BOREL SECTIONS OF MANY-VALUED MAPS

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§1. Introduction

1.1. Let T and X be nonempty sets. The many-valued map $\Gamma: T \to 2^X$ will be identified with its graph in $T \times X$,

$$\Gamma(t) = \{x \in X: (t, x) \in \Gamma\} \ \forall t \in T.$$

The map $\varphi: T \to X$ is called a section or a selector of Γ if $\varphi(t) \in \Gamma(t)$ for all $t \in T$.

1.2. Let X be a topological space,* $(T, \mathscr{A})$ be a measurable space. The map $\varphi: T \to X$ is said to be measurable if $\varphi^{-1}(F) \in \mathscr{A}$ for every closed set $F \subset X$. If $(T, \mathscr{A})$ is a topological space with a Borel σ-algebra, then the measurable map $\varphi: T \to X$ is named Borel.

1.3. From the Lusin – Yankov theorem [1, 2] follows the existence of a measurable section when T and X are Borel sets in Polish spaces,† Γ is a Souslin set, $\Gamma(t)$ is nonempty for all $t \in T$, and the σ-algebra $\mathscr{A}$ is generated by Souslin sets. We recall that a continuous image of a Polish space is called a Souslin set (Souslin sets are also called analytic or A-sets). Various variants and generalization of the theorem mentioned are known; certain new results of similar type have been obtained in [3, 4]. However, from these theorems it is not possible to extract the existence of Borel sections.

1.4. In 1939 Novikov [5] proved that if Γ is a Polish Borel set, then the points t for which $\Gamma(t)$ is nonempty and closed form a set that is the complement of a Souslin set (CA-set). Two important corollaries stem from this result.

COROLLARY A [5]. If Γ is a Borel set and all $\Gamma(t)$ are closed, then the projection of Γ onto T is a Borel set.

COROLLARY B [5].‡ We assume that all $\Gamma(t)$ are nonempty and closed. Then the following statements are equivalent: 1) Γ is a Borel set; 2) a denumerable family of Borel sections $\varphi_n: T \to X$, $n = 1, 2, \ldots$, exists such that $\{\varphi_n(t): n = 1, 2, \ldots\}$ is dense in $\Gamma(t)$ for each $t \in T$.

1.5. It is very well known that Novikov's theorem does not carry over to the case when T and X are arbitrary Polish spaces. As a matter of fact, let $T = [0, 1]$, X be a Baer space,** $f: X \to T$ be a continuous map, and $f(X)$ be a Souslin but not Borel set in T. Then $\Gamma = f^{-1}$ is closed in $T \times X$; consequently, all $\Gamma(t)$ are closed in X, but the projection of Γ onto T is not a Borel set.

1.6. Another approach exists to the investigation of measurable (and, in particular, Borel) sections of many-valued maps. For every $B \subset X$ we set

$$\Gamma^{-1}(B) = \{t \in T: \Gamma(t) \cap B \neq \varnothing\}.$$

Rokhlin [7] showed that if $(T, \mathscr{A})$ is a measurable space, X is a complete separable metric space, $\Gamma(t)$ is nonempty and closed for all $t \in T$, and $\Gamma^{-1}(G) \in \mathscr{A}$ for every open set $G \subset X$, then a measurable section exists for Γ. Analogous results were obtained later by Sion [8], Kuratowski and Ryll-Nardzewski [9], and Castaing [10].

1.7. Now let the the σ-algebra $\mathscr{A}$ be complete relative to some σ-finite measure and X be a complete separable metric space with metric ρ and let all $\Gamma(t)$ be nonempty and closed. In this case Castaing [10, 11], using a result of Debreu [12], established the equivalence of the following statements: (a) $\Gamma \in \mathscr{A} \otimes \mathscr{B}$, where $\mathscr{B}$

*The topological spaces being considered in the paper are assumed to be Hausdorff.

†A topological space homeomorphic to a complete separable metric space is called a Polish space.

‡See [6, p. 94] for the proof of this result of Novikov.

**See survey [6], e.g., about Baer space.

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denotes a Borel σ-algebra in X; (b) $\Gamma^{-1}(F)\in\mathscr{A}$ for all closed $F\subset X$; (c) $\Gamma^{-1}(G)\in\mathscr{A}$ for all open $G\in X$; (d) $\Gamma^{-1}(B)\in\mathscr{A}$ for all $B\in\mathscr{B}$; (e) the function $t\to\rho(x, \Gamma(t))=\inf\{\rho(x, y): y\in\Gamma(t)\}$ is measurable for every $x\in X$; (f) Γ possesses a denumerable family of measurable sections $\varphi_n: T\to X$ (n = 1, 2, ...) such that $\{\varphi_n(t): n = 1, 2, \ldots\}$ is dense in $\Gamma(t)$ for all $t\in T$. The condition of denseness of $\mathscr{A}$ is essential here.

1.8. In §2 we obtain a Borel analog of this theorem in the case when T is a Borel set in a Polish space and X is a metric space being the union of a denumerable set of compacta. This result is a corollary of Novikov's theorem and is proved with the help of assertion A.

The main result of the paper is the theorem in §3, stating that if $\Gamma(t)$ are convex metrizable compacta in a locally convex space, contained in set X, while Γ is a Borel set in $T\times X$, then under certain conditions on X the set $\{(t, x): x$ is an extreme point of $\Gamma(t)\}$ is Borel too.

§2. Borel Sections

2.1. We shall say that a topological space X enters into class σMK if it is representable in the form $X=\bigcup_{n=1}^{\infty} K_n$, where all K_n are metrizable compacta. It is easy to see that $X\in\sigma MK$ if and only if X is a continuous image of a closed subset of the real line. (As a matter of fact, the sufficiency of this condition is obvious, while the necessity follows from the fact that every metrizable compactum is a continuous image of a Cantor set.) Into class σMK enter, in particular, separable metrizable locally compact spaces and weak adjoints to separable metrizable locally convex spaces.

2.2. LEMMA (on Projection). Let T be a Borel set in a Polish space and let $X\in\sigma MK$. If a set $\Gamma\subset T\times X$ is Borel and $\Gamma(t)$ is closed for every $t\in T$, then the projection $\pi(\Gamma)$ of set Γ onto T is a Borel set.

Proof. We have $X = f(X_1)$, where X_1 is a closed subset of the real line and the map $f: X_1\to X$ is continuous. Further, T is Borel isomorphic to some Borel set $T_1\subset[0, 1]$, i.e., $T = g(T_1)$, where $g: T_1\to T$ is a one-to-one Borel map, and the map g^{-1} is Borel too (see §37 in [13]). Then $g\otimes f: T_1\times X_1\to T\times X$ is a Borel map; consequently, $\Gamma_1=(g\otimes f)^{-1}(\Gamma)$ is a Borel set in $T_1\times X_1$. In addition,

$$\Gamma_1(t_1)=f^{-1}[\Gamma(f(t_1))] \quad \forall t_1\in T_1,$$

and from the continuity of f it follows that all the sets $\Gamma_1(t_1)$, $t_1\in T_1$, are closed in X_1. According to statement A the projection $\pi_1(\Gamma_1)$ of set Γ_1 onto T_1 is a Borel set, but $\pi(\Gamma) = g(\pi_1(\Gamma_1))$. This completes the proof since g is a Borel isomorphism.

2.3. Proposition. Let T be a Borel set in a Polish space, X be a metric space with metric ρ, being the union of a denumerable set of compacta, and all $\Gamma(t)$ be nonempty and closed. Then the following statements are equivalent:

(a) Γ is a Borel set;

(b) $\Gamma^{-1}(F)$ is a Borel set for every closed $F\subset X$;

(c) $\Gamma^{-1}(K)$ is a Borel set for every compactum $K\subset X$;

(d) $\Gamma^{-1}(G)$ is a Borel set for every open $G\subset X$;

(e) $t\to\rho(x, \Gamma(t))=\inf\{\rho(x, y): y\in\Gamma(t)\}$ is a Borel function on T for every $x\in X$;

(f) $(t, x)\to\rho(x, \Gamma(t))$ is a Borel function on $T\times X$;

(g) there exists a denumerable family $\varphi_n: T\to X$ (n = 1, 2, ...) of Borel sections of Γ such that $\{\varphi_n(t): n = 1, 2, \ldots\}$ is dense in $\Gamma(t)$ for all $t\in T$.

For $X = \mathbf{R}^n$ this proposition was proved as Theorem 1.6 in [14].

Proof. (a) ⇒ (b). $\Gamma^{-1}(F)$ is a projection onto T of the set $\Gamma'=(T\times F)\cap\Gamma$ and, since $\Gamma'(t)=F\cap\Gamma(t)$ is closed, it remains to apply the lemma.

(b) ⇒ (c). Obvious.

(c) ⇒ (d). This follows from the fact that an open set in X is representable as a union of a denumerable set of compacta and

$$\Gamma^{-1}\left(\bigcup_{k=1}^{\infty} B_k\right)=\bigcup_{k=1}^{\infty}\Gamma^{-1}(B_k).$$

(d) ⇒ (e). We have $\{t : \rho(x, \Gamma(t)) < \alpha\} = \Gamma^{-1}(U(x, \alpha))$ for every $\alpha > 0$, where $U(x, \alpha)$ is an open ball of radius α with center at x.

(e) ⇒ (f). This follows from the equality

$$\rho(x, \Gamma(t)) = \inf_n (\rho(x, x_n) + \rho(x_n, \Gamma(t))),$$

where the sequence $\{x_n\}$ is everywhere dense in X.

(f) ⇒ (a). We have $\Gamma = \{(t, x) : \rho(x, \Gamma(t)) = 0\}$.

Thus, we have verified the equivalence of statements (a)-(f).

(g) ⇒ (d). This follows from the obvious equality $\Gamma^{-1}(G) = \bigcup_{n=1}^{\infty} \varphi_n^{-1}(G)$.

(a) ⇒ (g). Let $X = \bigcup_{i=1}^{\infty} K_i$, where the K_i are compacta. We set $\Gamma_i = \Gamma \cap (T \times K_i)$ and by T_i we denote the projection of Γ_i onto T. We have $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i$, $T = \bigcup_{i=1}^{\infty} T_i$. Further, $T_i = \Gamma^{-1}(K_i)$, consequently, T_i is a Borel set. We consider Γ_i as the many-valued map $T_i \to 2^{K_i}$. We wish to show that statement (g) is valid for Γ_i. Let us assume that this has already been proved, i.e., there exists a sequence $\varphi_{in} : T_i \to K_i$ ($n = 1, 2, \ldots$) of Borel sections of Γ_i such that $\{\varphi_{in}(t) : n = 1, 2, \ldots\}$ is dense in $\Gamma_i(t)$ for all $t \in T_i$. Then the family of maps

$$\varphi_n(t) = \begin{cases} \varphi_{1n}(t) & \text{for } t \in T_1, \\ \varphi_{2n}(t) & \text{for } t \in T_2 \setminus T_1, \\ \ldots\ldots\ldots & \\ \varphi_{in}(t) & \text{for } t \in T_i \setminus (T_1 \cup \ldots \cup T_{i-1}), \\ \ldots\ldots\ldots\ldots & \end{cases}$$

satisfies the conditions of statement (g) for Γ. Thus, it remains to verify that (g) holds for Γ_i, but this follows easily from the cited result of Rokhlin. The proposition has been proved completely.

2.4. Remarks. 1°. The implication (a) ⇒ (g) holds for an arbitrary $X \in \sigma MK$. As a matter of fact, if $X = \bigcup_{i=1}^{\infty} K_i$, where the K_i are metrizable compacts, then $\Gamma_i = \Gamma \cap (T \times K_i)$ and its projection T_i onto T is a Borel set and everything is derivable from the proposition as applied to the map $\Gamma_i : T_i \to 2^{K_i}$. We need this fact later on.

2°. From the statements in the proposition it does not follow that $\Gamma^{-1}(B)$ is a Borel set for every Borel $B \subset X$. We present a counterexample. Let $X = T \times Y$, where $T = Y = [0, 1]$, B is a Borel set (of type G_δ) in X, but its projection $\pi(B)$ onto T is not a Borel set. The set $\Gamma = \{(t, x) : x = (t, y),\ t \in T,\ y \in Y\}$ is Borel and $\Gamma(t) = \{t\} \times Y$ is closed in X for every $t \in T$. At the same time, $\Gamma^{-1}(B) = \{t : (t, y) \in B \text{ for some } y \in Y\} = \pi(B)$.

§3. Maps with Values in Convex Compacta and Extreme Points

3.1. Let X be a subset of a locally convex space (LCS) $\mathfrak{X}$, provided with an induced topology. We shall examine sets X having the following properties.

1°. $X \in \sigma MK$, and there exists a sequence of metrizable compacta K_i such that $X = \bigcup_{i=1}^{\infty} K_i$; every convex metrizable compactum in X is contained in one of the K_i.

2°. On X there exists a Borel function p whose restriction $p|K$ on every convex metrizable compactum $K \subset X$ is a continuous strictly convex function on K.

Examples of such X are a metrizable compactum in an arbitrary LCS $\mathfrak{X}$, as well as the whole space $\mathfrak{X}$, when $\mathfrak{X}$ is a weak adjoint to a separable metrizable LCS E. In the first case

$$p(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{\langle x, x_n' \rangle^2}{1 + q(x_n')},$$

where the sequence $\{x_n'\} \subset \mathfrak{X}^*$ separates the points of X, $q(x') = \max\{|\langle x, x' \rangle| : x \in X\}$. In the second case we can take

$$p(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{\langle e_n, x \rangle^2}{1 + q_n(e_n)},$$

where the sequence $\{e_n\}$ is dense in E, $q_1 \le q_2 \le \ldots$ is a sequence of prenorms defining the topology in E.

3.2. By ex K we denote the set of extreme points of a convex compactum K.

THEOREM (on Extreme Points). Let T be a Borel set in a Polish space, X be a set in a LCS $\mathfrak{X}$, possessing properties 1° and 2°, Γ be a Borel set in T × X, and Γ(t) is a nonempty convex metrizable compactum for every $t\in T$. Then the set $\Gamma_{ex}=\{(t,x)\in T\times X: x\in \mathrm{ex}\,\Gamma(t)\}$ is Borel too.

This theorem generalizes Theorem 1.7 in [14] in which $X=\mathfrak{X}=\mathbf{R}^n$. Closely related questions were studied in [15, 16].

3.3. Proof. We consider a linear space $\mathscr{E}$ of real functions ξ on X, whose restrictions $\xi|K_i$ are continuous functions on K_i. In $\mathscr{E}$ we introduce the topology of uniform convergence on compacta K_i (i = 1, 2, ...), defined by the sequence of prenorms $\|\xi\|_i=\max\{|\xi(x)|:x\in K_i\}$. It is easy to see that $\mathscr{E}$ is complete. Let us show that it is separable. For each i there exists a denumerable family Φ_i of continuous functions on K_i, everywhere dense in $C(K_i)$. From the Titze – Uryson lemma it follows that every function from Φ_i can be continued up to some function $\xi\in\mathscr{E}$. The union over i of all thus-obtained denumerable families of functions on X is everywhere dense in $\mathscr{E}$. Thus $\mathscr{E}$ is a Polish space.

As is well known (see [17], for example), for every metrizable convex compactum K

$$\mathrm{ex}\,K=\{x\in K: p_K(x)=\hat{p}_K(x)\},$$

where p_K is an arbitrary continuous strictly convex function on K, $\hat{p}_K(x)=\inf\{h(x):h\in H(K)\}$; $H(K)$ is a set of continuous affine functions on K, majorizing p_K. From the Titze – Uryson lemma and property 1° of space X it follows easily that every continuous function on K can be continued up to some function $\xi\in\mathscr{E}$. Therefore,

$$\mathrm{ex}\,\Gamma(t)=\{x\in\Gamma(t) : p(x)=\hat{p}(t, x)\} \quad \forall t\in T,$$

where $\hat{p}(t, x)=\inf\{\xi(x):\xi\in\mathscr{H}(t)\}$ $\forall (t, x)\in\Gamma$, $\mathscr{H}=\{(t, \xi):\xi|\Gamma(t)$ is an affine function and $\xi(x)\geqslant p(x)\ \forall x\in \Gamma(t)\}\subset T\times\mathscr{E}$.

Let $\{\varphi_n\}$ be a denumerable family of Borel sections of Γ and let $\{\varphi_n(t): n = 1, 2, \ldots\}$ be dense in Γ(t) for all $t\in T$ (see Remark 1° in §2). For any positive integers m and n and for any rational number r, 0 ≤ r ≤ 1, we define the set

$$\mathscr{H}_{mnr}=\{(t, \xi)\in T\times\mathscr{E}:\xi(r\varphi_m(t)+(1-r)\varphi_n(t))=r\xi(\varphi_m(t))+(1-r)\xi(\varphi_n(t)),\ \xi(\varphi_n(t))\geqslant p(\varphi_n(t))\}.$$

It is easy to see that $\mathscr{H}=\bigcap_{m,n,r}\mathscr{H}_{mnr}$. Let us show that $\mathscr{H}_{mnr}$ is a Borel set in $T\times\mathscr{E}$. It is evident that for this it is sufficient to verify that $F(t, \xi) = \xi(\varphi(t))$ is a Borel function on $T\times\mathscr{E}$ for any Borel map $\varphi: T\to X$. We have $F = F_2\circ F_1$, where the maps $F_1: T\times\mathscr{E}\to X\times\mathscr{E}$ and $F_2: X\times\mathscr{E}\to\mathbf{R}$ are given by the equalities $F_1(t, \xi) = (\varphi(t), \xi)$ and $F_2(x, \xi) = \xi(x)$. The map F_1 is obviously Borel. Further, the restriction of F_2 on set $K_i\times\mathscr{E}$ is continuous. As a matter of fact, if $(x_k, \xi_k)\subset K_i\times\mathscr{E}$ and $(x_k, \xi_k)\to(x, \xi)$, then $\xi(x_k)\to\xi(x)$, since $\xi|K_i$ is continuous and, consequently,

$$|\xi_k(x_k)-\xi(x)| \leqslant |\xi_k(x_k)-\xi(x_k)|+|\xi(x_k)-\xi(x)| \leqslant \|\xi_k-\xi\|_i+|\xi(x_k)-\xi(x)|\to 0$$

as $k\to\infty$. Then F_2 is a Borel function on $X\times\mathscr{E}$, and F is a Borel function on $T\times\mathscr{E}$. Thus, $\mathscr{H}_{mnr}$ is a Borel set and, consequently, $\mathscr{H}$ is a Borel set as well.

The set

$$\mathscr{L}=\{(t, x, \xi):x\in\Gamma(t),\ \xi\in\mathscr{H}(t)\}\subset T\times X\times\mathscr{E}$$

is Borel since it is homeomorphic to the Borel set

$$\{(t_1, x, t_2, \xi)\in\Gamma\times\mathscr{H}:t_1=t_2\}\subset(T\times X)\times(T\times\mathscr{E}).$$

The sets

$$\mathscr{L}_n=\mathscr{L}\cap\{(t, x, \xi) : \xi(x)-p(x)\leqslant 1/n\} \quad (n=1, 2, \ldots)$$

are Borel too; consequently, their projections $\varkappa(\mathscr{L}_n)$ onto T × X are Souslin sets.* Then Γ_{ex} too is a Souslin set since $\Gamma_{ex}=\bigcap_{n=1}^{\infty}\varkappa(\mathscr{L}_n)$. On the other hand, $\Gamma\setminus\Gamma_{ex}$ is a Souslin set, being the image of the Borel set

$$\mathscr{M}=\{(t, x, y):x\in\Gamma(t),\ y\in\Gamma(t),\ x\neq y\}\subset T\times X\times X$$

relative to the continuous map $(t, x, y)\to(t, (x+y)/2)$.

*This follows from the fact that $X\in\sigma MK$ and from the corresponding result for Polish spaces [13].

Thus, Γ_{ex} and its complement $\Gamma \setminus \Gamma_{ex}$ are Souslin sets. We set $\Gamma_i = \Gamma \cap (T \times K_i)$. Then $\Gamma_i \cap \Gamma_{ex}$ and $\Gamma_i \setminus \Gamma_{ex} = \Gamma_i \cap (\Gamma \setminus \Gamma_{ex})$ are Souslin sets without points in common and, since their union Γ_i is a Borel set in a Polish space, $\Gamma_i \cap \Gamma_{ex}$ is a Borel set. Since this is true for each i, the set $\Gamma_{ex} = \bigcup_{i=1}^{\infty} (\Gamma_i \cap \Gamma_{ex})$ also is Borel.

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