Periodica Mathematica Hungarica Vol. 31 (2), (1995), pp. 105–112

ESSENTIAL IDEALS IN $C(X)$

F. AZARPANAH (Ahvaz)

Abstract

It is shown that X is finite if and only if $C(X)$ has a finite Goldie dimension. More generally we observe that the Goldie dimension of $C(X)$ is equal to the Souslin number of X. Essential ideals in $C(X)$ are characterized via their corresponding z-filters and a topological criterion is given for recognizing essential ideals in $C(X)$. It is proved that the Fréchet z-filter (cofinite z-filter) is the intersection of essential z-filters. The intersection of ideals O_x where x runs through nonisolated points in X is the socle of $C(X)$ if and only if every open set containing all nonisolated points is cofinite. Finally it is shown that if every essential ideal in $C(X)$ is a z-ideal then X is a P-space.

Introduction

A nonzero ideal E in a commutative ring R is called *essential* if it intersects every nonzero ideal nontrivially. This concept was first introduced in [7] and plays an important role in the structure theory of noncummutative Notherian rings, see [5] or [13]. One of the central notions in the context of $C(X)$, the ring of continuous real valued functions on a completely regular Hausdorff space X, is that of a prime ideal. It turns out that every prime ideal in $C(X)$ is either an essential ideal or a maximal one, therefore the study of essential ideals in $C(X)$ is worthwhile. We note that for any ideal B in $C(X)$ (or more generally in any commutative semiprime ring), the ideal $B \oplus Ann(B)$, where $Ann(B) = \{f \in C(X) : fB = (0)\}$ is the annihilator of B in $C(X)$, is an essential ideal in $C(X)$. Hence an ideal B in a commutative semiprime ring is an essential ideal if and only if $Ann(B) = (0)$. This immediately shows that an ideal B in $C(X)$ is essential if and only if $B \cap C^*(X)$ is an essential ideal in $C^*(X)$, and also if A is an essential ideal in $C^*(X)$, then $AC(X)$ is an essential ideal in $C(X)$. The intersection of all essential ideals in any commutative ring R is the *socle*, the sum of all minimal ideals of R, see [5], p. 59, or

Mathematics subject classification numbers, 1991. Primary 54C40; Secondary 13A18.
Key words and phrases. Socle, P-space, Fréchet-filter, essential ideal, annihilator, nowhere dense, Goldie dimension, Souslin number.

0031-5303/95/\$5.00

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

[11], p. 62 and for a topological characterization of the socle of $C(X)$, see [10]. It is interesting to note that the set of isolated points in a completely regular Hasudorff space X is dense if and only if the socle of $C(X)$ is essential in $C(X)$, see [10], 2.1. We note that in this case the socle is the smallest essential ideal in $C(X)$.

A set $\{B_i\}_{i\in I}$ of nonzero ideals in $C(X)$ is said to be *independent* if $B_i \cap (\sum_{i\neq j\in I} B_j) = (0)$, i.e., $\sum_{i\in I} B_i = \oplus_{i\in I} B_i$. Then we say $C(X)$ has a *finite Goldie dimension*, if every independent set of nonzero ideals is finite, and if $C(X)$ has no finite Goldie dimension, then the Goldie dimension of $C(X)$, denoted by $dim\, C(X)$ is the smallest cardinal number c such that every independent set of nonzero ideals in $C(X)$ has cardinality less than or equal to c. An ideal A in $C(X)$ is said to be *uniform* if any two nonzero ideal contained in A intersect nontrivially. These ideals are related to finite Goldie dimension, see [5] or [13]. We observe that an ideal in $C(X)$ is uniform if and only if it is a minimal ideal.

The smallest cardinal number c such that every family of pairwise disjoint nonempty open subsets of X has cardinality less than or equal to c, is called the *Souslin number*, or *cellularity* of the space X and is denoted by $S(X)$, see [1]. If $S(X)$ is countable, then we say that X has the *Souslin property*. We show that for any completely regular Hausdorff space X, the Souslin number of X is equal to the Goldie dimension of $C(X)$. We also show that the z-filter corresponding to the socle (Fréchet z-filter) is the intersection of all essential z-filters. We study the relations of essential ideals with z-ideals, prime ideals and prove that if every essential ideal in $C(X)$ is z-ideal, then X is a P-space.

For a large class of topological spaces, including compact Hausdorff ones, we show that the socle of $C(X)$ is the intersection of the essential ideals O_x, where x runs through the set of nonisolated points in X and $O_x = \{f \in C(X) : Z(f)$ is a neighborhood of x$\}$, see [3]. In this paper $C(X)$ is the ring of continuous real valued functions on a completely regular Hausdorff space X and the reader is referred to [3], for undefined terms and notations.

1. Uniform ideals

The following results shows that the set of uniform ideals in $C(X)$ and the set of minimal ideals coincide.

PROPOSITION 1.1. *If A is an ideal in $C(X)$, then the following are equivalent.*
(i) A is a uniform ideal in $C(X)$.
(ii) For any two nonzero elements $f, g \in A$, $fg \neq 0$.
(iii) A is a minimal ideal in $C(X)$.

PROOF. (i)⇒(ii) We note $(f) \cap (g) \neq (0)$ implies that $\exists h_1, h_2 \in C(X)$ such that $fh_1 = gh_2 \neq 0$. This shows that $fgh_1h_2 \neq 0$ and therefore $fg \neq 0$.

(ii)⇒(iii) By Proposition 3.1 of [10], it is sufficient to show that there exists a fixed isolated point $x \in X$ such that $X - \{x\} \subseteq Z(f)$, $\forall f \in A$. Now let $0 \neq f \in A$, x and y be two distinct elements in $X - Z(f)$ and G, H be two disjoint open sets containing x and y respectively. Then by complete regularity of X, there are elements g_1, g_2 in $C(X)$ such that $g_1(X - G) = g_2(X - H) = \{0\}$ and $g_1(x) = g_2(y) = 1$. Clearly $g_1 f$ and $g_2 f$ are nonzero elements of A and $g_1 f g_2 f = 0$, a contradiction. Next, suppose that for distinct nonzero elements $f_1, f_2 \in A$ there are distinct elements x_1 and x_2 in X such that $X - \{x_1\} \subseteq Z(f_1)$ and $X - \{x_2\} \subseteq Z(f_2)$. Then we have $f_1 f_2 = 0$ which contradicts (ii).

(iii)$\Rightarrow$(i) This is trivial. ■

2. Goldie dimension versus Souslin number

It is well-known that if a commutative ring R (in fact any ring) has a finite Goldie dimension, then there is an integer $n > 0$ such that a direct sum of nonzero ideals in R has always m terms, where $m \leq n$ and there is a direct sum of uniform ideals (with n terms) which is essential in R, see [5] and [13]. We observe that any reasonable finiteness condition on a ring R, such as Noetherianness or having Krull-dimension, in the sense of Gabriel and Rentschler, see [5] or [13], immediately implies that R must have a finite Goldie dimension. The next result shows that X must be finite if the Goldie dimension of $C(X)$ is finite. Of course we could obtain this result from the equality of the Souslin number and the Goldie dimension but we give a different proof in this case because of the importance of finiteness of Goldie dimension.

Proposition 2.1. *X is finite if and only if $C(X)$ has a finite Goldie dimension.*

Proof. Suppose that the Goldie dimension of $C(X)$ is finite, then there is an essential ideal which is a finite direct sum of uniform ideals. Since each uniform ideal is minimal in $C(X)$, this essential ideal is the socle of $C(X)$ and we also note that the cardinality of the set of isolated points is the same as the cardinality of the set of minimal ideals in $C(X)$, see [10]. This shows that the set of isolated points in X is finite and dense and therefore it must be X. The converse is obvious. ■

Next we prove our main result of this section.

Theorem 2.2 *If X is an infinite space, then $dim\, C(X) = S(X)$.*

Proof. Let $dim\, C(X) = c$ and $\oplus_{i \in I} B_i$ be a direct sum of ideals in $C(X)$, where $|I|$, the cardinality of I, is less than or equal to c. Now for each $i \in I$, let $0 \neq f_i \in B_i$, then $f_i f_j = 0$, when $i \neq j$. Hence $\big(X - Z(f_i)\big) \cap \big(X - Zf_j)\big) = \emptyset$, and this implies that $F = \{X - Z(f_i) : i \in I\}$ is a collection of disjoint open sets in X, i.e., $S(X) \geq c$. Now let $\{G_k : k \in K\}$ be any collection of disjoint open sets in X, then for all $k \in K$, there exists $0 \neq f_k \in C(K)$ such that $f_k(X - G_k) = \{0\}$. Now we put $B_k = (f_k)$, $\forall k \in K$ and claim that $\{B_k\}_{k \in K}$ is an independent set of nonzero ideals in $C(X)$. If we prove our claim, then we are through, for in that case $dim\, C(X) = c \geq |K|$, i.e., $c \geq S(X)$. Therefore we must show that $B_k \cap (\sum_{k \neq r \in K} B_r) = (0)$.

Now let $f \in B_k \cap (\sum_{k \neq r \in K} B_r)$, then $f = f_k g = f_{r_1} g_1 + f_{r_2} g_2 + \cdots + f_{r_n} g_n$, where$g, g_i \in C(X)$, $f_k \in B_k$ and $f_{r_i} \in B_{r_i}$, $i = 1, 2, \ldots, n$, and $k \neq r_i$, for all $i = 1, 2, \ldots, n$. But clearly $f_k f_{r_i} = 0$ for every $i = 1, 2, \ldots, n$ implies that $f_k^2 g = 0$, i.e., $f^2 = 0$ and therefore $f = 0$. ■

REMARK 2.3. It is a celebrated question whether the product of two spaces with Souslin property has the Souslin property. This statement is independent of the usual axioms of set theory, see [8]. It is interesting to note that for completely regular spaces this becomes equivalent to an algebraic question, namely if $C(X)$ and $C(Y)$ have countable Goldie dimensions, then is the Goldie dimension of $C(X \times Y)$ countable?

3. Essential ideals in $C(X)$

In this section we characterize essential ideals via a topological property and then using this we note that any ideal containing a prime ideal in $C(X)$ (this is called a *pseudo-prime* ideal, see [4]) is either essential or a maximal ideal generated by an idempotent which is also a minimal prime ideal.

THEOREM 3.1. *If E is a nonzero ideal in $C(X)$, then the following are equivalent.*

(i) E intersects every nonzero z-ideal in $C(X)$ nontrivially.
(ii) E is an essential ideal in $C(X)$.
(iii) $Ann(E) = (0)$.
(iv) $\cap Z[E]$ is a nowhere dense subset of X.

PROOF. (i)⇒(ii) Let J be a nonzero ideal in $C(X)$, then $Z^{-1}[Z[J]] = \{f \in C(X) : Z(f) \in Z[J]\}$ is a z-ideal, therefore $\exists 0 \neq f \in E \cap Z^{-1}[Z[J]]$. Hence for some $g \in J$ we have $Z(f) = Z(g)$. This means that $0 \neq fg \in J \cap E$.

(ii)⇒(iii) It is clear that $\left(Ann(E) \cap E\right)^2 = (0)$, implies that $Ann(E) \cap E = (0)$. Hence $Ann(E) = (0)$.

(iii)⇒(iv) Suppose the interior of $\cap Z[E]$ is nonempty and is denoted by $U = int \cap Z[E]$. Now there exists a nonzero element $f \in C(X)$ such that $f(X - U) = \{0\}$. Thus for every $g \in E$ we have $fg = 0$, i.e., $Ann(E) \neq (0)$, a contradiction.

(iv)⇒(i) Let I be a nonzero ideal and $0 \neq g \in I$, then $X - Z(g)$ is open set and clearly $\left(X - Z(g)\right) \cap \left(X - \cap Z[E]\right) \neq \emptyset$, this implies that there is $f \in E$ such that $\left(X - Z(g)\right) \cap \left(X - Z(f)\right) \neq \emptyset$, therefore $Z(gf) \neq X$, i.e., $0 \neq gf \in E \cap I$. ■

REMARK 3.2. Part (iv) of the previous result is an effective criterion for recognizing the essential ideals in $C(X)$. One can easily see that every free ideal is an essential ideal and a principal ideal (f) is an essential ideal in $C(X)$ if and only if $Z(f)$ is nowhere dense. If $x \in X$ is a nonisolated point, then again by the same criterion we see that O_x is an essential ideal. We also note that if X is an infinite space and V is an open set such that $\overline{V} \neq X$, then $E = \{f \in C(X) : \overline{V} \subseteq Z(f)\}$ is a z-ideal which is not essential.

COROLLARY 3.3. *Every pseudoprime ideal in $C(X)$ is either an essential ideal or a maximal ideal which is at the same time a minimal prime ideal.*

PROOF. Let A be a pseudoprime ideal in $C(X)$ and P be a prime ideal such that $A \supset P$. Then it is sufficient to show that P is either essential or a maximal ideal with the required property. We know that $\cap Z[P]$ is either empty or a singleton. Therefore by our criterion if P is a nonessential ideal, then $\cap Z[P]$ is singleton $\{x\}$ and again by our criterion x must be an isolated point in X. Thus $P \subseteq M_x$ and $M_x = eC(X)$, where e is the idempotent in $C(X)$ such that $e(x) = 0$, $e(X - \{x\}) = \{1\}$. Now $e(1-e) = 0 \in P$ implies that $e \in P$, for $1-e \notin P$ (we note that $1-e \in P$ implies $1-e \in M_x$ which is impossible). This shows that $P = M_x = eC(X)$. Finally to show that P is a minimal prime, let Q be a prime ideal such that $Q \subseteq P$, then $e(1-e) = 0$ implies that $e \in Q$, i.e., $Q = P$. ■

COROLLARY 3.4. *X is finite if and only if $C(X)$ has no proper essential ideals.*

PROOF. If X is finite, then we are through. Conversely each maximal ideal M_x is nonessential ideal and hence x is an isolated point in X by our criterion. Thus X is a discrete space and again by our criterion every proper ideal must be fixed. Hence X is compact which shows that X must be finite. ■

COROLLARY 3.5. *X is a discrete space if and only if the set of essential ideals and the set of free ideals coincide.*

PROOF. Evident by our criterion. ■

Next we give a natural definition of an essential z-filter.

DEFINITION. A z-filter F in a space X is called an *essential z-filter* if $F \cap F' \neq \{X\}$ for every nontrivial z-filter F'. The following result shows that the essential z-filters behave like the z-ultrafilters and prime z-filters.

PROPOSITION 3.6. *(i) If E is an essential ideal in $C(X)$, then $Z[E]$ is an essential z-filter.*

(ii) If F is an essential z-filter, then $Z^{-1}[F] = \{f \in C(X) : Z(f) \in F\}$ is an essential ideal in $C(X)$.

PROOF. (i) Let F be a nontrivial z-filter, then $E \cap Z^{-1}[F] \neq (0)$, i.e., $Z[E] \cap F \neq \{X\}$.

(ii) By our Theorem 3.1, it is sufficient to show that $Z^{-1}[F] \cap I \neq (0)$, where I is a nonzero z-ideal. But this is clear, for $F \cap Z[I] \neq \{X\}$ implies that $\exists f \neq 0$, which $f \in Z^{-1}[F] \cap I$. ■

Next we note that if every essential ideal in $C(X)$ is a z-ideal, then by Corollary 3.3, every prime ideal becomes a z-ideal and therefore by [3], 14 B.3, X is a P-space. But we give a direct proof of this fact.

PROPOSITION 3.7. *X is a P-space if and only if every essential ideal in $C(X)$ is a z-ideal.*

PROOF. Let every essential ideal in $C(X)$ be a z-ideal, then we show that $C(X)$ is a regular ring, see [3], 4j. Now let $f \in C(X)$, then $E = (f^2) \oplus Ann(f^2)$ is an essential ideal and $Z(f) = Z(f^2) \in Z[E]$ implies that $f \in E$. Hence $f = f^2g + h$, where $g \in C(X)$, $h \in Ann(f^2)$. We claim that $h = 0$, for we note that $Z(fh) = Z(f^2h) = X$, i.e., $fh = 0$. Hence $h(f^2g + h) = 0$ and this shows that $h^2 = 0$, i.e., $h = 0$. Thus $f = f^2g$, i.e., $C(X)$ is a regular ring. The converse is well-known. ■

4. Essential ideals and prime ideals in $C(X)$

In Corollary 3.3, we have noted that any nonmaximal prime ideal in $C(X)$ is essential. The following result shows that for an infinite space X, there is always an essential ideal in $C(X)$ which is not a prime ideal.

PROPOSITION 4.1. *If X is an infinite space, there is an essential ideal in $C(X)$ which is not a prime ideal.*

PROOF. We consider two cases.

Case 1. Let X have more than one non-isolated points, say x and y. Now define $E = \{f \in C(X) : \{x, y\} \subseteq Z(f)\}$, then $\cap Z[E] = \{x, y\}$ and therefore by our criterion, E is essential. Now there are elements $f_1, f_2 \in C(X)$ such that $f_1(x) = f_2(y) = 0$ and $f_1(y) = f_2(x) = 1$, then $f_1 f_2 \in E$ but $f_i \notin E$, $i = 1, 2$, i.e., E is not prime ideal.

Case 2. Suppose X has at most one nonisolated point, say $x \in X$, if there exists such a point. Then if S is the socle of $C(X)$ we have $\cap Z[S] = \{x\}$, or $\emptyset$ if x does not exist, see the proof of Corollary 3.6 in [10]. Thus by our criterion S is essential. We claim that S is not a prime ideal, for let $A = \{x_1, x_2, \ldots, x_n, \ldots\}$, $B = \{y_1, y_2, \ldots, y_n, \ldots\}$ be two disjoint infinite open subsets of X such that $x \in A \cup B$. Define $f, g \in C(X)$ by $f(X - A) = \{0\}$, $f(x_k) \neq 0$, $g(X - B) = 0$ and $g(y_k) \neq 0$, $k = 1, 2, \ldots, n, \ldots$. Then $(X - A) \cup (X - B) = X$ implies that $fg = 0$, but $f \notin S$, $g \notin S$, for by [10], Proposition 3.3, $S = \{h \in C(X) : X - Z(h) \text{ is finite}\}$, i.e., S is not a prime ideal. ■

COROLLARY 4.2. *If X is an infinite space, then there is an essential ideal which is a z-ideal but not a prime ideal.* ■

5. Essential ideals and the socle of $C(X)$

The socle S of $C(X)$ is characterized in [10] as the set of all functions which vanish everywhere except on a finite number of points of X. We know that in any commutative ring the socle is the intersection of all essential ideals. In this section we prove a similar topological result, namely theorem 5.3. We recall that if X is any set and $x \in X$, then $F = \{A \subseteq X : x \in A\}$ is an ultrafilter called a *principal filter at* x and if X is an infinite set then the set F consisting of all confinite subsets of X is called *cofinite filter*, or the *Fréchet filter*. It is easy to see that every ultrafilter is either principal or contains the Fréchet filter. If $S \neq 0$ is the socle of $C(X)$, then we call $Z[S]$ the *Fréchet z-filter* (we note that every element of $Z[S]$ is confinite in X) and therefore every z-ultrafilter is either fixed (principal) at some isolated point in X or contains the Fréchet Z-filter, but not both. We need the following lemma:

LEMMA 5.1. *Let H be the set of nonisolated points of X and $f \in \cap_{x\in H} O_x$, then $Z(f)$ is open.*

PROOF. Clearly $H \subseteq int\, Z(f)$ and therefore $Z(f) - int\, Z(f) = G$ consists only of isolated points and is open. Hence $Z(f) = G \cup int\, Z(f)$ is also open. ■

The next result shows that for a compact space X, the socle of $C(X)$ is an intersection of certain essential ideals.

PROPOSITION 5.2. *If S is the socle of $C(X)$ and H is the set of nonisolated points of X, then $S = \cap_{x\in H} O_x$ if and only if every open set in X containing H is finite.*

PROOF. Let $S = \cap_{x\in H} O_x$ and G be an open set containing H. Then G is both open and closed set. Now there exist $f \in C(X)$ such that $f(G) = \{0\}$ and $f(X - G) = \{1\}$. Hence $f \in O_x$, $\forall x \in H$, i.e., $f \in S$ which means that $X - Z(f)$ is finite. Conversely since each O_x, $\forall x \in H$ is essential, we have $S \subseteq \cap_{x\in H} O_x$. Now let $f \in \cap_{x\in H} O_x$, then by the previous Lemma, $Z(f)$ is open and hence by our hypothesis, $X - Z(f)$ is finite, i.e., $f \in S$. ■

The following result shows that a topological space X has isolated points if and only if the intersection of essential z-filters is nontrivial.

THEOREM 5.3. *If S is the socle of $C(X)$, then $Z[S] = \cap_E Z[E]$, where E runs over the set of all essential ideals of $C(X)$.*

PROOF. We note that S is a z-ideal, see [10], and $S = \cap_E E$, where E runs over the set of all essential ideals of $C(X)$, therefore $Z(f) \in Z[S]$ implies that $f \in S = \cap_E E$, i.e., $f \in E$ and $Z(f) \in Z[E]$, for every essential ideal E. Conversely, let $Z(f) \in \cap_E Z[E]$ and put $F = X - Z(f)$. Then if H is the set of nonisolated points of X we have $H \subseteq Z(f)$, for $Z(f) \in \cap_{x\in H} Z[O_x]$. We also note that $f \in \cap_{x\in H} O_x$ for each O_x is a z-ideal. Now by Lemma 5.1, $Z(f)$ is open and therefore

F is both open and closed set and consists only of isolated points. Then define $F_K = \{g \in C(X) : F - Z(g) \text{ is finite}\}$. We observe that every finite subset A of F is both open and closed subset of X, therefore if we define $g_A \in C(X)$ such that $g_A(F - A) = \{0\}$ and $g_A(X - (F - A)) = \{1\}$, then $g_A \in F_K$. Now $\cap_A Z(g_A) = \emptyset$, where A runs over the collection of all finite subsets of F. This means that F_K is a free ideal and hence is an essential ideal in $C(X)$. Hence $Z(f) \in Z[F_K]$, i.e., $F - Z(f)$ is finite. But $F - Z(f) = X - Z(f)$ and this means that $f \in S$. Thus $Z[S] = \cap_E Z[E]$. ■

ACKNOWLEDGEMENT: I would like to thank professor O.A.S. Karamzadeh for his advice on this paper.

REFERENCES

[1] R. ENGELKING, *General Topology*, PWN polish scientific publishers, 1977.

[2] N.J. FINE, L. GILLMAN and J. LAMBEK, *Rings of quotients of rings of functions*, McGill Univ. Press, Montreal, 1966, MP # 635.

[3] L. GILLMAN and M. JERISON, *Rings of continuous functions*, Springer-Verlag, 1976.

[4] L. GILLMAN, Convex and Pseudo prime ideals in $C(X)$, *General topology and its applications*, Proceedings of the 1988 Northeast Conference, New York (1990), 87–95.

[5] K.R. GOODEARL and R.B. WARFIELD, JR., *An introduction to noncommutative Noetherian rings*, Cambridge Univ. Press, 1989.

[6] M. HENRIKSEN AND M. JERISON, The space of minimal prime ideals of commutative rings, *Trans. Amer. Math. Soc.* **115** (1965), 110-130.

[7] R.E. JOHNSON, The extended centralizer of a ring over a module, *Proc. Amer. Math. Soc.* **2** (1951), 891–895.

[8] I. JUHÁSZ, A. VERBEEK and N. S. KROONBERG, *Cardinal functions in topology*, Math. Center Tracts, **34**, Amsterdam, 1971.

[9] O.A.S. KARAMZADEH, On the classical Krull dimension of rings, *Fund. Math.* **117** (1983), 103–108.

[10] O.A.S. KARAMZADEH and M. Rostami, On the intrinsic topology and some related ideals of $C(X)$, *Proc. Amer. Math. Soc.* **93**(1) (1985), 179–184.

[11] J. LAMBEK, *Lectures on rings and modules*, Blaisdell Publishing Company, 1966.

[12] M. MANDELKER, Supports of continuous functions, *Trans. Amer. Math. Soc.* **156** (1971), 73–83.

[13] J.C. MCCONNEL and J. C. ROBSON, *Noncommutative Noetherian rings*, Wiley-Interscience, New York, 1987.

(Received: March 1, 1994)
(In final form: January 12, 1995)

DEPARTMENT OF MATHEMATICS
AHVAZ UNIVERSITY
AHVAZ
IRAN