LITERATURE CITED

1. P. Hall, "On the embedding of a group in a join of given groups," J. Australian Math. Soc., 17, No. 4, 434-495 (1974).
2. L. A. Bokut', "Some embedding theorems for rings and semigroups," Sibirsk. Mat. Zh., 4, No. 3, 500-518 (1963).
3. A. I. Shirshov, "Some algorithmic problems for Lie algebras," Sibirsk. Mat. Zh., 2, 291-296 (1962).
4. Dnestr Notebook. Unsolved Problems in the Theory of Rings and Modules, Kishinev (1968).
5. O. Kegel, "Zur Nilpotenz gewissen assoziativer Ringe," Math. Ann., 149, 258-260 (1962-63).
6. J. Bergman, "The diamond lemma for ring theory," (Preprint).
7. L. A. Bokut', "Undecidability of the equality problem and subalgebras of finitely presented Lie algebras," Izv. Akad. Nauk SSSR, Ser. Mat., 36, No. 6, 1173-1219 (1972).

VARIETIES OF GENERALIZED STANDARD AND GENERALIZED ACCESSIBLE ALGEBRAS

G. V. Dorofeev

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In this paper we shall consider algebras over an associative and commutative ring Φ with unity and containing 1/6.

For the variety of generalized accessible algebras and certain of its subvarieties, considered in the present paper, we shall use the following notation:

$\mathcal{GAcc}$ - variety of generalized accessible algebras,

$\mathcal{Acc}$ - variety of accessible algebras,

$\mathcal{GSt}$ - variety of generalized standard algebras,

$\mathcal{St}$ - variety of standard algebras,

$\mathcal{Comm}$ - variety of commutative algebras,

$\mathcal{Alt}$ - variety of alternative algebras,

$\mathcal{Jord}$ - variety of Jordan algebras,

$\mathcal{Ass}$ - variety of associative algebras,

$\mathcal{AssComm}$ - variety of commutative associative algebras.

Variety $\mathcal{GAcc}$ was defined in 1969 in [1], variety $\mathcal{Acc}$ in 1956 in [2], variety $\mathcal{GSt}$ in 1968 in [3], and variety St in 1948 in [4].

These varieties are related by the set-theoretical inclusions shown in the diagram on the next page (Fig. 1).

Moreover, the ordered set φ of the varieties shown on this diagram is a sub-semilattice of the lattice of all varieties of algebras relative to the operation of intersection.

The basic result of the present paper is the proof of the assertion that this ordered set forms a sublattice in the lattice of all varieties of algebras relative to the operations of union and intersection. In particular, there hold the equations

$$\mathcal{GAcc} = \mathcal{Comm} + \mathcal{Alt}, \quad \mathcal{GSt} = \mathcal{Jord} + \mathcal{Alt}.$$

We mention that for variety $\mathcal{Acc}$ and its subvarieties, the corresponding assertion was proven in [5].

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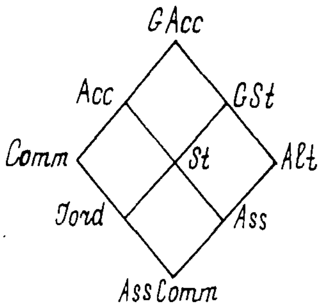

Fig. 1

Thus, the variety of generalized accessible algebras is the minimal variety containing all commutative and all alternative algebras. In other words, some identity holds in every generalized accessible algebra if and only if it holds in every commutative and in every alternative algebra. Whence follows the validity for generalized accessible rings of a host of assertions proven in [6, 7, 8]. Moreover, it turns out that a free generalized accessible algebra is the subdirect sum of free commutative and free alternative algebras, and that each primitive accessible algebra is either commutative or **alternative** [9].

Similarly, the variety of generalized standard algebras is the minimal variety containing all Jordan and all alternative algebras, and a free generalized standard algebra is the subdirect sum of a free Jordan and a free alternative algebra, while each primitive generalized standard algebra is either Jordan or alternative.

1°. Let A be an algebra over ring Φ. For any elements $x,y,z \in A$ we set

$$[x,y] = xy - yx, \quad x \circ y = xy + yx,$$
$$(x,y,z) = (xy)z - x(yz),$$
$$S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y).$$

Function $f(t_1,\dots,t_n)$, defined on algebra A and assuming values in this algebra, is called skewsymmetric in arguments $t_{i_1},\dots,t_{i_k}$, if it changes sign upon any transposition of these arguments.

Since $1/2 \in \Phi$, function f is then skewsymmetric in any collection of arguments if and only if it vanishes each time two of these arguments assume identical values. We shall denote the set of functions skewsymmetric in arguments $t_{i_1},\dots,t_{i_k}$ by SS $(t_{i_1},\dots,t_{i_k})$.

In algebra A we shall consider the following subsets:

A' is the ideal generated by all commutators $[x,y]$,

$K(A)$ is the ideal generated by all (right) alternators (x,y,y),

$S(A)$ is the ideal generated by all elements of the form $S(x,y,z)$,

$V(A) = \{v \in A \mid (v,x,y) \in KC(v,x,y)\}$ is the alternative center.

Moreover, we shall denote by $N(A)$, $Z(A)$, and $C(A)$, as usual, the respective associative center, commutative center, and associative-commutative center (or, simply, center), and by $D(A) = (A,A,A) + (A,A,A)A$, the associator ideal of algebra A.

Algebra A is called elastic if in it there holds the identity

$$(x,y,x) = 0, \tag{1}$$

or, in linearized form,

$$(x,y,z) \in \text{SS}\,(x,z). \tag{2}$$

An elastic algebra is called standard if in it there hold the identities

$$S(x,y,z)=0, \tag{3}$$

$$(xy,t,z)+(zx,t,y)+(zy,t,x)=0, \tag{4}$$

and generalized standard if in it there hold the identities

$$S(xy,z,x)=S(x,y,z)x, \tag{5}$$

$$(xy,t,z)+(zx,t,y)+(zy,t,x)=[z,(t,y,x)]+([t,y],x,z), \tag{6}$$

$$(zt)D_{x,y}=(zD_{x,y})t+z(tD_{x,y}), \tag{7}$$

where $wD_{x,y}=y(xw)-x(yw)+(x,w,y)+(wx)y-(wy)x$.

An algebra is called accessible if in it there hold the identities

$$(x,y,z)+(z,x,y)-(x,z,y)=0, \tag{8}$$

$$([x,y],z,t)=0, \tag{9}$$

and generalized accessible if in it there hold identity (1) and

$$[(x,x,y),z]=0, \tag{10}$$

$$3([x,y],z,t)=-[x,(y,z,t)]+[y,(z,t,x)]+2[z,(t,x,y)]-2[t,(x,y,z)]. \tag{11}$$

As was shown in [8], identities (5) and (7) in the definition of a generalized standard algebra are redundant while, in the definition of a generalized accessible algebra, identity (11) can be replaced by the simpler identity

$$([x,y],z,t)+(z,[x,y],t)=0. \tag{12}$$

The set-theoretical inclusions among the varieties at issue, shown on Fig. 1, are obvious or are readily proven [3, 8, 9]. We now consider the intersections of these sets.

Since $1/3\in\Phi$, it then immediately follows from identity (8) that an accessible alternative algebra is associative and, therefore, $Acc\cap Alt=Ass$. Exactly the same way, from identity (3) follows the equation $St\cap Alt=Ass$. Furthermore, in an accessible algebra the right side of identity (6) equals 0, which follows from identity (9) and the results of [5]. Consequently, an accessible generalized standard algebra is standard and, therefore, $Acc\cap GSt=St$. The coincidence of the remaining intersections with those shown on Fig. 1 is obvious, and we shall formulate the assertion thus obtained in the following form.

Assertion 1. The ordered set G of the varieties represented on Fig. 1 is a sub-semilattice in the semilattice of all varieties of algebras relative to the operation of intersection.

2°. Let A be a generalized accessible algebra over ring Φ, let $V=V(A)$, $Z=Z(A)$. In algebra A there hold the following identities:

$$(xy,z,t)-(x,yz,t)+(x,y,zt)=x(y,z,t)+(x,y,z)t, \tag{13}$$

$$([x,y],z,t)-(x,[y,z],t)+(x,y,[z,t])=[x,(y,z,t)]+[(x,y,z),t], \tag{14}$$

$$(x, y\circ z, t) - (x, y\circ z, t) + (x, y, z\circ t) = x\circ(y,z,t) + (x,y,z)\circ t, \tag{15}$$

$$\mathcal{S}(x,y,z) \in \text{ss}\,(x,y,z) \tag{16}$$

$$[xy, z] = x[y,z] + [x,z]y + \mathcal{S}(x,y,z), \tag{17}$$

$$2\mathcal{S}(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y], \tag{18}$$

$$\mathcal{S}(x\circ y, z, t) = x\circ\mathcal{S}(y,z,t) + y\circ\mathcal{S}(x,z,t), \tag{19}$$

$$\mathcal{S}([u,v], x, y) = 3([u,v], x, y), \tag{20}$$

$$\mathcal{S}((x,y,y), z, t) = 0, \tag{21}$$

$$([u,v], (x,y,y), z) = 0, \tag{22}$$

$$(x[u,v], y, x) = (x, [u,v], y)x, \tag{23}$$

$$([u,v]x, y, x) = x(x, [u,v], y), \tag{24}$$

$$(xy, [u,v], x) = (x, y, [u,v])x, \tag{25}$$

$$([u,v], yx, x) = x([u,v], y, x), \tag{26}$$

$$([u,v]y, x, x) = [u,v](y,x,x). \tag{27}$$

Identity (13) holds in any algebra, identities (14)-(19) hold in any elastic algebra: identity (16) follows from the linearization identity $(x,x,x) = 0$ and the others are proven in [10, 8]. Identity (20) follows immediately from the definition of $\mathcal{S}(x,y,z)$ since, by identity (12), the commutator $[u,v]$ lies in the alternative center; analogously, identity (21) follows from (18) since, by virtue of (10), alternator (x,y,y) lies in the commutative center. Identity (22) follows from (20) and (21), with (16) taken into account. Identities (23)-(26) are proven in [6].

Let us prove identity (27). Denoting commutator $[u,v]$ for brevity by m and using identity (13), we shall have

$$(my, x, x) = (m, yx, x) - (m, y, x^2) + m(y,x,x) + (m,y,x)x.$$

But, by virtue of identities (16), (20), and (19),

$$3(m,y,x^2) = \mathcal{S}(m,y,x^2) = x\circ\mathcal{S}(m,y,x) = 3x\circ(m,y,x),$$

and then, using (26), we obtain

$$(my, x, x) = x(m, y, x) - x\circ(m, y, x) + m(y,x,x) + (m,y,x)\circ x,$$

whence follows identity (27).

In what follows we shall use identities (1), (10), (12), (16), and (20) without, as a rule, any particular mention.

LEMMA 1. In algebra A there holds the identity

$$\mathcal{S}(xy, z, t) - \mathcal{S}(x, yz, t) + \mathcal{S}(x, y, zt) = x\mathcal{S}(y,z,t) + \mathcal{S}(x,y,z)t.$$

Proof. We denote the left side of the identity to be proven by h; then, using successively identities (19), (20), and (14), and taking (16) into account, we obtain:

$$\begin{aligned} 2h &= S(x\circ y,z,t)-S(x,y\circ z,t)+S(x,y,z\circ t)+ \\ &+ S([x,y],z,t)-S(x,[y,z],t)+S(x,y,[z,t])= \\ &= x\circ S(y,z,t)+y\circ S(x,z,t)-y\circ S(x,z,t)- \\ &- z\circ S(x,y,t)+z\circ S(x,y,t)+t\circ S(x,y,z)+ \\ &+ 3([x,y],z,t)-3(x,[y,z],t)+3(x,y,[z,t])= \\ &= x\circ S(y,z,t)+t\circ S(x,y,z)+3[x,(y,z,t)]+3[(x,y,z),t]. \end{aligned}$$

However, it follows from identity (10) that

$$[x,(y,z,t)]\in ss(y,z,t),$$

and, therefore,

$$3[x,(y,z,t)]+3[(x,y,z),t]=[x,s(y,z,t)]+[s(x,y,z),t].$$

Consequently, $2h = 2xS(y,z,t)+2S(x,y,z)t$, QED.

LEMMA 2. In algebra A there holds the identity

$$[[r,s],t](x,y,y)=0.$$

Proof. We denote commutator $[r,s]$ by m, and alternator (x,y,y) by p. By identity (17)

$$[m,t]p = [mp,t]-m[p,t]+S(m,p,t)=[mp,t],$$

as a corollary of (10) and (21); moreover, by virtue of (27),

$$mp = m(x,y,y) = (mx,y,y)\in Z.$$

Therefore,

$$[m,t]p = [mp,t] = 0,$$

which also proves Lemma 2.

COROLLARY 1. In algebra A there holds the identity

$$S(x,y,z)(t,u,u) = 0.$$

LEMMA 3. For any elements $r,s,t,u\in A$,

$$[[r,s],t]u\in V.$$

Proof. We denote the binary commutator $[[r,s],t]$ by n. We first note that, by identity (27) and Lemma 2,

$$(nu,x,x)=n(u,x,x)=0,$$

and, consequently, for any $y\in A$ $(nu,x,y)\in ss(x,y)$ and, since $([n,u],x,y)\in ss(x,y)$ by virtue of (12), then $(n\circ u,x,y)\in ss(x,y)$, i.e.,

$$(n\circ u,x,y)+(n\circ u,y,x)=0. \tag{28}$$

We now transform associator $(y,n\circ u,x)$, using identity (15) and the linearization, flowing from (23) and (24), of the identity

$$(x\circ[r,s],y,x)=(x,[r,s],y)\circ x:$$

$$(y, n\circ u, x) = (y\circ n, u, x) + (y, n, u\circ x) - y\circ(n,u,x) - (y,n,u)\circ x$$
$$= -(x\circ n, u, y) + (y,n,u)\circ x + (x,n,u)\circ y +$$
$$+ (y,n,u\circ x) - y\circ(n,u,x) - (y,n,u)\circ x = (x\circ n, y, u) + (y, n, u\circ x)$$

by virtue of (28). Moreover, again using the linearization of the aforementioned identity, we obtain

$$(x\circ n, y, u) = -(u\circ n, y, x) + (x,n,y)\circ u + (u,n,y)\circ x,$$

and we transform the second term by using identities (20) and (19):

$$3(y, n, u\circ x) = S(y,n,u\circ x) = u\circ S(y,n,x) + x\circ S(y,n,u) = 3u\circ(y,n,x) + 3x\circ(y,n,u).$$

From the expression we have obtained for the associator $(y, n\circ u, x)$, again using identity (28), we shall now have

$$(y, n\circ u, x) = -(u\circ n, y, x) = (n\circ u, x, y).$$

From this equation and from identity (28), it readily follows that $n\circ u \in V$, but commutator $[n,u]$ also lies in V, and therefore the sum $2nu = n\circ u + [n,u] \in V$, which also proves Lemma 3.

COROLLARY 1. For any elements $x, y, z, t \in A$

$$S(x,y,z)t \in V.$$

LEMMA 4. Let P be the ideal generated in algebra A by alternator $p = (x,y,y)$, and let $r, s, t \in A$. Then,

$$[[r,s],t]P = 0.$$

Proof. Each element of ideal P is the sum of elements of the form $w = pT_1T_2 \dots T_n$, where T_i is either a right multiple R_v or a left multiple L_v by some element $v \in A$.

When $n = 0$ the equation $mw = 0$ is valid thanks to Lemma 2. We denote the binary commutator $[[r,s],t]$ by m, and, when $n = 1$, for any $u \in A$ we shall have, remembering that $p \in Z$,

$$m(up) = m(pu) = (mp)u = 0 \tag{29}$$

by identity (22) and Lemma 2.

Consider n = 2. Using Eq. (29) and the linearization of identity (25), for any $u, v \in A$ we obtain

$$m[(p,u)v] = [m(pu)]v - (m, pu, v) = (pu, m, v) = -(vu, m, p) + (p,u,m)v + (v,u,m)p.$$

The first two terms on the right of the equation we have just obtained are equal to 0 by identity (22), while, from identity (20) and Corollary 1 to Lemma 2, it follows that the third term too equals 0. Therefore, it is also the case that

$$m[(pu)v] = 0. \tag{30}$$

We now note that, by virtue of Lemma 2,

$$m(x,y,z) \in \mathrm{ss}\ (x,y,z), \tag{31}$$

and then, taking into account that $p \in Z$ and using identities (30) and (29), we obtain

$$m[u(pv)] = m[(up)v] - m(u,p,v) = m(p,u,v) = m[(pu)v] - m[p(uv)] = 0. \tag{32}$$

We make the induction hypothesis: for any binary commutator m and any $k<n$

$$m(pT_1 T_2 \ldots T_k) = 0. \tag{33}$$

When n = 3, Eq. (33) is true by virtue of (29), (30), and (32); denoting $pT_1 T_2 \ldots T_{k-2}$ by q, we shall have $w = pT_1 T_2 \ldots T_k = qT_{k-1} T_k = qT_u T_v$, so we consider the four possible cases.

If $T_u = R_u$, $T_v = R_v$ then, using the induction hypothesis and the linearization of identity (25), we obtain

$$m[(qu)v] = [m(qu)]v - (m,qu,v) = (qu,m,v) = -(vu,m,q) + (q,u,m)v + (v,u,m)q.$$

However,

$$(vu,m,q) = (m,q,vu) = (mq)(vu) - m[q(vu)] = 0,$$

by the induction hypothesis and, analogously, $(q,u,m) = 0$. Moreover, if follows from identities (20) and (18) and the induction hypothesis that

$$6(v,u,m)q = 2S(v,u,m)q = 0.$$

Consequently,

$$m(qR_u R_v) = 0. \tag{34}$$

If $T_u = R_u$, $T_v = L_v$ then, using the induction hypothesis and identities (17) and (34), we obtain

$$m[v(qu)] = m[(qu)v] - m[qu,v] =$$

$$= [m,v](qu) - [m(qu),v] + S(m,qu,v) = 3(m,qu,v) = 3[m(qu)]v - 3m[(qu)v],$$

and, as a consequence of (34),

$$m(qR_u L_v) = 0. \tag{35}$$

If $T_u = L_u$ and $T_v = R_v$ then, by virtue of (35) and (31),

$$m[(uq)v] = m(u,q,v) + m[u(qv)] = m(q,v,u) = 0, \tag{36}$$

thanks to (34).

Finally, when $T_u = L_u$ and $T_v = L_v$, then equation

$$m[v(uq)] = 0 \tag{37}$$

is proven exactly the same as (36).

Equations (34)-(37) complete the induction and, therewith, the proof of the lemma.

COROLLARY 1. For any elements $x,y,z \in A$,

$$S(x,y,z)P = 0.$$

LEMMA 5. Let P be the ideal generated in algebra A by the alternator $p = (x,y,y)$. Then, for any element $w \in P$ and any $z,t \in A$, $S(w,z,t) = 0$.

Proof. Just as in the proof of Lemma 4, we shall assume that $w = pT_1 T_2 \ldots T_n$, and proceed by induction on n. When $n = 0$ the equation

$$S(pT_1 T_2 \ldots T_n, z,t) = 0 \tag{38}$$

assumes the form $S(p,z,t)=0$, and is valid thanks to identity (21). Denoting $pT_1T_2\dots T_{n-1}$ by q we shall have $w=qT_n=qT_v$.

If $T_v=L_v$ then, by Lemma 1,

$$S(vq,z,t)=S(z,t,vq)=S(z,tv,q)-S(zt,v,q)+zS(t,v,q)+S(z,t,v)q=0$$

by Corollary 1 to Lemma 4 and the induction hypothesis. If, however, $T_v=R_v$, then, again by Lemma 1,

$$S(qv,z,t)=S(t,qv,z)=S(tq,v,z)+S(t,q,vz)-tS(q,v,z)-S(t,q,v)z=0$$

by what has been proven as well as the induction hypothesis.

This also completes the proof of Lemma 5.

COROLLARY 1. For any element $T^n=T_1T_2\dots T_n$ of the multiplication algebra of algebra A,

$$s((x,y,z)T^n,t,u)\in KC(x,y,z).$$

LEMMA 6. In algebra A there holds the identity

$$S((x,y,z),t,u)=(S(x,y,z),t,u).$$

Proof. By virtue of the Corollary 1 to the Lemma 5 (when n = 0),

$$S((x,y,z),t,u)=S((y,z,x),t,u)=S((z,x,y),t,u),$$

and since $S(x,y,z)\in V$ then

$$3(S(x,y,z),t,u)=S(S(x,y,z),t,u)=S((x,y,z)+(y,z,x)+(z,x,y),t,u)=3S((x,y,z),t,u),$$

whence follows the assertion of the lemma.

LEMMA 7. Ideal $S=S(A)$ of algebra A coincides with the Φ submodule

$$B=S(A,A,A)+S(A,A,A)A.$$

Proof. Since $B\subset S$ it suffices to convince ourselves that B is an ideal of algebra A. But, by Lemma 1, for any $x,y,z,t\in A$,

$$tS(x,y,z)=S(tx,y,z)-S(t,xy,z)+S(t,x,yz)-S(t,x,y)z\in B,$$

and, by Lemma 6,

$$[S(x,y,z)t]u=S(x,y,z)(tu)+(S(x,y,z),t,u)=S(x,y,z)(tu)+S((x,y,z),t,u)\in B.$$

Finally,

$$u[S(x,y,z)t]=[uS(x,y,z)]t-(u,S(x,y,z),t)\in B$$

by what has been proven and by Lemma 6.

COROLLARY 1. Ideal S is contained in the alternative center V.

For the proof it suffices to use Lemma 6 and Corollary 1 to Lemma 3.

LEMMA 8. For any element $T^n=T_1T_2\dots T_n$ of the multiplication algebra of algebra A,

$$S((x,y,z)T^n,t,u)=(S(x,y,z)T^n,t,u).$$

Proof. Just as in the proof of Lemma 6, we remark that, by virtue of Corollary 1 to Lemma 5,

$$S((x,y,z)T^n, t, u) = S((y,z,x), T^n, t, u) = S((z,x,y)T^n, t, u),$$

and since, from Corollary 1 to Lemma 7, $S(x,y,z)T^n \in V$, the proof of Lemma 8 proceeds as in the previous case considered.

We formulate the results obtained in Lemmas 1, 6, and 7 as a separate assertion.

Assertion 2. Let A be a generalized accessible algebra over commutative and associative ring Φ with unity, containing 1/6. Then, for any elements $x, y, z, t, u \in A$ and any element $T^n = T_1 T_2 \ldots T_n$ of multiplication algebra of algebra A,

a) $S(xy, z, t) - S(x, yz, t) + S(x, y, zt) = xS(y,z,t) + S(x,y,z)t$,

b) $S((x,y,z)T^n, t, u) = (S(x,y,z)T^n, t, u)$,

c) $S(x, y, (z,t,u)) = (x, y, S(z,t,u))$.

3°. Let $\mathfrak{M}$ be some variety of algebras over ring Φ, A a free algebra of this variety with set of generators $X = \{x_1, x_2, \ldots\}$, and , on algebra A , let there be given the trilinear function $U(x, y, z)$ possessing the following properties: for any elements $x, y, z, t, u \in A$ and any element $R^n = R_1 R_2 \ldots R_n$ of the algebra of right multipliers of algebra A,

a) $U(xy, z, t) - U(x, yz, t) + U(x, y, zt) = xU(y,z,t) + U(x,y,z)t$,

b) $U((x,y,z)R^n, t, u) = (U(x,y,z)R^n, t, u)$,

c) $U(x, y, (z,t,u)) = (x, y, U(z,t,u))$.

(When n = 0, we assume, as usual, that for any $x \in A$ $xR^n = x$.)

LEMMA 9. Let U be the ideal generated in algebra A by the value of function $U(x,y,z)$ Then, set C of elements of the form

$$U(x,y,z)R_{x_1} R_{x_2} \ldots R_{x_n},$$

where x, y, z are monomials of generators of set X , $x_1, x_2, \ldots, x_n \in X$, and $n \geq 0$, generates U as a Φ module.

Proof. We prove initially that ideal U coincides with set B of linear combinations of elements of the form $U(x,y,z) + U(t,u,v)w$.

By property b), when n = 0, for any $x,y,z,t,u \in A$ we have:

$$[U(x,y,z)t]u = U(x,y,z)(tu) + (U(x,y,z), t, u) = U(x,y,z)(tu) + U((x,y,z), t, u) \in B,$$

whence it follows that B is a right ideal in A . Moreover, from property a) it immediately follows that

$$tU(x,y,z) = -U(t,x,y)z + U(tx,y,z) - U(t,xy,z) + U(t,x,yz) \in B,$$

and then, using property b), we obtain

$$u[U(x,y,z)t] = u[U(xy,z,t) - U(x,yz,t) + U(x,y,zt)] - u[xU(y,z,t)].$$

By what has been proven, the first three terms on the right of this equation lie in B, and for the second, by property c), we have

$$u[xU(y,z,t)] = (ux)U(y,z,t) - (u, x, U(y,z,t)) = (ux)U(y,z,t) - U(u, x, (y,z,t)).$$

Thus, each element of ideal U is a linear combination of elements of the form $U(x,y,z)$ and $U(x,y,z)t$, where it can obviously be assumed that x, y, z, and t are monomials.

We now consider an arbitrary element $w \in U$ of the form

$$w = U(x,y,z)R_{t_1}R_{t_2} \ldots R_{t_n},$$

where $x, y, z, t_1, t_2, \ldots, t_n$ are monomials. If not all the t_i belong to X then, having chosen among them the monomial $t_i = uv$ of maximal degree, we denote $R_{t_1} \ldots R_{t_{i-1}}$ by R^{i-1}, and $R_{t_{i+1}} \ldots R_{t_n}$ by R^{n-i}. Then, using property b), we obtain:

$$w = [U(x,y,z)R^{i-1}(uv)]R^{n-i} = ([U(x,y,z)R^{i-1}u]v)R^{n-i} - (U(x,y,z)R^{i-1},u,v)R^{n-i} =$$
$$= U(x,y,z)R^{i-1}R_u R_v R^{n-i} - [U((x,y,z)R^{i-1},u,v)]R^{n-i}.$$

This equation, as is easily seen, provides the possibility of induction on the maximal degree of the monomials $t_1, \ldots, t_n$, to show that element w is a linear combination of elements of set C. It is clear now that each element of ideal U is also linearily expressed in terms of set C.

Lemma 9 is proven.

LEMMA 10. Let F be a free nonassociative algebra over ring Φ with set of free generators X. Then, set C of elements of the form

$$(x,y,z)R_{x_1}R_{x_2} \ldots R_{x_n},$$

where x, y, and z are monomials of generators of set X, $x_1, x_2, \ldots, x_n \in X$ and $n \geq 0$, is a basis of associator ideal $D = D(F)$ of algebra F.

Proof. It follows from identity (13) that associator (x, y, z), as a trilinear function of its terms x, y, and z, possesses properties a)-c) and, having applied Lemma 9 to associator ideal D, we find that set C generates ideal D as a Φ module. It remains to prove the linear independence of set C over Φ.

Let

$$\sum_i \alpha_i (x_i, y_i, z_i) R_{x_{1i}} R_{x_{2i}} \ldots R_{x_{n_i i}} = 0 \tag{39}$$

be a linear combination of elements of set C with nonzero coefficients. In Eq. (39) we consider the term having the greatest total degree in associator (x_i, y_i, z_i), and let term

$$w = \alpha(x,y,z)R_{x_1}R_{x_2} \ldots R_{x_n}$$

have, among them, the greatest degree in the third term of the associator. Having expanded in Eq. (39) all the associators, we present its left side in the form of sum of monomials, one of which will be the monomial $v = \alpha(\ldots((x(yz))x_1)x_2)\ldots)x_n$ occurring as the expansion of the associator in w. We now show that monomial v cannot be contracted with the remaining monomials.

Indeed, for the construction of an element of set C, in the expansion of which there occurs monomial v, we must initially split off in it some number of right multiples by the elements $x_i \in X$, i.e., present monomial v in the form

$$v = \alpha \left[(\ldots((x(yz))x_1)x_2 \ldots) x_i \right] R_{x_{i+1}} \ldots R_{x_n} \quad (i=1,2,\ldots,n),$$

and thereafter present monomial $u = (\ldots((x(yz))x_1)x_2\ldots)x_i$ as the first or second term of an appropriate associator. Since $x_i \in X$, then, when $i \geq 2$, monomial u can be presented only as the first term of the associator

$$((\ldots((x(yz))x_1)x_2\ldots)x_{i-2}, x_{i-1}, x_i), \tag{40}$$

and, when i = 1, only as the first term of associator

$$(x, yz, x_1). \tag{41}$$

However, associators (40) and (41) have total degree strictly larger than that of associator (x,y,z) occurring in w and, therefore, do not occur as associators in the terms on the left of Eq. (39).

Consequently, we must split off in monomial v all right multiples by elements $x_i \in X$:

$$v = [x(yz)]\, R_{x_1} R_{x_2} \ldots R_{x_n}.$$

But then, monomial $u = x(yz)$ can be presented either as the second term of an associator occurring in w or, with $x = st$, as the second term of associator (s, t, yz); however, the second possibility also contradicts the choice of term w.

Thus, monomial v cannot in fact be contracted with the other monomials in Eq. (39), so that Eq. (39) is untrue. The contradiction thus obtained completes the proof of Lemma 10.

LEMMA 11. Let function $U(x,y,z)$ possess, in variety $\mathfrak{M}$, properties a)-c), and

$$v = \sum \alpha_i (x_i, y_i, z_i) T_{u_{1i}} T_{u_{2i}} \ldots T_{u_{n_i i}}, \tag{42}$$

where $x_i, y_i, z_i, u_{1i}, u_{2i}, \ldots, u_{n_i i}$ are monomials, and, moreover, arbitrary elements of the associator ideal of free nonassociative algebra F with set of free generators X. Then if v = 0 in algebra F, in each algebra A of variety $\mathfrak{M}$, there holds the equation

$$U(v) = \sum_i \alpha_i U(x_i, y_i, z_i) T_{u_{1i}} T_{u_{2i}} \ldots T_{u_{n_i i}} = 0.$$

Proof. Considering (x,y,z) as a trilinear function possessing properties a)-c), we can according to Lemma 9, express element v in the form of a linear combination of elements of set C, using, for this, only tranformations permissible by properties a)-c). The identical sequence of transformations can also be applied to element $U(v) \in A$ since, by hypothesis, function $U(x,y,z)$ also possesses properties a)-c) in variety $\mathfrak{M}$. With this, at each step the elements, having been obtained in algebra A, are "U rings" of the corresponding elements in algebra F.

If, now, v = 0 in algebra F then, by virtue of Lemma 10, with the last transformation in F we obtain a linear combination with zero coefficients; consequently, in algebra A we obtain a "U ring" of these linear combinations which, naturally, also has zero coefficients. This also means that in algebra A there holds the equation $U(v) = 0$.

LEMMA 12. If, in free alternative algebra G with set of free generators X, there holds the identity $v=0$, where element v has the form of (42), then, in free generalized accessible algebra A with set of free generators X, there holds the identity $S(v)=0$.

Proof. In algebra G let hold the identity $v=0$; this means that element v, considered as an element of free nonassociative algebra F, lies in the ideal of algebra F generated by associators of the forms (x,y,y) and (x,x,y). In other words, in algebra F there holds the identity:

$$\sum_i \alpha_i (x_i, y_i, z_i) T_{u_{1i}} T_{u_{2i}} \cdots T_{u_{n_i i}} = \sum_j \beta_j \left[(p_j, q_j, q_j') + (p_j, q_j', q_j) \right] T_{v_{1j}} T_{v_{2j}} \cdots T_{v_{e_j j}} +$$

$$+ \sum_k \gamma_k \left[(r_k, r_k', s_k) + (r_k', r_k, s_k) \right] T_{w_{1k}} T_{w_{2k}} \cdots T_{w_{m_k k}},$$

in which all the elements of F are monomials.

According to Assertion 2, function $S(x,y,z)$ in algebra A possesses properties a)-c) and, consequently, by Lemma 11 there holds in A the identity:

$$\sum_i \alpha_i S(x_i, y_i, z_i) T_{u_{1i}} T_{u_{2i}} \cdots T_{u_{n_i i}} = \sum_j \beta_j \left[S(p_j, q_j, q_j') + S(p_j, q_j', q_j) \right] T_{v_{1j}} T_{v_{2j}} \cdots T_{v_{e_j j}} +$$

$$+ \sum_k \gamma_k \left[S(r_k, r_k', s_k) + S(r_k', r_k, s_k) \right] T_{w_{1k}} T_{w_{2k}} \cdots T_{w_{m_k k}}.$$

However, in algebra A function $S(x,y,z)$ is skewsymmetric and, therefore, the right side of the obtained equation equals 0 whence we also obtain that $S(v)=0$. QED.

LEMMA 13. In free generalized accessible algebra A over ring Φ with set of free generators X, the intersection of ideal $S = S(A)$ with alternator ideal $K = K(A)$ equals 0.

Proof. Let w be an arbitrary element of the intersection $S \cap K$; by Lemma 9, element w is presented in the form

$$w = \sum_i \alpha_i S(x_i, y_i, z_i) R_{t_{1i}} R_{t_{2i}} \cdots R_{t_{n_i i}},$$

and, since $w \in K$, then w is also presented in the form

$$w = \sum_j \beta_j (p_j, q_j, q_j) T_{u_{1j}} T_{u_{2j}} \cdots T_{u_{n_j j}}.$$

Thus, in algebra A we have the identity

$$\sum_i \alpha_i S(x_i, y_i, z_i) R_{t_{1i}} R_{t_{2i}} \cdots R_{t_{n_i i}} = \sum_j \beta_j (p_j, q_j, q_j) T_{u_{1j}} T_{u_{2j}} \cdots T_{u_{n_j j}} \tag{43}$$

Identity (43) also holds in free alternative algebra G as in the homomorphic image of algebra A by ideal K; but in algebra G this identity assumes the form

$$3 \sum_i \alpha_i (x_i, y_i, z_i) R_{t_{1i}} R_{t_{2i}} \cdots R_{t_{n_i i}} = 0, \tag{44}$$

since $S(x_i, y_i, z_i) = 3(x_i, y_i, z_i)$.

After having applied Lemma 12 to identity (44), we obtain the identity

$$3 \sum_i \alpha_i S(x_i, y_i, z_i) R_{t_{1i}} R_{t_{2i}} \cdots R_{t_{n_i i}} = 0,$$

valid in algebra A. But this identity means that w = 0, QED.

LEMMA 14. In free generalized accessible algebra A, the intersection of ideal L, generated in A by the left sides of identities (2), (8), and (9), with the alternator ideal $K=K(A)$, equals 0.

Proof. We shall show that ideal L actually coincides with ideal S. Indeed, the inclusion $S\subset L$ is obvious since the left side of identity (8), by virtue of identity (2) which does hold in algebra A, equals $S(x, y, z)$ and, for the proof of the inverse inclusion, it suffices to use identity (20). It just remains to apply Lemma 13.

THEOREM 1. Ordered set G of varieties of algebras over ring Φ is a sublattice in the lattice of all varieties of algebras over ring Φ relative to the operations of intersection and union.

Proof. By virtue of assertion 1 there remains to consider only unions of the given varieties. Thanks to the results of [3] there hold the equations

$$\mathit{Jord} + \mathit{Ass} = \mathit{St}, \quad \mathit{Comm} + \mathit{Ass} = \mathit{Comm} + \mathit{St} = \mathit{Acc}. \tag{45}$$

Lemma 14 means, obviously, that

$$\mathit{Acc} + \mathit{Alt} = \mathit{GAcc}. \tag{46}$$

It follows from Eqs. (46) and (45) that

$$\mathit{Comm} + \mathit{Alt} = \mathit{Comm} + \mathit{Ass} + \mathit{Alt} = \mathit{Acc} + \mathit{Alt} = \mathit{GAcc} \tag{47}$$

From this and from the inclusion

$$\mathit{Acc} + \mathit{GSt} \supset \mathit{Comm} + \mathit{GSt} \supset \mathit{Comm} + \mathit{Alt}.$$

follows the equation

$$\mathit{Acc} + \mathit{GSt} = \mathit{Comm} + \mathit{GSt} = \mathit{GAcc}. \tag{48}$$

Moreover, by virtue of the modularity of the lattice of varieties of algebra, we obtain from (46) that

$$\mathit{St} + \mathit{Alt} = \mathit{Acc} \cap \mathit{GSt} + \mathit{Alt} = (\mathit{Acc} + \mathit{Alt}) \cap \mathit{GSt} = \mathit{GSt}, \tag{49}$$

and, finally, by virtue of (45) and (49),

$$\mathit{Jord} + \mathit{Alt} = \mathit{Jord} + \mathit{Ass} + \mathit{Alt} = \mathit{St} + \mathit{Alt} = \mathit{GSt}. \tag{50}$$

Equations (47)-(50), together with assertion 1, also complete the proof of Theorem 1.

4°. In this section we derive some corollaries of Theorem 1.

COROLLARY 1. The variety of generalized accessible (generalized standard) algebras over ring Φ is the minimal variety containing the variety of commutative (Jordan) algebras and alternative algebras over ring Φ.

In other words, some identity holds in each generalized accessible (generalized standard) algebra over ring Φ if and only if it holds in each commutative (Jordan) and each alternative algebra over ring Φ.

The first assertion of this corollary is immediately contained in Theorem 1. For the proof of the second assertion we remark that the mapping $\mathfrak{M} \mapsto T(\mathfrak{M})$ putting into correspondence with each variety $\mathfrak{M}$ of algebras over ring Φ its ideal $T(\mathfrak{M})$ of identities in a free nonassociative algebra over Φ, is an anti-isomorphism between the lattice of varieties and the lattice of T-ideals, so that the equation $\mathfrak{P}=\mathfrak{M}+\mathfrak{N}$ is equivalent to the equation $T(\mathfrak{P})=T(\mathfrak{M})\cap T(\mathfrak{N})$.

For the varieties at issue, we find that the assertions

$$f\in T(GAcc) \text{ and } [f\in T(Comm) \text{ and } f\in T(Alt)]$$

and

$$f\in T(GSt) \text{ and } [f\in T(Jord) \text{ and } f\in T(Alt)]$$

are, respectively, equivalent.

COROLLARY 2. Each primitive generalized accessible (generalized standard) algebra over ring Φ is either commutative (Jordan) or alternative.

Indeed, from Corollary 1 we obtain that, in a free accessible algebra, $A'\cap K(A)=0$ and, consequently, $A'\cdot K(A)=0$. This equation also holds in any generalized accessible algebra. Therefore, in the primitive algebra, either $A'=0$, or $K(A)=0$.

We deal analogously with the generalized standard algebra A.

COROLLARY 3. Let A be a free generalized accessible algebra over ring Φ. Then

1) algebra A is a subdirect sum of a free commutative and a free alternative algebra;

2) alternative center $V(A)$ of algebra A coincides with the commutator ideal A';

3) commutative center $Z(A)$ of algebra A strictly contains alternator ideal $K(A)$.

Indeed, algebra A, by virtue of Theorem 1, is the subdirect sum of its own factor algebras A/A' and $A/K(A)$, i.e., a free commutative and a free alternative algebra.

Furthermore, if element f belongs to A' it can then be presented in the form

$$f=\sum_i \alpha_i [x_i,y_i]\, T_{u_{1i}} T_{u_{2i}} \cdots T_{u_{n_i i}},$$

and, considering f as an element of a free nonassociative algebra, we find that, in every commutative and in every alternative algebra there hold the identities

$$(f,x,y)=(x,y,f)=(y,f,x).$$

But then, these identities also hold in every generalized accessible algebra; consequently, $f\in V(A)$ and $A'\subset V(A)$. The converse inclusion follows from the fact that in a free commutative ring A/A' the alternative center equals 0.

Finally, the inclusion $K(A)\subset Z(A)$ is proven the same way as the inclusion $A'\subset V(A)$. The converse inclusion, in the given case, is untrue, since, by virtue of the results of [11], element $[(x,y,z),t]^8$ lies in the commutative center $Z(A)$ but does not lie in the alternator ideal $K(A)$ since, in the contrary case, in free alternative algebra $A/K(A)$ there would hold the identity $[(x,y,z),t]^8=0$.

COROLLARY 4. Let A be an arbitrary generalized accessible algebra over ring Φ. Then, in algebra A there hold the Kleinfeld identities

$$([x,y]^4,z,t)-[x,y]([x,y]^2,z,t)-([x,y]^2,z,t)[x,y]=0,$$

and if algebra A has a system of generators of three elements, then in it there holds the identity

$$([x,y]^2, z, t) = 0.$$

Indeed, these assertions are valid in each commutative and in each alternative algebra, the latter for algebras with three generators [12, 13].

COROLLARY 5. Let $GAcc_n$ be the variety of algebras generated by a free generalized accessible algebra over ring Φ with n generators. Then, there hold the strict inclusions

$$GAcc_1 \subset GAcc_2 \subset GAcc_3 \subset GAcc_4.$$

Indeed, each generalized accessible algebra with two generators in accessible [6] and, therefore, $GAcc_3 \neq GAcc_2$ and, from the example constructed in [14] and from Corollary 4, it follows that $GAcc_4 \neq GAcc_3$. Finally, the noncoincidence of varieties $GAcc_1$ and $GAcc_2$ or, what amounts to the same thing, of varieties Acc_1 and Acc_2, follows from the fact that a free accessible algebra with one generator is commutative: this readily follows by induction from identity (17) which, in an accessible algebra, takes the form

$$[xy, z] = x[y, z] + [x, z]y.$$

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LITERATURE CITED

1. E. Kleinfeld, M. H. Kleinfeld, and F. Kosier, "The structure of generalized accessible rings," Bull. Amer. Math. Soc., 75, 415-417 (1969).
2. E. Kleinfeld, "Standard and accessible rings," Can. J. Math., 8, 335-340 (1956).
3. R. D. Schafer, "On generalized standard algebras," Proc. Nat. Acad. Sci. USA, 60, 73-74 (1968).
4. A. A. Albert, "Power associative rings," Trans. Amer. Math. Soc., 64, 552-593 (1948).
5. G. V. Dorofeev and S. V. Pchelintsev, On Varieties of Standard and Accessible Algebras [in Russian] (in printing).
6. G. V. Dorofeev, "Identities of generalized accessible rings," Sibirsk. Mat. Zh., 17, No. 4 (1976).
7. G. V. Dorofeev, "Kleinfeld identities in generalized accessible rings," Mat. Zametki, 19, No. 2, 291-297 (1976).
8. I. P. Shestakov, "Generalized standard rings," Algebra i Logika, 13, No. 1, 88-103 (1974).
9. E. Kleinfeld, M. H. Kleinfeld, and F. Kosier, "A generalization of commutative and alternative rings," Can. J. Math., 22, 348-362 (1970).
10. I. P. Shestakov, "On certain classes of noncommutative Jordan rings," Algebra i Logika, 10, No. 4, 407-448 (1971).
11. G. V. Dorofeev, "Centers of nonassociative rings," Algebra i Logika, 12, No. 5, 530-549 (1973).
12. E. Kleinfeld, "Simple alternative rings," Ann. Math., 58, No. 3, 545-547 (1953).
13. G. V. Dorofeev, "Alternative rings with three generators," Sibirsk. Mat. Zh., 4, No. 5, 1029-1048 (1963).
14. G. V. Dorofeev, "One example in the theory of alternative rings," Sibirsk. Mat. Zh., 4, No. 5, 1049-1052 (1963).