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A Propositional Logic with Explicit Fixed Points

Abstract. This paper studies a propositional logic which is obtained by interpreting implication as formal provability. It is also the logic of finite irreflexive Kripke Models.

A Kripke Model completeness theorem is given and several completeness theorems for interpretations into Provability Logic and Peano Arithmetic.

0.0 The strange tale of the Formalist who lost Modus Ponens

Imagine a formalist. He has become convinced that any philosophy of mathematics worth being taken seriously, must explain the meaning of the logical constants. Yet he clings to the tenets of formalism. He sets out to produce a hybrid: formal semantics.

Consider the translation $*$ from $\mathscr{L}$, the language of Propositional Logic (PL), to $\mathscr{L}_\Box$, the language of Modal Propositional Logic (MPL). It is given by:

$$p_i \mapsto^* \Box p_i$$
$$\bot \mapsto^* \Box\bot$$
$$(A \wedge B) \mapsto^* (A^* \wedge B^*)$$
$$(A \vee B) \mapsto^* (A^* \vee B^*)$$
$$(A \to B) \mapsto^* \Box(A^* \to B^*).$$

Our formalist stipulates: the formalistic meaning of C is to be the classical meaning of C^*, where $\Box$ is interpreted as provability in some fixed formal system. For definiteness he restricts his attention to Peano Arithmetic (PA). An interpretation of $\mathscr{L}$ into $\mathscr{L}_{PA}$ is just $f_\bigcirc{}^*$, where f is some interpretation for Provability Logic (PrL) of $\mathscr{L}_\Box$ into $\mathscr{L}_{PA}$ (for more details see §5). Formal Validity will be defined as: A is *formally valid* iff for every interpretation g $PA \vdash A^g$. The formalist proceeds to formalize. He looks for a logic — even before conception he calls it: Formal Propositional Logic (FPL) — satisfying: $\vdash_{FPL} A$ iff A is formally valid. By Solovay's Completeness Theorem for Provability Logic (PrL), this is equivalent to: $\vdash_{FPL} A$ iff $\vdash_{Prl} A^*$.

The logic he finds turns out to have Full Explicit Fixed Points, i.e. for any $A(p, \vec{q})$ there is a $B(\vec{q})$ s.t. $B(\vec{q}) \dashv\vdash_{FPL} A(B(\vec{q}), \vec{q})$.

For this luxury however a price has to be paid. The Fixed Point Theorem readily yields an explicit Liar Sentence. Yet $\bot$ should not be derivable. So one of the steps in the usual derivation of $\bot$ must be blocked. It turns out to be Modus Ponens. The reason is that PA non$\vdash$ $Prov \ulcorner \underline{A} \urcorner \to A$

in general, e.g. PA *non*$\vdash$ $Prov\ulcorner \underline{0} = \underline{1}\urcorner \rightarrow \underline{0} = \underline{1}$. This, of course, is part of Gödel's Second Incompleteness Theorem.

After some soulsearching our formalist decides to follow the tortoise and repudiate Modus Ponens, for — he argues — what can one expect, to have a formal proof that $A^* \rightarrow B^*$, is no evidence whatsoever that if A^* is true, then B^* is true.

Only lately he is having some trouble expressing the thought ....

0.1 Motivation

A more serious reason than speculation about strange formalists to be interested in logics like *FPL* is the study of notions like *Prov*, formal provability as coded in *PA*, which naturally occur in the metamathematics of formal theories.

0.2 Contents

In the following I will provide a formalization of *FPL* and prove the associated completeness theorem.

The notion of interpretation described in 0.0 is not the only one which yields *FPL* and possibly not even the most interesting one. *FPL* turns out to be also the logic of Σ_1^0-sentences of *PA* by interpreting the atoms as Σ_1^0-sentences, $\perp$ as $0 = 1$ and treating conjunction, disjunction and implication as before.

§1 contains the Kripke Model theoretic preliminaries for the development of *FPL*. §2 gives *FPL* plus associated Kripke Model completeness theorem. §3 has the basic facts about *FPL*. §4 studies an extension of *FPL* which is complete for a certain infinite matrix. §5 finally gives three interpretations into *PA* plus associated completeness theorems.

0.3 Some partly philosophical remarks

0.3.1. *On a certain equation.* Let *IPL* be Intuitionistic Logic. We have:

$$\frac{IPL}{S_4} = \frac{FPL}{PrL}.$$

The reason is that $*$ is a Gödel translation for *IPL* in S_4. We have:

$$\vdash_{IPL} A \quad \text{iff} \quad \vdash_{S_4} A^*$$

as well as:

$$\vdash_{FPL} A \quad \text{iff} \quad \vdash_{PrL} A^*.$$

The analogy gains substance when one considers that the axioms of S_4 are valid for true, real or rigid provability. The crucial rule $\dfrac{\Box A}{A}$, which is blocked in *PrL*, is justified as follows: suppose we have a proof p of A, then A, *or else p would not be a true proof of A.* This argument suggests that what a proof is, cannot be fixed completely by a set of 'formal' properties; there must be at least some semantical properties.

0.3.2 *Liar Paradox and Provability Paradox.* With the usual derivation of the Liar Paradox in *IPL* from the postulated rules $\dfrac{L}{\neg L}$ (1) and $\dfrac{\neg L}{L}$ (2) corresponds via $*$ the S_4 derivation of the (true) Provability Paradox from the postulated rules $\dfrac{G}{\Box\neg G}$ (3) and $\dfrac{\Box\neg G}{G}$ (4). In *FPL* there is an explicit sentence L satisfying (1) and (2), likewise in *PrL* there is an explicit sentence G satisfying (3) and (4). In both cases paradox is blocked by failure of Modus Ponens resp. $\dfrac{\Box A}{A}$.

0.4 Prerequisites

§1–4 are quite selfcontained. The reader only needs a basic understanding of Kripke Models for *IPL*, see e.g. [4] or [10]. §5 requires an understanding of the main results of *PrL* see e.g. [2], [9], [7], [8].

0.5 Acknowledgements

The use of the work of the pioneers of Provability Logic should be evident.

For the development of the Kripke Model theory of §1, [6] has been my guide. §4 is related to [3]. The present work seems a companion to [5], [2] where the relation between *PrL* and *S* is studied.

1. Basic Propositional Logic

1.1. Language

Let $\boldsymbol{P}:=\{p_0, p_1, \ldots\}$ be the set of propositional variables. $\mathscr{L}$ is the smallest set s.t.:

$$\boldsymbol{P}\subseteq\mathscr{L};\ \bot\in\mathscr{L};\ A, B\in\mathscr{L}\Rightarrow(A\wedge B),\ (A\vee B),\ (A\to B)\in\mathscr{L}.$$

1.2 The Theory

The theory Basic Propositional Logic *BPL* is given by the following groups of ruleschemes:

group I:

$$\wedge I\quad \frac{A\quad B}{(A\wedge B)}\qquad \wedge E\quad \frac{(A\wedge B)}{A}\ \frac{(A\wedge B)}{B}$$

$$\vee I\quad \frac{A}{(A\vee B)}\ \frac{B}{(A\vee B)}\qquad \bot\mathrm{E}\quad \frac{\bot}{A}$$

$$\text{Tr}\qquad \frac{(A\to B)(B\to C)}{(A\to C)}$$

(Transitivity)

$\wedge$I f $\qquad \dfrac{(A\to B)(A\to C)}{(A\to(B\wedge C))}$

(formalized $\wedge$I)

$\vee$E f $\qquad \dfrac{(A\to C)(B\to C)}{((A\vee B)\to C)}$

(formalized $\vee$E)

group II: $\to$I $\qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{(A\to B)}$ $\qquad\qquad \vee$E $\qquad \dfrac{\begin{matrix} & [A] & [B]\\ & \vdots & \vdots\\ (A\vee B) & C & C\end{matrix}}{C}$

A rulescheme is considered here as a set of rules. A rule always contains individual elements of $\mathscr{L}$, e.g. $\dfrac{p_0\#p_1}{(p_0\wedge p_1)}$ is a rule and $\dfrac{p_0\#p_1}{(p_0\wedge p_1)} \in \dfrac{A\,B}{(A\wedge B)}$.

We may add to our system a set of additional rules $\mathbf{R}$ of the form $\dfrac{A_1\ldots A_k}{B}$. Such rules will be called normal rules. When $\Gamma\subseteq\mathscr{L}$, $\Gamma\vdash_{\mathbf{R}} A$ means: A is derivable from Γ, with the rules of $BPL+\mathbf{R}$.
We say:

$\vdash_{\mathbf{R}} A$	for: $\emptyset\vdash_{\mathbf{R}} A$
$\Gamma\ \vdash A$	for: $\Gamma\vdash_{\emptyset} A$
$\vdash A$	for: $\emptyset\vdash_{\emptyset} A$
$A\dashv\vdash_{\Gamma,\mathbf{R}} B$	for: $\Gamma, A\vdash_{\mathbf{R}} B$ and $\Gamma, B\vdash_{\mathbf{R}} A$
$A\dashv\vdash_{\mathbf{R}} B$	for: $A\dashv\vdash_{\emptyset,\mathbf{R}} B$
$A\dashv\vdash_{\Gamma} B$	for: $A\dashv\vdash_{\Gamma,\emptyset} B$
$A\dashv\vdash B$	for: $A\dashv\vdash_{\emptyset,\emptyset} B$

1.3 A few basic facts about *BPL*

1.3.1 *Derived and underived schemes.* The following are easily seen to be derived schemes:

$\wedge$E f: $\qquad \dfrac{(A\to(B\wedge C))}{(A\to B)} \qquad \dfrac{(A\to(B\wedge C))}{(A\to C)}$

$\vee$I f: $\qquad \dfrac{(A\to B)}{(A\to(B\vee C))} \qquad \dfrac{(A\to C)}{(A\to(B\vee C))}$

$\to$I f: $\qquad \dfrac{((A\wedge B)\to C)}{(A\to(B\to C))}$

The following are not derived ruleschemes as can be seen from the completeness theorem (1.10):

$\to$E $\qquad \dfrac{A \# A \to B}{B}$

(or Modus Ponens)

$\to$E f $\qquad \dfrac{(A \to (B \to C))}{((A \wedge B) \to C)}$

1.3.2 *Substitution of equivalents.* Let a propositional context $C[\ ]$ be defined as usual. The following can be proved by easy inductions:

i) SE (Substitution of Equivalents) : $A \dashv\vdash_{\Gamma,\mathbf{R}} B \Rightarrow C[A] \dashv\vdash_{\Gamma,\mathbf{R}} C[B]$

ii) S E f (formalized S E) : $A \leftrightarrow B \vdash C[A] \leftrightarrow C[B]$

SE justifies us to be careless with brackets, so we will be.

1.4 Models

A (Kripke) Model K is a structure $\langle W, \lhd, f\rangle$ where W is a set of "worlds"; $\lhd$ a binary transitive relation on W; f a function from W in the subsets of $\mathbf{P}$ s.t. $w \lhd w' \Rightarrow f(w) \subseteq f(w')$.

1.4.1 *Satisfaction.* Let $K = \langle W, \lhd, f\rangle$ be a Model. $\Vdash_K \subseteq W \times \mathscr{L}$ is the smallest relation s.t.:

— $p_i \in f(w) \Rightarrow w \Vdash_K p_i$

— $w \Vdash_K A$ and $w \Vdash_K B \Rightarrow w \Vdash_K A \wedge B$

— $w \Vdash_K A$ or $w \Vdash_K B \Rightarrow w \Vdash_K A \vee B$

— $(\forall w' \lhd w\ \ w' \Vdash_K A \Rightarrow w' \Vdash_K B) \Rightarrow w \Vdash_K (A \to B)$.

Define further for $\Gamma \subseteq \mathscr{L}$:

$\Gamma \Vdash_K A :\Leftrightarrow \forall w \in W (w \Vdash_K \Gamma \Rightarrow w \Vdash_K A)$.

We write $K \Vdash A$ or $\Vdash_K A$ for $\emptyset \Vdash_K A$.

1.4.2 Fact. For all $A \in \mathscr{L}$ and $w, w' \in W$: $w \Vdash_K A$ and $w' \lhd w \Rightarrow w' \Vdash_K A$.

1.4.3 *Closure under rules.* We say that K is closed under a rule of the form $\dfrac{A_1 \ldots A_k}{B}$ if $\{A_1, \ldots, A_k\} \Vdash_K B$.

Let $\mathbf{R}$ be a set of normal rules. We say that K is closed under $\mathbf{R}$ if K is closed under every element of $\mathbf{R}$.

Define: $\Gamma \Vdash_{\mathbf{R}} A :\Leftrightarrow$ for every $\mathbf{R}$-closed K $\ \Gamma \Vdash_K A$.

We have:

1.4.4 Soundness Theorem. $\Gamma \vdash_{\mathbf{R}} A \Rightarrow \Gamma \Vdash_{\mathbf{R}} A$.

PROOF. Routine. □

As a preliminary for the completeness theorem we will give a connection between 'formalized' and 'unformalized'.

1.5 DEFINITION. Let $\Gamma \subseteq \mathscr{L}$. Define: $\mathbf{R}_\Gamma := \left\{\frac{A_1 \dots A_k}{B} \mid \Gamma \vdash_{\mathbf{R}} (A_1 \wedge \wedge \dots A_k) \to B\right\}$.

1.5.1 FACT. $\mathbf{R} \subseteq \mathbf{R}_\Gamma$.

PROOF. If $\frac{A_1 \dots A_k}{B} \in \mathbf{R}$ then $\Gamma \vdash_{\mathbf{R}} (A_1 \wedge \dots A_k) \to B$. □

1.6 THEOREM. *For all* $\Gamma \subseteq \mathscr{L}$; $k \in N$; $A_1 \dots A_k$, $B \in \mathscr{L}$: $\Gamma \vdash_{\mathbf{R}} (A_1 \wedge \wedge \dots A_k) \to B \Leftrightarrow \Gamma, A_1 \dots A_k \vdash_{\mathbf{R}_\Gamma} B$.

PROOF. "$\Rightarrow$". By definition.
"$\Leftarrow$". By induction on the length of the proof.
Case i). The length of the proof is 0, i.o.w. $B \in \Gamma, A_1 \dots A_k$. Trivial.
Case ii). The last rule applied is of group I or of $\mathbf{R}_\Gamma$. Say it is $R = \frac{B_1 \dots B_s}{C}$.
We have:

$$\Gamma, A_1 \dots A_k \vdash_{\mathbf{R}_\Gamma} B_1, \dots \Gamma, A_1 \dots A_k \vdash_{\mathbf{R}_\Gamma} B_s.$$

By Induction Hypothesis and $\wedge$I f:

$$\Gamma \vdash_{\mathbf{R}} (A_1 \wedge \dots A_k) \to (B_1 \wedge \dots B_s).$$

When $R \in \mathbf{R}_\Gamma$ we have by definition:

$$\Gamma \vdash_{\mathbf{R}} (B_1 \wedge \dots B_s) \to C.$$

When R is of group I this follows by $\wedge$E and $\to$I.
By *Tr* we find:

$$\Gamma \vdash_{\mathbf{R}} (A_1 \wedge .. A_k) \to C.$$

Case iii). The last rule applied is $\to$I. We have:

$$\Gamma, A_1 \dots A_k \vdash_{\mathbf{R}_\Gamma} B \to C \quad \text{from} \quad \Gamma, A_1 \dots A_k, B \vdash_{\mathbf{R}_\Gamma} C.$$

By Induction Hypothesis:

$$\Gamma \vdash_{\mathbf{R}} (A_1 \wedge \dots A_k \wedge B) \to C.$$

Using the derived rule $\to$I f we find:

$$\Gamma \vdash_{\mathbf{R}} (A_1 \wedge \dots A_k) \to (B \to C).$$

Case iv). The last rule applied is $\vee$E. We have $\Gamma, A_1 \dots A_k \vdash_{\mathbf{R}_\Gamma} D$ from

$$\Gamma, A_1 \dots A_k \vdash_{\mathbf{R}_\Gamma} B \vee C; \quad \Gamma, A_1 \dots A_k, B \vdash_{\mathbf{R}_\Gamma} D; \quad \Gamma, A_1, \dots, A_k, C \vdash_{\mathbf{R}_\Gamma} D.$$

By Induction Hypothesis:

$$\Gamma \vdash_{\mathbf{R}} (A_1 \wedge .. A_k) \to (B \vee C); \; \Gamma \vdash_{\mathbf{R}} (A_1 \wedge .. A_k \wedge B) \to D;$$
$$\Gamma \vdash_{\mathbf{R}} (A_1 \wedge .. A_k \wedge C) \to D.$$

By $\vee$E f we find:

$$\Gamma \vdash_{\mathbf{R}} \big((A_1 \wedge .. A_k \wedge B) \vee (A_1 \wedge .. A_k \wedge C)\big) \to D.$$

One easily shows:

$$\Gamma \vdash_{\mathbf{R}} \big((A_1 \wedge \,..\, A_k) \wedge (B \wedge C)\big) \to \big((A_1 \wedge \,..\, A_k \wedge B) \vee (A_1 \wedge \,..\, A_k \wedge C)\big).$$

So by *Tr*:

$$\Gamma \vdash_{\mathbf{R}} \big((A_1 \wedge \,..\, A_k) \wedge (B \vee C)\big) \to D.$$

Using:

$$\frac{(A_1 \wedge \,..\, A_k) \to (A_1 \wedge \,..\, A_k)\quad (A_1 \wedge \,..\, A_k) \to B \vee C}{(A_1 \wedge \,..\, A_k) \to \big((A_1 \wedge \,..\, A_k) \wedge (B \vee C)\big)} \wedge\mathrm{I\,f}$$

we find using *Tr* again:

$$\Gamma \vdash_{\mathbf{R}} (A_1 \wedge \,..\, A_k) \to D.$$

□

1.7. Definition. $\Gamma \subseteq \mathscr{L}$ is called **R**-prime if whenever $\Gamma \vdash_{\mathbf{R}} A$ we have $A \in \Gamma$ and whenever $\Gamma \vdash_{\mathbf{R}} A \vee B$ we have $\Gamma \vdash_{\mathbf{R}} A$ or $\Gamma \vdash_{\mathbf{R}} B$. Moreover Γ must be consistent.

1.8 Fact. If $\Gamma\ non{\vdash_{\mathbf{R}}} A$ then there is an **R**-prime $\Gamma' \supseteq \Gamma$ s.t. $\Gamma'\ non{\vdash_{\mathbf{R}}} A$.

Proof. Routine. □

1.9 Theorem. *If $\Gamma\ non{\vdash_{\mathbf{R}}} A$ then there is an* **R***-closed model* $K = \langle W, \lhd, f\rangle$ s.t. $K \Vdash \Gamma$ *and* $K\ non{\Vdash} A$.

Proof. Take $W := \{[\Delta] \mid \Delta \supseteq \Gamma,\ \Delta$ is **R**-prime$\}$, where $[\Delta]$ stands for $\langle \emptyset, \Delta\rangle$. We use $[\Delta]$ rather than Δ itself to avoid confusion between $\Delta \Vdash_K A$ as a world and $\Delta \Vdash_K A$ as a theory.

$$[\Delta] \lhd [\Delta'] :\Leftrightarrow \Delta' \text{ is } \mathbf{R}_\Delta \text{ prime.}$$

$$f([\Delta]) := \mathbf{P} \cap \Delta.$$

Note that:

$$[\Delta] \lhd [\Delta'] \Rightarrow \Delta \subseteq \Delta' \Rightarrow f([\Delta]) \subseteq f([\Delta']).$$

It is easy to verify that $\lhd$ is transitive so K is a genuine model.

Claim. $\{B \mid [\Delta] \Vdash_K B\} = \Delta$.

The proof is by induction on the length of B. The only problematic case is $\to$:

Suppose $(D \to E) \in \Delta$. Then $\dfrac{D}{E} \in \mathbf{R}_\Delta$.

So for every $\mathbf{R}_\Delta$-prime Δ': $D \in \Delta' \Rightarrow E \in \Delta'$. By Induction Hypothesis: for every $[\Delta'] \rhd [\Delta]([\Delta'] \Vdash_K D \Rightarrow [\Delta'] \Vdash_K E)$, i.e. $[\Delta] \Vdash_K (D \to E)$.

Now suppose $(D \to E) \notin \Delta$, then by theorem 1.6 $\Delta, D \not\vdash_{\mathbf{R}_\Delta} E$. Extend $\Delta \cup \{D\}$ to an $\mathbf{R}_\Delta$-prime Δ' s.t. $\Delta'\ non{\vdash_{\mathbf{R}_\Delta}} E$. We have $[\Delta'] \rhd [\Delta]$. By Induction Hypothesis:

$$[\Delta'] \Vdash_K D \quad\text{and}\quad [\Delta']\ non{\Vdash_K} E. \quad\text{So}\quad [\Delta]\ non{\Vdash_K} (D \to E).$$

It follows easily that:

- K is **R**-closed
- $K \Vdash \Gamma$, because each $\Delta \supseteq \Gamma$
- $K\ non{\Vdash} A$, because there is an **R**-prime Δ s.t. $\Delta \supseteq \Gamma$ and $A \notin \Delta$. □

1.10 COMPLETENESS THEOREM. $\Gamma \vdash_{\mathbf{R}} A \Leftrightarrow \Gamma \Vdash_{\mathbf{R}} A$.

PROOF. Combine 1.4.4 and 1.9. □

1.11 REMARK. By inspection of the proof, we can see that we have the completeness theorem also for the class of models $K = \langle W, \lhd, f \rangle$ s.t. $w \lhd w' \lhd w \Rightarrow w = w'$.

1.12 COROLLARY: Completeness Theorem for Intuitionistic Propositional Logic. *Let Intuitionistic Propositional Logic (IPL) be BPL* $+\to$E. *Then*: $\Gamma \vdash_{IPL} A \Leftrightarrow$ *for all reflexive* K: $\Gamma \Vdash_K A \Leftrightarrow$ *for all* K *where* $\lhd$ *is a weak partial order* $\Gamma \Vdash_K A$.

PROOF. "$\Rightarrow$" is clear.

"$\Leftarrow$". It is sufficient to show that the model constructed in 1.9 is reflexive when $\mathbf{R} = \to$E. We have to show that any $\to$E-prime $\Delta \supseteq \Gamma$ is $\mathbf{R}_\Delta$-prime. But that is immediate using $\to$E. □

1.13 LEMMA. *Suppose* Γ *non*$\vdash A$, Γ *finite. Then there is a finite* $K_0 = \langle W_0, \lhd_0, f_0 \rangle$, *where* $w \lhd_0 w' \lhd_0 w \Rightarrow w = w'$ s.t. $K_0 \Vdash \Gamma$ *and* K_0 *non*$\Vdash A$.

PROOF. Consider the model $K = \langle W, \lhd, f \rangle$ constructed in 1.9 s.t. $K \Vdash \Gamma$ and K *non*$\Vdash A$ (for $\mathbf{R} = \emptyset$). Let Δ be the set of subformulas of the elements of $\Gamma \cup \{A\}$. (Each formula is a subformula of itself).

Define:

- $\tilde{w} := \{B \in \Delta \mid w \Vdash_K B\}$
- $W_0 := \{\tilde{w} \mid w \in W\}$
- for $a, b \in W_0$: $a \lhd_0 b :\Leftrightarrow a \subseteq b$ and if $(E \to F) \in a, E \in b$ then $F \in b$
- $f_0(a) = a \cap P$

It is easy to see that K_0 is a model, that K_0 is finite and that $a \lhd_0 a' \lhd_0 a \Rightarrow a = a'$.

Claim. for every $B \in \Delta$: $w \Vdash_K B \Leftrightarrow \tilde{w} \Vdash_{K_0} B$. The proof is by induction on the length of B. The only problematic case is $\to$.

Let $B \equiv (E \to F) \in \Delta$.

"$\Rightarrow$". Suppose $w \Vdash_K (E \to F)$ and $a \rhd_0 \tilde{w}$ and $a \Vdash_{K_0} E$. There that $o = \tilde{u}$. By Induction Hypothesis: $u \Vdash_K E$, hence $E \in \tilde{u}$ i.e. $E \in a$. Moreover $(E \to F) \in \tilde{\omega}$, $a \rhd_0 \tilde{w}$, so $F \in a$. Hence $u \Vdash_K F$. Again by Induction Hypothesis: $\tilde{u} \Vdash_{K_0} F$ i.e. $a \Vdash_{K_0} F$.

"$\Leftarrow$". Suppose $\tilde{w} \Vdash_{K_0} E \to F$, $w' \rhd w$, $w' \Vdash_K E$. Clearly $\tilde{w}' \rhd_0 \tilde{w}$ and by Induction Hypothesis: $\tilde{w}' \Vdash_{K_0} E$, hence $\tilde{w}' \Vdash_{K_0} F$. □

1.14 COROLLARY. *If* Γ *is finite we have*

(i) $\Gamma \vdash A \Leftrightarrow$ *for all finite models* K: $\Gamma \Vdash_K A$

(ii) $\Gamma \vdash_{IPL} A \Leftrightarrow$ *for all finite* K, *where* $\lhd$ *is a weak partial order*: $\Gamma \Vdash_K A$.

PROOF. From 1.13. Observe that the construction in 1.13 preserves reflexivity. □

2. Formal Propositional Logic

2.1 THE SYSTEM *FPL*. Let Löb's Rule L be: $\dfrac{(\top\to A)\to A}{\top\to A}$,

where $\top :\equiv (\bot\to\bot)$. *FPL* is $BPL+L$.

2.2 COMPLETENESS THEOREM FOR *FPL*. *Let $\Gamma\subseteq\mathscr{L}$, Γ finite, then: $\Gamma\vdash_{FPL} A \Leftrightarrow$ for every finite irreflexive K $\Gamma\Vdash_K A$.*

PROOF.
"$\Rightarrow$". We will prove more generally that every K s.t. $\lhd$ is reverse wellfounded (i.e. $\rhd$ is wellfounded) is closed under L.

Consider $w\in W$. Suppose that every $u\rhd w$ is closed under L and suppose $w\Vdash_K(\top\to A)\to A$.

For $u\rhd w$ we have $u\Vdash_K(\top\to A)\to A$ by monotonicity. u is closed under L so $u\Vdash_K \top\to A$. Moreover if $u\Vdash_K\top\to A$, then $u\Vdash_K A$ because $w\Vdash_K(\top\to A)\to A$. So $u\Vdash_K A$ for any $u\rhd w$. So $w\Vdash_K\top\to A$. By bar induction we find that K is closed under L.
"$\Leftarrow$" Suppose Γ *non*$\vdash_{FPL} A$. Let K be the model constructed in 1.9 s.t. K is L-closed and $K\Vdash\Gamma$ and K *non*$\Vdash A$. Let K_0 be the model constructed in 1.13. We have $K_0\Vdash\Gamma$ and K_0 *non*$\Vdash A$. Note that we do not know whether K_0 is closed under L. Define $K_1=\langle W_0,\lhd_1,f_0\rangle$, where $a\lhd_1 b :\Leftrightarrow (a\lhd_0 b$ and $a\neq b)$. By the "$\Rightarrow$" part of our proof we see that K_1 is automatically closed under L. Let Λ be as in 1.13. We prove:

For all $B\in\Lambda$, for all $w\in W$: $w\Vdash_K B\Leftrightarrow\tilde w\Vdash_{K_0}B\Leftrightarrow\tilde w\Vdash_{K_1}B$.

We already have the first equivalence. We prove the second one with induction on the length of B. The only non trivial case is again $\to$.

Suppose $B=(E\to F)\in\Lambda$.

I) If $\tilde w\Vdash_{K_0}E\to F$, then for any $\tilde u\rhd_0\tilde w$ $(\tilde u\Vdash_{K_0}E\Rightarrow\tilde u\Vdash_{K_0}F)$. By Induction Hypothesis: for every $\tilde u\rhd_0\tilde w$ $(\tilde u\Vdash_{K_1}E\Rightarrow\tilde u\Vdash_{K_1}F)$. So certainly: for any $\tilde u\rhd_1\tilde w$ $(\tilde u\Vdash_{K_1}E\Rightarrow\tilde u\Vdash_{K_1}F)$. Thus: $\tilde w\Vdash_{K_1}(E\to F)$.

II) If $\tilde w$ *non*$\Vdash_{K_0}E\to F$, then there is a $\tilde u\rhd_0\tilde w$ with $(\tilde u\Vdash_{K_0}E$ and $\tilde u$ *non*$\Vdash_{K_0}F)$.

case i) There is such a $\tilde u$ s.t. $\tilde u\neq\tilde w$. Then $\tilde u\rhd_1\tilde w$ and by Induction Hypothesis $\tilde u\Vdash_K E$ and $\tilde u$ *non*$\Vdash_{K_1}F$. So $\tilde w$ *non*$\Vdash_{K_1}E\to F$.

case ii) The only such $\tilde u$ is $\tilde w$ itself. We have:

a) For every $\tilde u\rhd_1\tilde w$: $\tilde u\Vdash_{K_0}E\Rightarrow\tilde u\Vdash_{K_0}F$

b) $\tilde w\Vdash_{K_0}E$ and $\tilde w$ *non*$\Vdash_{K_0}F$ and $\tilde w\rhd_0\tilde w$.

Consider a $u\rhd w$ with $u\Vdash_K E\to F$. It follows that:

c) $\tilde u\rhd_0\tilde w$

d) $\tilde u\Vdash_{K_0}(E\to F)$ (because $(E\to F)\in\Lambda$), $\tilde w$ *non*$\Vdash_{K_0}E\to F$

e) $\tilde u\rhd_1\tilde w$ (by (c), (d))

f) $\tilde u\Vdash_{K_0}E$ (by (b), (c))

g) $\tilde u\Vdash_{K_0}F$ ((a), (e), (f))

h) $u\Vdash_K F$ $(F\in\Lambda)$

$u\rhd w$ with $u\Vdash_K(E\to F)$ was arbitrary, so:

i) $w\Vdash_K(E\to F)\to F$.

By:

$$\dfrac{\dfrac{\top}{E\to\top} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\top}{E\to\top}\quad [\top\to F]}{E\to F}}{(\top\to F)\to(E\to F)} \quad (E\to F)\to F}{(\top\to F)\to F}}{\top\to F}}{E\to F}$$

we find:

j) $w \Vdash_K E\to F$ $\big((\mathrm{i})\big)$

Thus

k) $\tilde{w} \Vdash_{K_0} E\to F$ $\big((E\to F)\in \Lambda,\ (\mathrm{j})\big)$

Contradiction. □

3. Basic Facts about FPL

3.1 An alternative version of Löb's Rule

There is an alternative version of Löb's Rule L' which is interderivable with L over BPL. Namely:

$$\dfrac{\begin{matrix}[\top\to A]\\ \vdots\\ A\end{matrix}}{A}\ L'$$

PROOF.

$$\dfrac{\dfrac{\begin{matrix}[\top\to A]\\ \vdots\\ A\end{matrix}\ \pi}{(\top\to A)\to A}\to\mathrm{I}}{\top\to A}\ L \qquad\qquad \dfrac{\dfrac{[\top\to(\top\to A)]\quad(\top\to A)\to A}{\top\to A}\ \mathrm{Tr}}{\top\to A}\ L'$$

$$\begin{matrix}\top\to A\\ \vdots\\ A\end{matrix}\ \pi$$

3.2 DEFINITION. $FC(p_i)$, the set of *Formal Contexts* of p_i is the smallest set s.t.

i) $i\neq j \Rightarrow p_j \in FC(p_i)$

ii) $\perp \in FC(p_i)$
iii) $A, B \in \mathscr{L} \Rightarrow (A \to B) \in FC(p_i)$
iv) $A, B \in FC(p_i) \Rightarrow (A \wedge B), (A \vee B) \in FC(p_i)$.

3.3 EXAMPLE. $p_0 \wedge (p_1 \to p_0)$ is a formal context of p_1 and not of p_0.

3.4 THEOREM: More Substitution of Equivalents. *Let $C[p_0]$ be a formal context of p_0.*

i) SE$^+$: *FPL* is closed under: $\dfrac{A \leftrightarrow B \# C[A]}{C[B]}$

ii) SE$^+$f: *FPL* is closed under: $\dfrac{\top \to (A \leftrightarrow B)}{C[A] \leftrightarrow C[B]}$

PROOF. Induction on proof length or use Completeness Theorem. □

3.5 Unicity of Fixed Points in Formal Contexts

Suppose $C[p_0]$ is a formal context of p_0. We have:
i) $(A \dashv\vdash_{\Gamma,FPL} C[A]$ and $B \dashv\vdash_{\Gamma,FPL} C[B]) \Rightarrow A \dashv\vdash_{\Gamma,FPL} B$.
ii) (Formal Version of i): *FPL* is closed under:

$$\frac{A \leftrightarrow C[A] \quad B \leftrightarrow C[B]}{A \leftrightarrow B}$$

PROOF.
i)

$$\dfrac{\dfrac{A}{C[A]} \quad \text{(ex hypothesi)} \quad \dfrac{\dfrac{A}{B \to A} \quad \dfrac{\dfrac{\top}{A \to \top \mathit{non}[\top \to B]}}{A \to B}}{A \leftrightarrow B}}{\dfrac{\dfrac{C[B]}{B}\,\text{(ex hypothesi)}}{B}\,L'}\,\text{SE}^+$$

ii)

$$\dfrac{\dfrac{A \leftrightarrow C[A] \quad \dfrac{[\top \to (A \leftrightarrow B)] \ \text{SE}^+\text{f}}{C[A] \leftrightarrow C[B]}}{A \leftrightarrow C[B]}\,\text{Tr} \qquad B \leftrightarrow C[B]}{\dfrac{A \leftrightarrow B}{A \leftrightarrow B}\ L'.}\,\text{Tr}$$

□

3.6 FIXED POINT THEOREM. *For any $C[p_0]$ we have:*

$$C[\top] \dashv\vdash_{FPL} C[C[\top]].$$

Before proving 3.6, first we give two lemmas.

3.6.1. LEMMA. *Let $D[p_0]$ be a formal context of p_0. Then:*

$$D[\top] \dashv\vdash_{FPL} D[D[\top]].$$

PROOF. "$\vdash$"

Trivially we have: $D[\top] \dashv\vdash_{D[\top],FPL} \top$,

so by SE: $D[D[\top]] \dashv\vdash_{D[\top],FPL} D[\top]$.

Thus: $D[\top] \vdash_{FPL} D[D[\top]]$.

(Note that we did not use that $D[p_0]$ is formal, nor did we use L.)

"$\dashv$"

$$\dfrac{\dfrac{D[D[\top]] \qquad \dfrac{[\top\to D[\top]] \qquad \dfrac{\top}{D[\top]\to\top}}{\top\leftrightarrow D[\top]}}{D[\top]}\,\mathrm{SE}^+}{D[\top]}\,L'$$ □

3.6.2 DEFINITION. $SIC(p_i)$, the set of strictly informal contents of p_i, is the smallest set s.t.

i) $p_i \in SIC(p_i)$

ii) p_i does not occur in $A \Rightarrow A \in SIC(p_i)$

iii) $D, E \in SIC(p_i) \Rightarrow D \wedge E, D \vee E \in SIC(p_i)$.

3.6.3 EXAMPLE. $(p_0 \wedge p_1) \vee (p_1 \to p_2)$ is in $SIC(p_0)$ but not in $SIC(p_1)$.

3.6.4 LEMMA. *For every* $D \in \mathscr{L}$, $p_i \in \mathbf{P}$, *there is a* $p_k, p_l \in \mathbf{P}$, $E[p_k]\,\langle p_l\rangle \in \mathscr{L}$ s.t. $D \equiv E[p_i]\,\langle p_i\rangle$ *and* $E[p_k]\,\langle p_l\rangle \in FC(p_k)$ *and* $E[p_k]\,\langle p_l\rangle \in SIC(p_l)$.

PROOF. Routine. □

3.6.5 EXAMPLE. Let $D \equiv p_0 \vee (p_1 \to p_0)$. For p_0 choose p_2, p_3 and $E[p_2]\,\langle p_3\rangle \equiv p_3 \vee (p_1 \to p_2)$.

3.6.6 LEMMA. *If* $D[p_0] \in SIC(p_0)$, *then FPL is closed under* $\dfrac{D[A]}{D[\top]}$.

PROOF. Induction of $D[p_0]$. (The idea is that the relevant occurrences of A are only in positive places.) □

PROOF OF 3.6

"$\vdash$" By the same reasoning as the "$\vdash$" part of 3.6.1.

"$\dashv$" Write $C[p_0]$ as $D[p_0]\,\langle p_0\rangle$, where $D[p_0]\,\langle p_0\rangle$ is as in 3.6.4.

Apply 3.6.1 to $D[p_0]\,\langle\top\rangle$. We find:

$$C[\top] \equiv D[\top]\,\langle\top\rangle \dashv\vdash_{FPL} D[C[\top]]\langle\top\rangle.$$

By 3.6.6:

$$C[C[\top]] \equiv D[C[\top]]\,\langle C[\top]\rangle \vdash_{FPL} D[C[\top]]\,\langle\top\rangle.$$

So we have: $C[C[\top]] \vdash_{FPL} C[\top]$. □

3.7 REMARK. Of course we do not have *unicity* of fixed points in general as is seen in the case $C[p_0] \equiv p_0$.

We do get:

$A \dashv\vdash_{\Gamma,FPL} C[A] \Rightarrow A \vdash_{\Gamma,FPL} C[\top]$. Since $A \dashv\vdash_{\Gamma,A,FPL} \top$, we get $C[A] \dashv\vdash_{\Gamma A,FPL} C[\top]$. So $A, C[A] \vdash_{\Gamma,FPL} C[\top]$. Thus $A \dashv_{\Gamma,FPL} C[\top]$. So $C[\top]$ is the maximal fixed point w.r.t. $\vdash_{\Gamma,FPL}$.

3.8 EXAMPLE: "*The Liar*". $\top \to \bot$ is the unique A (modulo $\dashv\vdash_{FPL}$) s.t. $A \dashv\vdash_{FPL} \neg A (\equiv A \to \bot)$.

4. A "classical" version of *FPL*

4.1 DEFINITIONS. Let $\Gamma, \Delta \subseteq \mathscr{L}$.

i) $\mathrm{Sub}_\Delta = \{s | s\colon \mathbf{P} \to \Delta\}$

ii) — $p_i^s = s(p_i)$
— $(A \wedge B)^s = (A^s \wedge B^s)$
— $(A \vee B)^s = (A^s \vee B^s)$
— $(A \to B)^s = A^s \to B^s$

iii) $\Gamma^s = \{B^s | B \in \Gamma\}$

iv) $\Gamma \Vert\!= _{\Delta,\mathbf{R}} A\colon \Leftrightarrow$ for every $s \in Sub_\Delta$ $(\vdash_{\mathbf{R}} \Gamma^s \Rightarrow \vdash_{\mathbf{R}} A^s)$.

4.2 SOME FACTS.

i) Consider $s \in Sub_{\{\top\}}$. We have for any $A \in \mathscr{L}$:
$A^s \dashv\vdash_{IPL} \top$ or $A^s \dashv\vdash_{IPL} \bot$.

ii) Let *CPL* be Classical Propositional Logic. We have:
$\Gamma \vdash_{CPL} A \Leftrightarrow \Gamma \Vert\!=_{\{\top,\bot\},IPL} A$.

iii) $\Gamma \vdash_{CPL} A \Leftrightarrow \Gamma \Vert\!=_{\mathscr{L},CPL} A$.

PROOFS. Routine. □

Below we will do something analogous to the "classification" of *IPL* for *FPL*.

4.3 DEFINITIONS

i) — $\bot_\omega \colon\equiv \top (\equiv \bot \to \bot)$
— $\bot_0 \colon\equiv \bot$
— $\bot_{n+1} \colon\equiv \top \to \bot_n$

4.4 FACTS

i) $a, b \in \omega+1$ and $a < b \Rightarrow \bot_a \vdash_{FPL} \bot_b$ and $\bot_b$ *non*$\vdash_{FPL} \bot_a$.

ii) Let $a, b \in \omega+1$, then:
— $\bot_a \wedge \bot_b \dashv\vdash_{FPL} \bot_{\min(a,b)}$
— $\bot_a \vee \bot_b \dashv\vdash_{FPL} \bot_{\max(a,b)}$
— if $a \leqslant b$ $\bot_a \to \bot_b \dashv\vdash_{FPL} \top$
if $a > b$ $\bot_a \to \bot_b \dashv\vdash_{FPL} \bot_{b+1}$

iii) If $s \in Sub_{\{\bot\}}$, then for any $A \in \mathscr{L}$, there is an $a \in \omega+1$ s.t.:
$A^s \dashv\vdash_{FPL} \bot_a$.

PROOFS. All the proofs are easy. Let me just prove:

$$a > b \Rightarrow \bot_a \to \bot_b \dashv\vdash_{FPL} \bot_{b+1}.$$

"⊣"

$$\dfrac{\dfrac{\top}{\bot_a\to\top} \qquad \begin{array}{c}\bot_{b+1}\\ |||\\ \top\to\bot_b\end{array}}{\bot_a\to\top_b}\ \mathrm{Tr}$$

"⊢"

$$\begin{array}{c}
\dfrac{\bot_a\to\bot_b \qquad \dfrac{\dfrac{[\bot_b]}{\bot_{a-1}}\text{ by (i)}}{\bot_b\to\bot_{a-1}}}{\bot_a\to\bot_{a-1}}\\
|||\\
\dfrac{(\top\to\bot_{a-1})\to\bot_{a-1}}{\top\to\bot_{a-1}}\ L\\
|||\\
\dfrac{\dfrac{\bot_a}{\top\to\bot_a}\qquad \bot_a\to\bot_b}{\top\to\bot_b}\\
|||\\
\bot_{b+1}
\end{array}$$

□

4.5 Definitions

i) $\wedge$: $(\omega+1)^2\to(\omega+1)$ with $\wedge(a,b)=\min(a,b)$
$\vee$: $(\omega+1)^2\to(\omega+1)$ with $\vee(a,b)=\max(a,b)$
$\to$: $(\omega+1)^2\to(\omega+1)$ with $\to(a,b)=\begin{cases}\omega \text{ if } a\leqslant b\\ b+1 \text{ if } a>b.\end{cases}$

ii) An assignment f is a function $\mathbf{P}\to\omega+1$.

iii) Let f be an assignment. Define:
- $[\![p_i]\!]_f=f(p_i)$
- $[\![\bot]\!]_f=0$
- $[\![(A\wedge B)]\!]_f=\wedge([\![A]\!]_f,[\![B]\!]_f)$
- $[\![(A\vee B)]\!]_f=\vee([\![A]\!]_f,[\![B]\!]_f)$
- $[\![A\to B]\!]_f=\to([\![A]\!]_f,[\![B]\!]_f)$.

iv) Let $\Gamma\cup\{A\}\subseteq\mathscr{L}$. Define:
- $\Gamma\vDash_f^* A$: $\Leftrightarrow$((for every $B\in\Gamma$: $[\![B]\!]_f=\omega)\Rightarrow[\![A]\!]_f=\omega$)
- $\Gamma\vDash^* A$: for every assignment g $\Gamma\vDash_g A$
- $\Gamma\vDash_f A$: $\Leftrightarrow\inf(\{[\![B]\!]_f\mid B\in\Gamma\})\leqslant[\![A]\!]_f$
  (Note that $\inf(\emptyset)=\omega$)
- $\Gamma\vDash A$: $\Leftrightarrow$ for every assignment g: $\Gamma\vDash_g A$.

4.6 Facts

i) $\vDash_f^* A\Leftrightarrow\vDash_f A$
ii) $\vDash_f^* A\to B\Leftrightarrow\vDash_f A\to B\Leftrightarrow A\vDash_f B$
iii) $\Gamma\vDash_f A\Rightarrow\Gamma\vDash_f^* A$
iv) $\Gamma\vDash^* A\Leftrightarrow\Gamma\,||\!=_{\{\bot_a|a\in\omega+1\},FPL}A$
v) $\Gamma\vDash^* A\Leftrightarrow$ (for every $s\in Sub_{\mathscr{L}}$: $\vDash^*\Gamma^s\Rightarrow\vDash^* A^s$).

Proofs. i)–iv) are entirely routine. Let us do v).

"$\Rightarrow$" Suppose $\Gamma\vDash^* A$ and $\vDash^*\Gamma^s$. Consider an assignment g. We have: $\vDash_g^*\Gamma^s$. Define an assignment h as: $h(p_i)=[\![s(p_i)]\!]_g$. Then $\vDash_h^*\Gamma$, so $\vDash_h^* A$. Thus $\vDash_g^* A^s$.

"$\Leftarrow$" Suppose for every $s \in Sub_L \models^* \Gamma^s \Rightarrow \models^* A^s$. Let $\models_f^* \Gamma$. Define s with $s(p_i) = \bot_{f(p_i)}$. Then: $\models^* \Gamma^s$, so $\models^* A^s$, thus $\models_f^* A$. $\square$

4.7 FACTS

i) $A, A \to B \models^* B$

ii) $p_0, p_0 \to p_1 \not\models p_1$

iii) $\bot_1 \models^* \bot$, but not $\models^* \bot_1 \to \bot$.

iv) $\Gamma, A \models B \Rightarrow \Gamma \models A \to B$. $\square$

PROOFS. Routine.

Obviously $\models$ looks more like FPL than $\models^*$. Below we will axiomatize $\models$.

4.8 THEOREM. *Let* $\Gamma \cup \{A\} \subseteq \mathscr{L}$. *We have*: $\Gamma \models A \Leftrightarrow$ *for all finite, irreflexive, linear* K: $\Gamma \Vdash_K A$.

PROOF.
"$\Rightarrow$" Suppose $\Gamma \models A$. Let K be finite, irreflexive and linear, and $w \Vdash_K \Gamma$. Without loss of generality we may assume that w is the downmost node of K and that $W = \{1, \ldots N\}$ and that $m \lhd n \Leftrightarrow m > n$.

Define $a_K(A) = \begin{cases} \max\{k|1 \leqslant k \leqslant N \text{ and } k \Vdash_K A\} & \text{if there is such a } k \\ 0 & \text{else.} \end{cases}$

Define an assignment f_K by: $f_K(p_i) = a_K(p_i)$

We claim: $a_K(A) = \min([\![A]\!]_{f_K}, N)$.

The proof is by induction on the length of A. Suppose e.g. $A \equiv (B \to C)$. In case $[\![B]\!]_{f_K} \leqslant [\![C]\!]_{f_K}$ we have $a_K(B) \leqslant a_K(C)$ by Induction Hypothesis. So $N \Vdash B \to C$ or $a_K(B \to C) = N$.

When $[\![B]\!]_{f_K} > [\![C]\!]_{f_K}$ we have to consider two possibilities: $-[\![C]\!]_{f_K} \geqslant N$, then $[\![B \to C]\!]_{f_K} \geqslant N$. By Induction Hypothesis: $N \Vdash_K C$, so $N \Vdash B \to C$. $-[\![C]\!]_{f_K} < N$. Then by Induction Hypothesis: $a_K(C) < a_K(B)$. So $a_K(B \to C) = a_K(C) + 1 = [\![B \to C]\!]_{f_K} = \min([\![B \to C]\!]_{f_K}, N)$. We assumed $N \Vdash_K \Gamma$. So for $B \in \Gamma$ $a_K(B) = N$ and thus $[\![B]\!]_{f_K} \geqslant N$. Because $\Gamma \models A$, we have $[\![A]\!]_{f_K} \geqslant \min(\{[\![B]\!]_{f_K} | B \in \Gamma\}) \geqslant N$. So $N \Vdash_K A$.

"$\Leftarrow$" Suppose $\Gamma \not\models A$, then there is an f s.t. $\min(\{[\![B]\!]_f | B \in \Gamma\}) > [\![A]\!]_f$. Suppose $\min(\{[\![B]\!]_f | B \in \Gamma\}) = M$. We construct $K_f = \langle W, \lhd, g\rangle$. Let $W = \{1, \ldots, M\}$ and for $m, n \in W$: $m \lhd n$: $\Leftrightarrow n < m$. Take $p_i \in g(m)$: $:\Leftrightarrow f(p_i) \geqslant m$.

By the usual induction on the length of A we prove for $1 \leqslant m \leqslant M$: $:m \Vdash_{K_f} A \Leftrightarrow [\![A]\!]_f \geqslant m$. It follows that $M \Vdash_{K_f} \Gamma$, but M *non*$\Vdash_{K_f} A$.

4.9 DEFINITIONS. We shall consider an axiom as a rule with empty premiss.

i) $BPLL$ is $BPL + ((A \to B) \vee ((A \to B) \to A))$

ii) FPL^{CL} is $FPL + ((A \to B) \vee ((A \to B) \to A))$

or

$BPLL + L$

4.10 COMPLETENESS FOR $BPLL$ AND FPL^{CL}.

i) *Let* $\Gamma \cup \{A\} \subseteq \mathscr{L}$:
$\Gamma \vdash_{BPLL} A \Leftrightarrow$ *for all linear Kripke models* K: $\Gamma \Vdash_K A$. (*Here linear means: for any* $w, w' \in W$, $w \lhd w'$ *or* $w = w'$ *or* $w' \lhd w$).

ii) *Let* $\Gamma \subseteq \mathscr{L}$ *be* finite, $A \in \mathscr{L}$, *then*:
$\Gamma \vdash_{FPL^{CL}} A \Leftrightarrow$ *for all finite, irreflexive, linear* K $\Gamma \Vdash_K A$
$\Leftrightarrow \Gamma \vDash A$.

PROOF.

i) "$\Rightarrow$" The validity of $(A \to B) \vee (A \to B) \to A$ is routine.
"$\Leftarrow$" Suppose Γ *non*$\vdash_{BPLL} A$. Consider the model $K = \langle W, \lhd, f\rangle$ constructed in 1.9 Let $\Delta \supseteq \Gamma$ be $BPLL$-prime with $A \notin \Delta$. Let $K_1 := \langle W_1, \lhd_1, f_1\rangle$ where $W_1 = \{[\Delta'] \in W \mid [\Delta'] \rhd [\Delta]\}$ and $\lhd_1 = \lhd \restriction W_1 \times W_1$ and $f_1 = f \restriction W_1$.
Clearly Γ *non*$\Vdash_{K_1} A$, so it is sufficient to show that K_1 is linear. Consider $[\Delta_0], [\Delta_1] \in W_1$. The case that $\Delta_0 = \Delta$ or $\Delta_1 = \Delta$ is trivial, so assume $\Delta_0, \Delta_1 \neq \Delta$. We have:
$[\Delta] \lhd_1 [\Delta_0]$
$[\Delta] \lhd_1 [\Delta_1]$.
The case that $\Delta_0 = \Delta_1$ is trivial, so assume that for some C:
$[\Delta_0]$ *non*$\Vdash_{K_1} C$,
$[\Delta_1] \Vdash_{K_1} C$.
Suppose:
$[\Delta_0] \Vdash_{K_1} A \to B$ and $[\Delta_1] \Vdash_{K_1} A$.
We distinguish:
case i) $[\Delta_0]$ *non*$\Vdash_{K_1} A$, then $[\Delta]$ *non*$\Vdash_{K_1} (A \to B) \to A$, so $[\Delta] \Vdash_{K_1} A \to B$. So $[\Delta_1] \Vdash_{K_1} B$.
case ii) $[\Delta_0] \Vdash_{K_1} A$, then $[\Delta_0] \Vdash_{K_1} \top \to B$.
Well: $[\Delta] \Vdash_{K_1} (C \to B) \vee ((C \to B) \to C)$. Because $(\top \to B) \Vdash_{K_1} (C \to B)$ we have: $[\Delta_0] \Vdash_{K_1} (C \to B)$ and $[\Delta_0] \nVdash_{K_1} C$. So $[\Delta]$ *non*$\Vdash_{K_1} (C \to B) \to C$. So $[\Delta] \Vdash_{K_1} (C \to B)$. But $[\Delta_1] \Vdash_{K_1} C$. So $[\Delta_1] \Vdash_{K_1} B$.
So Δ_1 is $\mathbf{R}_{\Delta_0}$-prime and $[\Delta_1] \rhd_1 [\Delta_0]$

ii) "$\Rightarrow$" as in (i).
"$\Leftarrow$" use (i) and the fact the constructions in 1.13 and 2.2 preserve linearity. □

4.11 FACT. Let

$$\mathrm{DM}_1: \frac{\neg(A \wedge B)}{\neg A \vee \neg B} \qquad \mathrm{DM}_2: \frac{\neg A \vee \neg B}{\neg(A \wedge B)}$$

$$\mathrm{DM}_3: \frac{\neg(A \vee B)}{\neg A \wedge \neg B} \qquad \mathrm{DM}_4: \frac{\neg A \wedge \neg B}{\neg(A \vee B)}$$

i) $\mathrm{DM}_2, \mathrm{DM}_3, \mathrm{DM}_4$ are derived ruleschemes for BPL.

ii) DM_1 is derived for $BPLL$.

PROOF. i) Routine.
ii)

$$
\dfrac{(B\to\bot)\vee(B\to\bot)\to B \qquad \dfrac{[B\to\bot]}{(A\to\bot)\vee(B\to\bot)} \qquad \dfrac{\dfrac{\dfrac{A\to A \qquad \dfrac{\dfrac{\begin{array}{c}\neg(A\wedge B)\\ \equiv\\ (A\wedge B)\to\bot\end{array}}{A\to(B\to\bot)}{\to}\mathrm{I}f \qquad [(B\to\bot)\to B]}{A\to B}}{A\to A\wedge B}\wedge\mathrm{I}f \qquad A\wedge B\to\bot}{A\to\bot}}{(A\to\bot)\vee(B\to\bot)}}{\begin{array}{c}(A\to\bot)\vee(B\to\bot)\\ \equiv\\ \neg A\vee\neg B\end{array}}
$$

or via the completeness theorem. □

5. Interpretations

We will interprete *FPL* in Peano Arithmetic (*PA*) via *PrL* and directly.

5.1 The Language of Modal Propositional Logic (*MPL*)

Let $\mathbf{P}$, $\wedge$, $\vee$, $\to$, $\bot$ be as in the case of $\mathscr{L}$. $\Box$ is an additional logical constant. $\mathscr{L}_\Box$, the language of *MPL*, is the smallest set s.t.

- $\mathbf{P}\subseteq\mathscr{L}_\Box$, $\bot\in\mathscr{L}_\Box$
- $A, B\in\mathscr{L}_\Box\Rightarrow(A\wedge B)$, $(A\vee B)$, $(A\to B)$, $(\Box A)\in\mathscr{L}_\Box$.

5.2 Kripke Models for *MPL*

i) A (Kripke) Model for *MPL* is a structure $M=\langle W, \lhd, f\rangle$, where W is a set (of worlds), $\lhd$ a transitive binary relation on W and f a function $W\to\mathbf{P}$.

ii) $\vDash_M\subseteq W\times\mathscr{L}_\Box$ is the smallest relation s.t.:
- $p_i\in f(w)\Rightarrow w\vDash_M p_i$
- $(w\vDash_M A$ and $w\vDash_M B)\Rightarrow w\vDash_M(A\wedge B)$
- $(w\vDash_M A$ or $w\vDash_M B)\Rightarrow w\vDash_M(A\vee B)$
- $(w\vDash_M A\Rightarrow w\vDash_M B)\Rightarrow w\vDash_M(A\to B)$
- $(\forall w'\rhd w\ \ w'\vDash_M A)\Rightarrow w\vDash_M(\Box A)$.

iii) Define for $\Gamma\subseteq\mathscr{L}_\Box$:
- $w\vDash_M\Gamma :\Leftrightarrow\forall B\in\Gamma\ \ w\vDash_M B$
- $\Gamma\vDash_M A :\Leftrightarrow(\forall w\in W\ \ w\vDash_M\Gamma\Rightarrow w\vDash_M A)$
- $M\vDash A :\Leftrightarrow\ \vDash_M A :\Leftrightarrow\emptyset\vDash_M A$

iv)
- $\boldsymbol{K}$ is the class of all finite irreflexive Kripke Models for *BPL*
- $\boldsymbol{M}$ is the class of all finite irreflexive Kripke Models for *MPL*
- $\Gamma\vDash_{PrL}A$ means: for every $M\in\boldsymbol{M}$ $\Gamma\vDash_M A$.

5.3 Two Gödel Translations

We define:

i) $0\colon \mathscr{L} \to \mathscr{L}_\Box$ by:

$(p_i)^0 := \Box p_i$

$(\bot)^0 := (\Box\bot)$

$(A \wedge B)^0 := (A^0 \wedge B^0)$

$(A \vee B)^0 := (A^0 \vee B^0)$

$(A \to B)^0\colon \Box(A^0 \to B^0)$

ii) $1\colon \mathscr{L} \to \mathscr{L}_\Box$ by:

$(p_i)^1 := (p_i \wedge \Box p_i)$

$(\bot)^1 := \bot$

$(A \wedge B)^1 := (A^1 \wedge B^1)$

$(A \vee B)^1 := (A^1 \vee B^1)$

$(A \to B)^1 := \Box(A^1 \to B^1)$.

We have:

5.4 THEOREM. *For finite* $\Gamma \subseteq \mathscr{L}$, $A \in \mathscr{L}$:

i) $\Gamma \vdash_{FPL} A \Leftrightarrow \Gamma^0 = \{B^0 \mid B \in \Gamma\} \vdash_{Prl} A^0$

ii) $\Gamma \vdash_{FPL} A \Leftrightarrow \Gamma^1 \vdash_{Prl} A^1$.

PROOF. We know: for $\Gamma \subseteq \mathscr{L}$ finite, $A \in \mathscr{L}$: $\Gamma \vdash_{FPL} A \Leftrightarrow \Gamma \Vdash_{FPL} A$. And for $\Gamma' \subseteq \mathscr{L}_\Box$ finite, $A' \in \mathscr{L}_\Box$: $\Gamma' \vdash_{PrL} A' \Leftrightarrow \Gamma' \models_{PrL} A'$ (see [1], [9]). So it is sufficient to show for $\Gamma \subseteq \mathscr{L}$ finite, $A \in \mathscr{L}$:

i) $\Gamma \Vdash_{FPL} A \Leftrightarrow \Gamma^0 \Vdash_{PrL} A^0$

ii) $\Gamma \Vdash_{FPL} A \Leftrightarrow \Gamma^1 \Vdash_{PrL} A^1$

Case ii) is rather easy, so let us do case i):
It is sufficient to provide $\Phi\colon \boldsymbol{M} \to \boldsymbol{K}$ and $\Psi\colon \boldsymbol{K} \to \boldsymbol{M}$ s.t.:

$$\Phi(M) \Vdash A \Leftrightarrow M \models A^0$$

$$K \Vdash A \Leftrightarrow \Psi(K) \models A^0.$$

For assume $\Gamma \Vdash_{FPL} A$ and $M \models \Gamma^0$. Then $\Phi(M) \Vdash \Gamma$. So $\Phi(M) \Vdash A$. Conclude: $M \models A^0$. (Clearly $\Delta \models_{PrL} B \Leftrightarrow \forall M\colon M \models \Delta \Rightarrow M \models B$.) The other direction is similar.

Now let us construct Φ, Ψ. Consider $M = \langle W, \lhd, f\rangle \in \boldsymbol{M}$. Take $\Phi(M) = M^\Phi = \langle W^\Phi, \lhd^\Phi, f^\Phi\rangle$, where:

- $W^\Phi := \{w \in W \mid \exists w' \;\; w \lhd w'\}$
- for $w, w' \in W^\Phi$: $w \lhd_\Phi w' :\Leftrightarrow w \lhd w'$
- for $w \in W^\Phi$: $f^\Phi(w) = \bigcap \{f(w') \mid w' \rhd w, w' \in W\}$.

Consider on the other hand $K = \langle W, \lhd, f\rangle \in \boldsymbol{K}$. Take $\Psi(K) = K^\Psi = \langle W^\Psi, \lhd^\Psi, f^\Psi\rangle$ where:

- $W^\Psi := W \cup (W \times \{W\})$
- for $w, w' \in W^\Psi$: $w \lhd^\Psi w' :\Leftrightarrow \big((w, w' \in W \text{ and } w \lhd w')$ or $(w = \langle w'', W\rangle$ and $(w \lhd^\Psi w''$ or $w = w''))\big)$.
- for $w \in W^\Psi$: $f^\Psi(w) = \bigcup \{f(w') \mid w' \lhd^\Psi w, w' \in W\}$.

Now it is easy to see that Φ, Ψ have the derived properties (by induction on the length of A; note that $K^{\Psi\Phi} = K$). □

5.5 DEFINITION. Let f: $\mathbf{P} \to \mathscr{L}_{PA}$. Define:

i) $(p_i)^f := (fp_i)$
$(\bot)^f := (0 = 1)$
$(A \wedge B)^f := (A^f \wedge B^f)$
$(A \vee B)^f := (A^f \vee B^f)$
$(A \to B)^f := \Box(A^f \to B^f)\big(\equiv Prov(\ulcorner A^f \to B^f \urcorner)\big)$.

ii) $(p_i)^{0f} := \Box f(p_i)$
$(\bot)^{0f} := \Box(0 = 1)$
$(A \wedge B)^{0f} := (A^{0f} \wedge B^{0f})$
$(A \vee B)^{0f} := (A^{0f} \vee B^{0f})$
$(A \to B)^{0f} := \Box(A^{0f} \to B^{0f})$

iii) Let $g(p_i) := (f(p_i) \wedge \Box f(p_i))$, then $A^{1f} := A^g$.

5.6 THEOREM. *Let $\Gamma \subseteq \mathscr{L}$ be finite and $A \in \mathscr{L}$, then:*

i) $\Gamma \vdash_{FPL} A \Leftrightarrow \forall f\colon \mathbf{P} \to \Sigma_1^0 PA + \Gamma^f \vdash A^f$

ii) $\Gamma \vdash_{FPL} A \Leftrightarrow \forall f\colon \mathbf{P} \to \mathscr{L}_{PA}\ PA + \Gamma^{0f} \vdash A^{0f}$

iii) $\Gamma \vdash_{FPL} A \Leftrightarrow \forall f\colon \mathbf{P} \to \mathscr{L}_{PA}\ PA + \Gamma^{1f} \vdash A^{1f}$.

PROOF. The proofs of ii) and iii) are by combining 5.4 with Solovay's Completeness Theorem [9]. Note that: $\Gamma \vdash_{FPL} A \Leftrightarrow \vdash_{FPL} \bigwedge \Gamma \to A$, by an easy Kripke Model proof, and $PA + \Gamma^{if} \vdash A^{if} \Leftrightarrow PA \vdash \Box(\bigwedge \Gamma^{if} \to A^{if})$ $(i = 0, 1)$. Let us turn to the proof of i).

There are standard ways of reducing certain sentences to provably (in PA) equivalent Σ_1^0 sentences. It will be convenient to forget to mention these reductions. Alternatively one can read:
'$\{A \in \mathscr{L}_{PA} \mid \exists B \in \Sigma_1^0\ PA \vdash A \leftrightarrow B\}$' for '$\Sigma_1^0$' in the statement of the theorem.

"$\Rightarrow$" By induction on the length of the proof. L uses Löb's Rule for PA. $\to$I uses the fact that for $A \in \Sigma_1^0$: $PA \vdash A \to \Box A$.

"$\Leftarrow$" Suppose $\Gamma\ non\vdash_{FPL} A$. Let K be the finite irreflexive model of 2.2 such that $K \Vdash \Gamma$ and $K\ non\Vdash A$. Without loss of generality we may assume that $W = \{1, \ldots, N\}$ and that $1 \lhd 2, \ldots, 1 \lhd N$. '$m \lhd n$' can be birepresented in PA in the obvious way.

We now turn to the proof of Solovay's Completeness Theorem (see [9]). Solovay provides a (primitive) recursive function h s.t.:

i) $PA \vdash (h\colon N \to \{0, 1, \ldots N\})$

ii) $PA \vdash \big(h(m) \neq 0 \to h(m+1) \trianglerighteq h(m)\big)$

Let $l := \lim_{m\to\infty} h(m)$. (By i), ii) l exists).

iii) $(PA + l = i)$ is consistent for $i = 0, \ldots, N$.

iv) $PA \vdash \big(l = i \to \Box(l \rhd i)\big)$ for $i = 1, \ldots N$

v) $PA \vdash (l = i \to \neg\Box\neg l = j)$, for $j \rhd i$, for $i = 1, \ldots N$.

Clearly $i \trianglelefteq l$ is provably equivalent to $\exists m\, i \trianglelefteq h(m)$, a Σ_1^0 sentence. Define:

$$g(p_j) = \begin{cases} \bigvee\{\text{“}i \trianglelefteq l\text{”} \mid i \Vdash_K p_j\} & \text{if there is such an } i \\ \text{“}0 = 1\text{”} & \text{else.} \end{cases}$$

Clearly $g(p_j)$ is Σ_1^0. We claim:

I: $i \Vdash_K A \Rightarrow PA \vdash (l = i \to A^g)$

II: i non$\Vdash_K A \Rightarrow PA \vdash (l = i \to \neg A^g)$

The proof is by induction on A, simultaneously over I, II.

Suppose $i \Vdash_K A$:

a) $A \equiv p_i$. Clearly $PA \vdash l = i \to i \trianglelefteq l$
$\to g(p_i)$

b) Then $\wedge$, $\vee$ case is trivial.

c) $A \equiv (B \to C)$. We have:
$\forall j \rhd i\, j \Vdash_K B \Rightarrow j \Vdash_K C$.

Using the induction hypothesis and the fact that there are only finitely many $j \rhd i$, we find:

$PA \vdash (\forall j \rhd i\; l = j \to (B^g \to C^g))$

(Note that we use the induction hypothesis on II for B.)

By iv): $PA \vdash (l = i \to \Box l \rhd i)$. So $PA \vdash (l = i \to \Box(B^g \to C^g))$ i.e. $PA \vdash$ $\vdash (l = i \to A^g)$.

Suppose i non$\Vdash A$:

a′) Suppose $A \equiv p_j$. Consider i' s.t. $i' \Vdash_K p_j$, then $i' \underset{\neq}{\Delta} i$. Thus $PA \vdash (i = l \to i' \underset{\neq}{\trianglelefteq} l)$. So $PA \vdash i = l \to \neg g(p_j)$. (It there is no $i' \Vdash_K p_j$ the case is trivial).

b′) The $\wedge$, $\vee$, $\bot$ cases are trivial. In the $\bot$ case we use iii).

c′) Suppose $A \equiv (B \to C)$. So there is an $i' \rhd i$ with $i' \Vdash_K B$ and i' non$\Vdash_K C$. By Induction Hypothesis and propositional logic:
$PA \vdash l = i' \to \neg(B^g \to C^g)$.
By v): $PA \vdash (l = i \to \neg \Box \neg l = i')$.
So: $PA \vdash (l = i \to \neg \Box (B^g \to C^g))$
i.e.: $PA \vdash l = i \to \neg A^g$.

We find: $PA \vdash l = 1 \to \bigwedge \Gamma^g$
and: $PA \vdash l = 1 \to \neg A^g$.

By iii): $(PA + \Gamma^g + \neg A^g)$ is consistent.

So: $PA + \Gamma^g$ non$\vdash A^g$. □

5.7 Remark. Let for f: $\mathbf{P} \to \mathscr{L}_{PA}$, $(A)^{*f}$ denote the usual interpretation of $\mathscr{L}_\Box$ in $\mathscr{L}_{PA}$. It is easily seen that the proof of 5.6. i can be adapted to give:

$$(p_i \to \Box p_i) \wedge \Box(p_i \to \Box p_i) \mid i \in N\} \vdash_{PrL} A) \Leftrightarrow (\forall f\colon P \to \Sigma_1^0 \quad PA \vdash A^{*f}).$$

5.8 Corollary. *Let* $\mathbf{C} = \{\text{“}0 = 0\text{”}\} \cup \{\text{“}\neg con^n(PA)\text{”} \mid n \in N\}$. *Then for finite* $\Gamma \subseteq \mathscr{L}$. $A \in \mathscr{L}$:

$$\Gamma \vdash_{FPL^{CL}} A \Leftrightarrow \forall f\colon \mathbf{P} \to \mathbf{C}\; PA + \Gamma^f \vdash A^f.$$

Proof: Note that $(\bot_n)^f$ is provably equivalent to $\neg con^n(PA)$ and $(\bot_\infty)^f$ with $0 = 0$. □

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Received March 13, 1980; *revised June* 2, 1980.