] + O\left(\left|\frac{\ln^2\left(\beta + i\frac{z}{a}\right)}{\left(\beta + i\frac{z}{a}\right)^4}\right|\right),$$

$$X(\beta,t) = \mathrm{Re}\left[\frac{\beta}{(\beta+it)^2} + \frac{3 - 6\gamma + 4\gamma^2 + (3-2\nu)\beta}{(\beta+it)^3} + \frac{(1-2\gamma-\beta)(3-2\gamma)\ln 2(\beta+it)}{(\beta+it)^3}\right],$$

$$Y(\beta,t) = \mathrm{Re}\left[\frac{1}{\beta+it} + \frac{\pi^2/3 - 2 - (7-2\gamma)\gamma - (6-4\gamma)\ln 2(\beta+it)}{(\beta+it)^3} + \frac{1 - \gamma - \ln 2(\beta+it)}{(\beta+it)^2} + \frac{2(3-2\gamma)\ln^2 2(\beta_i t)}{(\beta+it)^3}\right], \quad (10)$$

where $\gamma = 0.57721\ldots$ is Euler's constant.

The asymptotics of relation (9) have the form

$$\frac{1-\nu}{4\mu bc} P = 1 - \frac{1}{2\beta^2} + O\left(\frac{1}{\beta^3}\right).$$

For comparison we give a table of values of the quantity $Q = \frac{1-\nu}{4\mu bc} P$ found for the case of the free surface of the cavity ($k = \infty$) by numerical solution of Eq. (8). It can be seen from the table that even for $\beta = 2$ the error in the asymptotic formula is already less than 3%.

Analysis of formulas (10) shows that in the neighborhood of the opening of the cavity (i.e., as $z \to 0$) the radial tensions are compressive, while the tangential tensions may happen to be dilating. This last circumstance may lead to the appearance of radial cracks and damage to the base in a neighborhood of the opening.

β	1.2	1.3	1.4	1.5	1.7	2.0
Q	0.56857	0.68682	0.75137	0.79447	0.85005	0.89758

In conclusion we remark that the suggested method makes it possible to carry out computations for other kinds of reinforcement, in particular for reinforcement modeled by a multilayered pipe.

Literature Cited

1. V. Z. Parton and P. I. Perlin, *Methods of the Mathematical Theory of Elasticity* [in Russian], Nauka, Moscow (1981).
2. P. Ya. Malits, "The contact problem for a half-space with a reinforced cavity," *Dinam. Sist.*, No. 3, 39–45 (1984).
3. M. V. Fedoryuk, *The Saddle-point Method* [in Russian], Nauka, Moscow (1977).