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SEMI-NORMAL SPACES AND SOME FUNCTIONS

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

1. Introduction

Arya and Bhamini [1] and Dorsett [7] have introduced the notion of semi-normal spaces by using semi-open sets due to Levine [9]. Recently, in [2], the concept of semi-generalized open sets has been introduced as a generalization of semi-open sets. In the present paper, we obtain further characterizations of semi-normal spaces by using semi-generalized open sets. Moreover, in order to obtain preservation theorems of semi-normal spaces, we introduce the concepts of pre sg-continuous functions and pre sg-closed functions.

2. Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let X be a space and A a subset of X. We denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *semi-open* [9] if there exists an open set U of X such that $U \subset A \subset \mathrm{Cl}(U)$. The complement of a semi-open set is said to be *semi-closed*. The family of all semi-open (resp. semi-closed) sets of X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{SC}(X)$). The intersection of all semi-closed sets containing A is called the *semi-closure* of A [3] and is denoted by $\mathrm{sCl}(A)$. The *semi-interior* of A, denoted by $\mathrm{sInt}(A)$, is defined to be the union of all semi-open sets contained in A.

DEFINITION 1. A subset A of a space X is said to be *semi-generalized closed* (briefly sg-closed) [2] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathrm{SO}(X)$.

Every semi-closed set is sg-closed but the converse is false [2, Example 3]. The complement of a sg-closed set is said to be *semi-generalized open* (briefly sg-open) [2]. A subset A is sg-open if and only if $F \subset \mathrm{sInt}(A)$ whenever $F \in \mathrm{SC}(X)$ and $F \subset A$ [2, Theorem 6].

DEFINITION 2. A function $:X \to Y$ is said to be *semi-continuous* [9] (resp. *irresolute* [4]) if $f^{-1}(V) \in \mathrm{SO}(X)$ for every open set V of Y (resp. $V \in \mathrm{SO}(Y)$).

It is obvious that semi-continuity is implied by both continuity and irresoluteness.

DEFINITION 3. A function $f: X \to Y$ is said to be *semi-closed* [10] (resp. *presemiclosed* [11]) if $f(F) \in \mathrm{SC}(Y)$ for every closed set F of X (resp. $F \in \in \mathrm{SC}(X)$).

DEFINITION 4. A function $f: X \to Y$ is said to be *sg-continuous* [12] (resp. *sg-irresolute* [12]) if $f^{-1}(F)$ is sg-closed in X for every closed (resp. sg-closed) set F of Y.

It was shown that semi-continuity implies sg-continuity but the converse is false [12, Example 3.4].

DEFINITION 5. A space X is said to be *semi-normal* [7] if for each pair of disjoint semi-closed sets A and B, there exist disjoint $U, V \in \mathrm{SO}(X)$ such that $A \subset U$ and $B \subset V$.

In [1], Arya and Bhamini called semi-normal spaces *s-normal*. However, in this paper, we shall use the term "*semi-normal*" in the sequel.

DEFINITION 6. A space X is said to be *semi-*$T_{\frac{1}{2}}$ [2] if every sg-closed set of X is semi-closed in X.

3. Semi-normal spaces

We shall obtain the further characterizations of semi-normal spaces by using sg-open sets and sg-closed sets.

THEOREM 1. *The following properties are equivalent for a space* X:

(a) X *is semi-normal;*

(b) *for each pair of disjoint* $A, B \in \mathrm{SC}(X)$, *there exists disjoint* sg-*open sets* U *and* V *such that* $A \subset U$ *and* $B \subset V$;

(c) *for each* $A \in \mathrm{SC}(X)$ *and each* $U \in \mathrm{SO}(X)$ *containing* A, *there exists a* sg-*open set* G *such that* $A \subset G \subset \mathrm{sCl}(G) \subset U$;

(d) *for each* $A \in \mathrm{SC}(X)$ *and each sg-open set* U *containing* A, *there exists* $G \in \mathrm{SO}(X)$ *such that* $A \subset G \subset \mathrm{sCl}(G) \subset \mathrm{sInt}(U)$;

(e) *for each* sg-*closed set* A *and each* $U \in \mathrm{SO}(X)$ *containing* A, *there exists* $G \in \mathrm{SO}(X)$ *such that* $A \subset \mathrm{sCl}(A) \subset G \subset \mathrm{sCl}(G) \subset U$;

(f) *for each* $A \in \mathrm{SC}(X)$ *and each* $U \in \mathrm{SO}(X)$ *containing* A, *there exists* $G \in \mathrm{SO}(X) \cap \mathrm{SC}(X)$ *such that* $A \subset G \subset U$.

PROOF. (a) $\Rightarrow$ (b). This is obvious since every semi-open set is sg-open.

(b) $\Rightarrow$ (c). Let $A \in \mathrm{SC}(X)$ and $U \in \mathrm{SO}(X)$ containing A. Then $A \cap \cap (X - U) = \emptyset$ and $X - U \in \mathrm{SC}(X)$. There exist sg-open sets G and V such that $A \subset G$, $X - U \subset V$, and $G \cap V = \emptyset$. Therefore, we have $A \subset G \subset X - - V \subset U$ and hence $\mathrm{sCl}(G) \subset \mathrm{sCl}(X - V) \subset U$ since $X - V$ is sg-closed and $U \in \mathrm{SO}(X)$. Consequently, we obtain $A \subset G \subset \mathrm{sCl}(G) \subset U$.

(c) $\Rightarrow$ (d). Let $A \in \mathrm{SC}(X)$ and U be a sg-open set containing A. We have $A \subset \mathrm{sInt}(U)$ [2, Theorem 6] and $\mathrm{sInt}(U) \in \mathrm{SO}(X)$. There exists a sg-open set V such that $A \subset V \subset \mathrm{sCl}(V) \subset \mathrm{sInt}(U)$. Put $G = \mathrm{sInt}(V)$, then we obtain $G \in \mathrm{SO}(X)$ and $A \subset G \subset \mathrm{sCl}(G) \subset \mathrm{sInt}(U)$.

(d) $\Rightarrow$ (e). Let A be any sg-closed set and $U \in \mathrm{SO}(X)$ containing A. Then, we have $\mathrm{sCl}(A) \subset U$ and $\mathrm{sCl}(A) \in \mathrm{SC}(X)$. Since every semi-open set is sg-open, there exists $G \in \mathrm{SO}(X)$ such that $A \subset \mathrm{sCl}(A) \subset G \subset \mathrm{sCl}(G) \subset$ $\subset U$.

(e) $\Rightarrow$ (f). Let $A \in \mathrm{SC}(X)$ and $U \in \mathrm{SO}(X)$ containing A. There exists $V \in \mathrm{SO}(X)$ such that $A \subset V \subset \mathrm{sCl}(V) \subset U$. Put $G = \mathrm{sCl}(V)$, then G is semi-open and semi-closed [6, Proposition 2.2] and $A \subset G \subset U$.

(f) $\Rightarrow$ (a). Let A and B be any pair of disjoint semi-closed sets. Then, we have $A \subset X - B \in \mathrm{SO}(X)$ and there exists $U \in \mathrm{SO}(X) \cap \mathrm{SC}(X)$ such that $A \subset U \subset X - B$. Now, put $V = X - U$, then we obtain $A \subset U$, $B \subset$ $\subset V \in \mathrm{SO}(X)$, and $U \cap V = \emptyset$. This shows that X is semi-normal.

4. Pre sg-continuous functions

In this section we introduce a new class of functions called pre sg-continuous functions.

DEFINITION 7. A function $f: X \to Y$ is said to be *pre sg-continuous* if $f^{-1}(F)$ is *sg-closed* in X for every $F \in \mathrm{SC}(Y)$.

It is obvious that $f: X \to Y$ is pre sg-continuous if and only if $f^{-1}(V)$ is sg-open in X for every $V \in \mathrm{SO}(Y)$. From Definitions 2, 4 and 7, for the properties of a function we obtain the following relations.

$$\begin{array}{ccccc} & & & & sg\text{-irresoluteness} \\ & & & & \Downarrow \\ & & \text{irresoluteness} & \Longrightarrow & \text{pre } sg\text{-continuity} \\ & & \Downarrow & & \Downarrow \\ \text{continuity} & \Longrightarrow & \text{semi-continuity} & \Longrightarrow & sg\text{-continuity} \end{array}$$

Diagram I

REMARK 1. By the three examples stated below we obtain the following properties:

(a) none of the implications in Diagram I are reversible;

(b) sg-irresoluteness, irresoluteness, and continuity are pairwise independent.

(c) pre sg-continuity and continuity are independent of each other;

(d) pre sg-continuity and semi-continuity are independent of each other.

EXAMPLE 1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and f: $:(X, \tau) \to (X, \tau)$ be a function defined as follows: $f(a) = f(b) = a$ and

$f(c) = c$. Then f is continuous but it is not pre sg-continuous since $\{a\} \in$ $\in \mathrm{SC}(X,\tau)$ and $f^{-1}(\{a\}) = \{a,b\}$ is not sg-closed in (X,τ).

EXAMPLE 2. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma =$ $= \{\emptyset, X, \{a\}, \{b,c\}\}$. Let $f:(X,\tau) \to (X,\sigma)$ be the identity function. Then f is irresolute but it is neither sg-irresolute nor continuous. There exists a sg-closed set $\{a,b\}$ in (X,σ) such that $f^{-1}(\{a,b\})$ is not sg-closed in (X,τ).

EXAMPLE 3. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$, and $\sigma = \{\emptyset, X,$ $\{a\}, \{a,b\}, \{a,c\}\}$. Let $f:(X,\tau) \to (X,\sigma)$ be the identity function. Then f is sg-irresolute but it is not semi-continuous since $f^{-1}(\{a,c\}) \notin \mathrm{SO}(X,\tau)$.

THEOREM 2. *If a function $f: X \to Y$ is pre* sg*-continuous and presemiclosed, then f is* sg*-irresolute.*

PROOF. Let K be any sg-closed set of Y and $U \in \mathrm{SO}(X)$ containing $f^{-1}(K)$. Since f is presemiclosed, it follows from [8, Theorem 3.5] that there exists $V \in \mathrm{SO}(Y)$ such that $K \subset V$ and $f^{-1}(V) \subset U$. Since K is sg-closed in Y, we have $\mathrm{sCl}(K) \subset V$ and hence $f^{-1}(\mathrm{sCl}(V)) \subset f^{-1}(V) \subset$ $\subset U$. Since f is pre sg-continuous, $f^{-1}(\mathrm{sCl}(V))$ is sg-closed in X and hence $\mathrm{sCl}(f^{-1}(K)) \subset \mathrm{sCl}\Big(f^{-1}(\mathrm{sCl}(V))\Big) \subset U$. This shows that $f^{-1}(K)$ is sg-closed in X. Therefore, f is sg-irresolute.

The following two corollaries are immediate consequences of Theorem 2.

COROLLARY 1 (Sundaram et al. [12]). *Every irresolute presemiclosed function is* sg*-irresolute.*

COROLLARY 2 (Sundaram et al. [12]). *Semi-$T_{\frac{1}{2}}$ spaces are preserved under irresolute presemiclosed surjections.*

PROPOSITION 1. *Let X be a semi-$T_{\frac{1}{2}}$ space. A function $f: X \to Y$ is pre* sg*-continuous if and only is f is irresolute.*

PROOF. Suppose that f is pre sg-continuous. Let K be any semi-closed set of Y. Then $f^{-1}(K)$ is sg-closed in X and hence $f^{-1}(K) \in \mathrm{SC}(X)$ since X is semi-$T_{\frac{1}{2}}$. Therefore, it follows from [4, Theorem 1.4] that f is irresolute. The converse is obvious.

COROLLARY 3 (Sundaram et al. [12]). *If $f: X \to Y$ is* sg*-irresolute and X is semi-$T_{\frac{1}{2}}$, then f is irresolute.*

THEOREM 3. *If $f: X \to Y$ is a pre* sg*-continuous presemiclosed injection and Y is a semi-normal space, then X is semi-normal.*

PROOF. Let A and B be any disjoint semi-closed sets of X. Since f is a presemiclosed injection, $f(A)$ and $f(B)$ are disjoint semi-closed sets of Y. By the semi-normality of Y, there exist disjoint $U, V \in \mathrm{SO}(Y)$ such that

$f(A) \subset U$ and $f(B) \subset V$. Since f is pre sg-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint sg-open sets containing A and B, respectively. It follows from Theorem 1 that X is semi-normal.

COROLLARY 4 (Arya and Bhamini [1]). *The inverse image of a semi-normal space under an irresolute presemiclosed injection is semi-normal.*

5. Pre sg-closed functions

In this section we introduce a new class of functions called pre sg-closed functions

DEFINITION 8. A function $f: X \to Y$ is said to be *pre* sg-*closed* (resp. *sg-closed* [5]) if $f(F)$ is sg-closed in Y for every semi-closed (resp. closed) set F of X.

By definition 3 and 8, we easily obtain the following diagram:

$$\begin{array}{ccccc} & & \text{presemiclosed} & \Longrightarrow & \text{pre } sg\text{-closed} \\ & & \Downarrow & & \Downarrow \\ \text{closed} & \Longrightarrow & \text{semi-closed} & \Longrightarrow & sg\text{-closed} \end{array}$$

Diagram II

REMARK 2. By the two examples stated below, we obtain the following properties:

(a) none of the implications in Diagram II are reversible;
(b) a continuous closed open surjection need not be pre sg-closed;
(c) closedness and pre sg-closedness are independent of each other;
(d) semi-closedness and pre sg-closedness are independent of each other.

EXAMPLE 4. Let $f: (X, \tau) \to (X, \sigma)$ be the same function as in Example 2. Then f is pre sg-closed but it is not semi-closed. Moreover, f^{-1} is presemiclosed but it is not closed.

EXAMPLE 5. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}\}$, $Y = \{a, b, c\}$, and $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = f(d) = a$, $f(b) = b$, and $f(c) = c$. Then f is a continuous closed open surjection. However, f is not pre sg-closed since $\{a\} \in \mathrm{SC}(X, \tau)$ and $f(\{a\})$ is not sg-closed in (Y, σ).

PROPOSITION 2. *If $f: X \to Y$ is an irresolute pre* sg-*closed function and A is a* sg-*closed set of X, then $f(A)$ is* sg-*closed in Y.*

PROOF. Let A be a sg-closed set of X and $V \in \mathrm{SO}(Y)$ containing $f(A)$. Since f is irresolute, we have $A \subset f^{-1}(V) \in \mathrm{SO}(X)$ and hence $\mathrm{sCl}(A) \subset f^{-1}(V)$. Since f is pre sg-closed and $\mathrm{sCl}(A) \in \mathrm{SC}(X)$, $f(\mathrm{sCl}(A))$ is

sg-closed in Y and $f(\mathrm{sCl}(A)) \subset V$. Therefore, we obtain $\mathrm{sCl}(f(A)) \subset \subset \mathrm{sCl}\Big(f(\mathrm{sCl}(A))\Big) \subset V$. This shows that $f(A)$ is sg-closed in Y.

PROPOSITION 3. *A surjective function $f\colon X \to Y$ is pre* sg*-closed if and only if for each subset B of Y and each $U \in \mathrm{SO}(X)$ containing $f^{-1}(B)$, there exists a* sg*-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

PROOF. *Necessity*. Suppose that f is pre sg-closed. Let B be any subset of Y and $U \in \mathrm{SO}(X)$ containing $f^{-1}(B)$. Put $V = Y - f(X - U)$. Then, V is sg-open in $Y, B \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any semi-closed set of X. Put $B = Y - f(F)$, then we have $f^{-1}(B) \subset X - F \in \mathrm{SO}(X)$. There exists a sg-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we obtain $f(F) = Y - V$ and hence $f(F)$ is sg-closed in Y. This shows that f is pre sg-closed.

In Example 5, (X, τ) is semi-normal, (Y, σ) is not semi-normal, and $f\colon (X, \tau) \to (Y, \sigma)$ is a closed irresolute surjection. Therefore, semi-normality is not preserved under closed irresolute surjections.

THEOREM4. *If $f\colon X \to Y$ is a pre* sg*-closed irresolute surjection and X is a semi-normal space, then Y is semi-normal.*

PROOF. Let F and K be any pair of disjoint semi-closed sets of Y. Since f is irresolute, $f^{-1}(F)$ and $f^{-1}(K)$ are disjoint semi-closed sets of X. By the semi-normality of X, there exist $U, V \in \mathrm{SO}(X)$ such that $f^{-1}(F) \subset U$, $f^{-1}(K) \subset V$, and $U \cap V = \emptyset$. By Proposition 3, there exist sg-open sets G and H such that $F \subset G$, $K \subset H$, $f^{-1}(G) \subset U$, and $f^{-1}(H) \subset V$. Since f is surjective and $U \cap V = \emptyset$, we have $G \cap H = \emptyset$. It follows from Theorem 1 that Y is semi-normal.

COROLLARY 5 (Arya and Bhamini [1]). *Semi-normality is preserved under presemiclosed irresolute surjections.*

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