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Bending of Uniformly Cantilever Rectangular Plates

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Abstract

An exact solution is given for the bending of uniformly loaded rectangular cantilever plates by using the idea of generalized simply supported edge together with the method of superposition. As illustrative examples, a square plate and a rectangular plate with the ratio of the clamped edge to the neighbouring free edge equal to two are solved numerically. The results are compared with those obtained from approximate methods to confirm the validity of the method presented.

I. Introduction

Rectangular cantilever plate has one edge clamped, three edges free with two free corners. To find an exact solution, which satisfies both the differential equation and all the boundary conditions including the two free corners, has long remained one of the most difficult problems in the theory of elastic thin plates. However, due to the importance of this problem in engineering, much work has been done. For example, the method of finite difference and energy method were used to effect an approximate solution. Moreover, Fourier integral was used to solve an infinitely long rectangular cantilever plate. Probably, L. V. Kantorovich was the earliest research worker who tried to solve an uniformly loaded square cantilever plate by using his own method. C. W. MacGregor solved the rectangular cantilever plate with the clamped edge infinitely long, and a concentrated load acting on the infinitely long free edge. His solution was verified nicely by experiments. D. L. Holl used the method of finite difference to get a solution of a cantilever plate, the ratio of the clamped edge to the adjacent free edge being equal to four and a concentrated load acting on the middle of the long free edge. J. J. Jaramillo made further calculations of the infinitely cantilever plate by placing the concentrated load respectively at distance ¼, ½, ¾ of the depth of the plate. W. A. Nash also used the method of finite difference for solving an uniformly loaded rectangular cantilever plate, the ratio of the clamped edge to the adjacent free edge being equal to two. Shih Tsun-tong was the first to use the generalized variational principle for the elastic thin plate. He attempted to get a solution of the same problem solved by Nash. Later, this variational principle was also used by H. J. Plass, Jr. and others to work out a solution for an uniformly loaded square cantilever plate. Recently, with the advent of computer the method of finite elements is used to attack this old problem, although much has to be improved.

From the works mentioned above, it can be seen that up to now for a rectangular cantilever plate no exact solution is available. This paper attempts to get an exact solution of this well-known problem. For this purpose, the concept of generalized simply supported edge and the method of superposition are used. Like the case of clamped edged rectangular plates, we shall be led to series of infinite simultaneous equations to be solved.

II. Generalized Simply Supported Edges

For an ordinary simply supported edge, such as $x = a$, the corresponding boundary conditions will be $W = 0$ and $M_x = 0$. Along a generalized simply supported edge such as $x = a$, the bending moment M_x still vanishes but the deflection W does not vanish. Accordingly, when setting occurs along an ordinary simply supported edge, it becomes a generalized simply supported edge. And along both kinds of simply supported edges there will be transverse forces acting.

The merit of adopting generalized simply supported edge is very plain. To start from it, we have only to eliminate the transverse forces along the edge to fulfil the boundary conditions of the free edge.

As some preparatory work, we shall solve the following several simple problems involving generalized simply supported edges.

(1) A rectangular plate has three simply supported edges and the fourth edge $y = b$ is a generalized simply supported edge, as shown in Fig. 1. Along this edge the deflections will be expressed by the sine series:

$$(W)_{y=b}=\sum_{m=1} a_m \sin \frac{m\pi x}{a}$$

The deflection surface of the plate is

$$W=\frac{1-\mu}{2}\sum_{m=1}\frac{a_m}{\sinh \alpha_m}\left\{\left(\frac{2}{1-\mu}+\alpha_m \coth \alpha_m\right)\cdot \sinh \frac{m\pi y}{a}-\frac{m\pi y}{a}\cdot \cosh \frac{m\pi y}{a}\right\}\sin \frac{m\pi x}{a} \qquad (2.1)$$

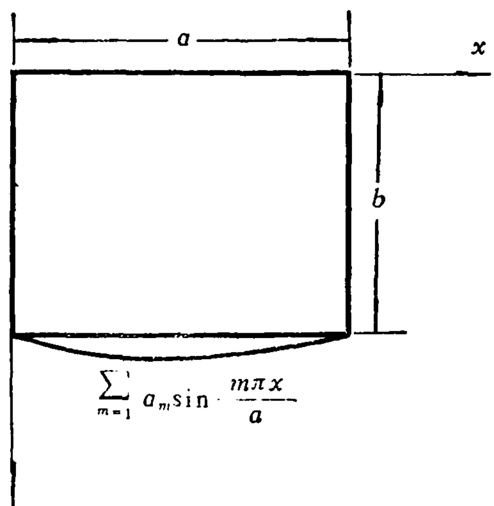

Fig 1

in which $\alpha_m = \frac{m\pi b}{a}$. Along the edge $y = b$ the transverse forces are

$$(V_y)_{y=b}=-D\left[\frac{\partial^3 W}{\partial y^3}+(2-\mu)\frac{\partial^3 W}{\partial y \partial x^2}\right]_{y=b}$$

$$=\frac{D}{2}(1-\mu)^2\sum_{m=1}\frac{m^3\pi^3}{a^3}a_m\left[\frac{3+\mu}{1-\mu}\coth \alpha_m+\frac{\alpha_m}{\sinh^2 \alpha_m}\right]\sin \frac{m\pi x}{a} \qquad (2.2)$$

Along the edge $x = a$, the transverse forces will be

$$(V_x)_{x=a}=-D\left[\frac{\partial^3 W}{\partial x^3}+(2-\mu)\frac{\partial^3 W}{\partial x \partial y^2}\right]_{x=a}$$

$$=-D\frac{1}{2}(1-\mu)^2\sum_{m=1}\frac{m^3\pi^3}{a^3}\cdot\frac{a_m}{\sinh \alpha_m}$$

$$\cdot\left[(\alpha_m \coth \alpha_m-2)\sinh \frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh \frac{m\pi y}{a}\right]\cos m\pi$$

Expressing the expression in the brackets in sine series, we get E_i

$$E_i=\frac{2}{b}\int_0^b\left[(\alpha_m \coth \alpha_m-2)\sinh \frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh \frac{m\pi y}{a}\right]\sin \frac{i\pi y}{b}\,dy$$

$$=\frac{4\sinh\alpha_m}{b}\cdot\frac{\cos i\pi}{\pi^4\left(\frac{m^2}{a^2}+\frac{i^2}{b^2}\right)^2}\cdot\frac{i^3\pi^3}{b^3}$$

Substituting this equation into the above equation, we have:

$$(V_x)_{x=a}=-D\frac{2(1-\mu)^2}{a^3}\pi^2\sum_{m=1}\frac{\alpha_m}{m}\cos m\pi\sum_{i=1}\frac{i^3\cos i\pi}{\left(\frac{b^2}{a^2}+\frac{i^2}{m^2}\right)^2}\sin\frac{i\pi y}{b} \tag{2.3}$$

Similarly, along the edge $x = 0$ we have the transverse forces:

$$(V_x)_{x=0}=-D\frac{2(1-\mu)^2}{a^3}\pi^2\sum_{m=1}\frac{\alpha_m}{m}\sum_{i=1}\frac{i^3\cos i\pi}{\left(\frac{b^2}{a^2}+\frac{i^2}{m^2}\right)^2}\sin\frac{i\pi y}{b} \tag{2.4}$$

Along the edge $y = 0$ the deflection surface has its slopes:

$$\left(\frac{\partial W}{\partial y}\right)_{y=0}=\pi\frac{1-\mu}{2a}\sum_{m=1}\frac{m a_m}{\sinh\alpha_m}\left[\frac{1+\mu}{1-\mu}+\alpha_m\coth\alpha_m\right]\sin\frac{m\pi x}{a} \tag{2.5}$$

The concentrated forces acting respectively at the corners (a, b) and $(0, b)$ are:

$$(R)_{\substack{x=a\\y=b}}=2D(1-\mu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)_{\substack{x=a\\y=b}}=D(1-\mu)^2\frac{\pi^2}{a^2}\sum_{m=1}a_m m^2\cos m\pi$$

$$\cdot\left[\frac{1+\mu}{1-\mu}\coth\alpha_m+\frac{\alpha_m}{\sinh^2\alpha_m}\right] \tag{2.6}$$

$$(R)_{\substack{x=0\\y=b}}=D(1-\mu)^2\cdot\frac{\pi^2}{a^2}\sum_{m=1}a_m m^2\left[\frac{1+\mu}{1-\mu}\coth\alpha_m+\frac{\alpha_m}{\sinh^2\alpha_m}\right] \tag{2.7}$$

When the generalized simply supported edge has its deflections symmetrical to the mid-point of the edge, we have to change $\sum_{m=1}$ to $\sum_{m=1,3,\ldots}$

(2) A rectangular plate has its two edges $y = 0, y = b$ which are simply supported and the other two edges $x = 0$, $x = a$ are generalized simply supported edges, the deflections of which are given by (Fig. 2):

$$(W)_{\substack{x=0\\x=a}}=\sum_{i=1}b_i\sin\frac{i\pi y}{b}$$

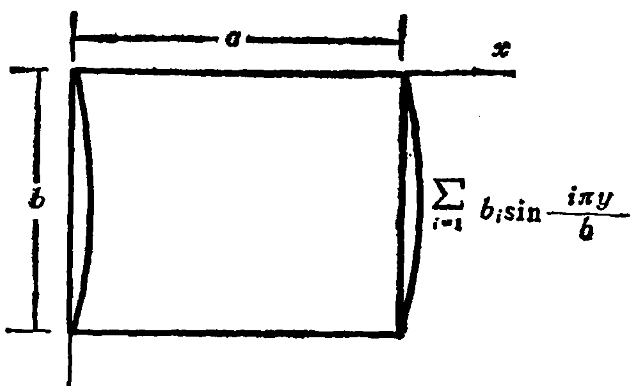

Fig 2

The deflection surface is given by:

$$W=\frac{1-\mu}{2}\sum_{i=1}b_i\left\{\frac{\cosh\beta_i-1}{\sinh\beta_i}\left[\left(\frac{\beta_i}{\sinh\beta_i}-\frac{2}{1-\mu}\right)\sinh\frac{i\pi x}{b}\right.\right.$$

$$+\frac{i\pi x}{b}\cosh\frac{i\pi x}{b}\Big]+\frac{2}{1-\mu}\cosh\frac{i\pi x}{b}-\frac{i\pi x}{b}\sinh\frac{i\pi x}{b}\Big\}\sin\frac{i\pi x}{b} \tag{2.8}$$

in which $\beta_i=\dfrac{i\pi a}{b}$.

Along the edge $x=a$, the transverse forces are equal to:

$$(V_x)_{x=a}=\frac{D}{2}(1-\mu)^2\sum_{i=1} b_i\frac{i^3\pi^3}{b^3}\cdot\frac{\cosh\beta_i-1}{\sinh\beta_i}\Big[\frac{3+\mu}{1-\mu}-\frac{\beta_i}{\sinh\beta_i}\Big]\sin\frac{i\pi y}{b} \tag{2.9}$$

Along the edge $y=b$, we have the transverse forces:

$$(V_y)_{y=b}=D\frac{4(1-\mu)^2}{b^3}\pi^2\sum_{i=1} b_i\frac{\cos i\pi}{i}\sum_{m=1,3\cdots}\frac{m^3}{\left(\dfrac{m^2}{i^2}+\dfrac{a^2}{b^2}\right)^2}\sin\frac{m\pi x}{a} \tag{2.10}$$

Along the edge $y=0$ the deflection surface has its slopes:

$$\left(\frac{\partial W}{\partial y}\right)_{y=0}=\frac{4}{b}\sum_{i=1}\frac{b_i}{i}\sum_{m=1,3\cdots} m\frac{(2-\mu)\dfrac{a^2}{b^2}+\dfrac{m^2}{i^2}}{\left(\dfrac{m^2}{i^2}+\dfrac{a^2}{b^2}\right)^2}\sin\frac{m\pi x}{a} \tag{2.11}$$

The concentrated force acting on the corner (a, b) is equal to

$$(R)_{\substack{x=a\\y=b}}=D(1-\mu)^2\frac{\pi^2}{b^2}\sum_{i=1} b_i\, i^2\Big[\frac{\cosh\beta_i-1}{\sinh\beta_i}\Big(\beta_i\coth\beta_i+\frac{1+\mu}{1-\mu}\Big)-\beta_i\Big]\cos i\pi$$

(2.12)

(3) There is a simply supported rectangular plate and along the edge $y=0$ are acting bending moments expressed by (Fig. 3):

$$M(x)=\sum_{m=1} E_m\sin\frac{m\pi x}{a}$$

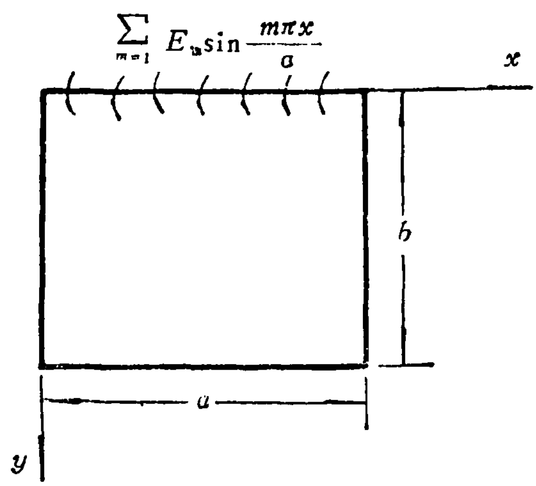

Fig 3

The deflection surface of the plate is

$$W=\frac{a^2}{2D\pi^2}\sum_{m=1}\frac{E_m}{m^2}\Big[-\frac{\alpha_m}{\sinh^2\alpha_m}\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}$$

$$+\coth \alpha_m \frac{m\pi y}{a} \cosh \frac{m\pi y}{a}\Big]\sin \frac{m\pi y}{a} \tag{2.13}$$

The transverse forces along the edge $y = b$ are:

$$(V_y)_{y=b} = -(1+\mu)\frac{\pi}{2a}\sum_{m=1}\frac{mE_m}{\sinh\alpha_m}\left[1+\frac{1-\mu}{1+\mu}\alpha_m \coth\alpha_m\right]\sin\frac{m\pi x}{a} \tag{2.14}$$

Along the edges $x = 0, x = a$ the transverse forces are:

$$(V_x)_{x=a} = \frac{2}{a}\sum_{i=1}\sum_{m=1}\frac{E_m i\left[\frac{b^2}{a^2} + (2-\mu)\frac{i^2}{m^2}\right]\cos m\pi}{m\left(\frac{b^2}{a^2}+\frac{i^2}{m^2}\right)^2}\sin\frac{i\pi x}{b} \tag{2.15}$$

$$(V_x)_{x=0} = \frac{2}{a}\sum_{i=1}\sum_{m=1}\frac{E_m i\left[\frac{b^2}{a^2} + (2-\mu)\frac{i^2}{m^2}\right]}{m\left(\frac{b^2}{a^2}+\frac{i^2}{m^2}\right)^2}\sin\frac{i\pi y}{b} \tag{2.16}$$

The deflection surface of the plate has the slope along one of the edges $y = 0$:

$$\left(\frac{\partial W}{\partial y}\right)_{y=0} = \frac{a}{2D\pi}\sum_{m=1}\frac{E_m}{m}\left[\operatorname{soth}\alpha_m - \frac{\alpha_m}{\sinh^2\alpha_m}\right]\sin\frac{m\pi x}{a} \tag{2.17}$$

The concentrated forces at the two corners $(a, b), (0, b)$ are:

$$(R)_{\substack{x=a\\y=b}} = -(1-\mu)\sum_{m=1}\frac{E_m}{\sinh\alpha_m}(\alpha_m \coth\alpha_m - 1)\cos m\pi \tag{2.18}$$

$$(R)_{\substack{x=0\\y=b}} = -(1-\mu)\sum_{m=1}\frac{E_m}{\sinh\alpha_m}(\alpha_m \coth\alpha_m - 1) \tag{2.19}$$

(4) A rectangular plate is simply supported and uniformly loaded, q being the load intensity. The transverse forces along the edge $x = a$ are:

$$(V_x)_{x=a} = -\frac{2bq}{\pi^2}\sum_{i=1,3\cdots}\frac{1}{i^2}\left|(3-\mu)\tanh\frac{\beta_i}{2} - (1-\mu)\frac{\frac{\beta_i}{2}}{\cosh^2\frac{\beta_i}{2}}\right|\sin\frac{i\pi y}{b} \tag{2.20}$$

Along the edge $y = 0$ the deflection surface of the plate has its slopes equal to:

$$\left(\frac{\partial W}{\partial y}\right)_{y=0} = \frac{2a^3q}{D\pi^4}\sum_{m=1,3\cdots}\frac{1}{m^4}\left[\tanh\frac{\alpha_m}{2} - \frac{\frac{\alpha_m}{2}}{\cosh^2\frac{\alpha_m}{2}}\right]\sin\frac{m\pi x}{a} \tag{2.21}$$

Along the edge $y = b$ the transverse forces are:

$$(V_y)_{y=b} = -\frac{2aq}{\pi^2}\sum_{m=1,3\cdots}\frac{1}{m^2}\left[(3-\mu)\tanh\frac{\alpha_m}{2} - (1-\mu)\frac{\frac{\alpha_m}{2}}{\cosh^2\frac{\alpha_m}{2}}\right]\sin\frac{m\pi x}{a}$$

The concentrated reactive force at the corner (a, b) is: (2.22)

$$(R)_{\substack{x=a\\y=b}} = \frac{4(1-\mu)}{\pi^3}qa^2\sum_{m=1,3\cdots}\frac{1}{m^3}\left(\tanh\frac{\alpha_m}{2} - \frac{\frac{\alpha_m}{2}}{\cosh^2\frac{\alpha_m}{2}}\right) \tag{2.23}$$

Having got the above four parts, we can superpose them to satisfy all the boundary conditions and the free corner conditions. However, only these four parts are not sufficient to realize the displacements of the two free corners (0, b), (a, b) as it should be. For the above four parts still keep these two corners fastened. Therefore, we have to introduce an additional part to be superposed.

Let the deflection surface be

$$W = ky \tag{2.24}$$

in which k is a constant to be determined. In fact this is a rotation of the plate as a rigid body with respect to the axis x. The angle of rotation can be expressed by:

$$\left(\frac{\partial W}{\partial y}\right)_{y=0} = k = \frac{4k}{\pi} \sum_{m=1,3\cdots} \frac{1}{m} \sin\frac{m\pi x}{a} \tag{2.25}$$

III. Uniformly Loaded Rectangular Cantilever Plates

Let there is a rectangular cantilever plate for which the edge $y = 0$ is fixed and the other three edges are free, as shown in Fig. 4. Our problem can be reduced to as follows:

Within the boundary of the plate we have to satisfy the equation

$$\nabla\nabla W = \frac{q}{D} \tag{a}$$

All the boundary conditions are:

$$(W)_{y=0} = \left(\frac{\partial W}{\partial y}\right)_{y=0} = 0 \tag{b}$$

$$\left.\begin{aligned} &\left(\frac{\partial^2 W}{\partial y^2} + \mu \frac{\partial^2 W}{\partial x^2}\right)_{y=b} = 0 \\ &\left[\frac{\partial^3 W}{\partial y^3} + (2-\mu) \frac{\partial^3 W}{\partial x^2 \partial y}\right]_{y=b} = 0 \end{aligned}\right\} \quad (c)$$

$$\left.\begin{aligned} &\left(\frac{\partial^2 W}{\partial x^2} + \mu \frac{\partial^2 W}{\partial y^2}\right)_{x=0,\ x=a} = 0 \\ &\left[\frac{\partial^3 W}{\partial x^3} + (2-\mu) \frac{\partial^3 W}{\partial x \partial y^2}\right]_{x=0,\ x=a} = 0 \end{aligned}\right\} \quad (d)$$

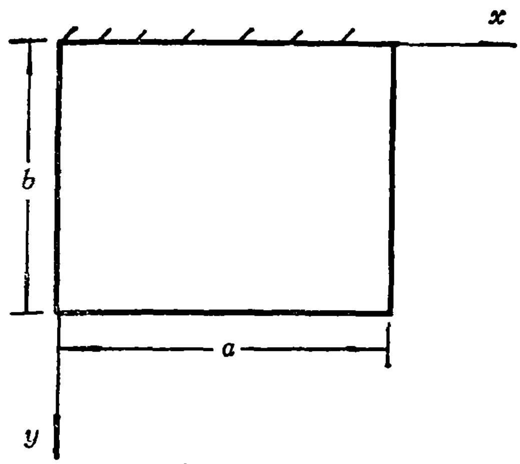

Fig. 4

The two free corners (a, b), (0, b) require

$$R = 2D(1-\mu)\left(\frac{\partial^2 W}{\partial x \partial y}\right) = 0 \tag{e}$$

The above five parts all satisfy the differential equation of the deflection surface of the plate. Now we have only to superpose them to fulfil the conditions of a clamped edge and three free edges together with two free corners. The solution thus obtained should belong to the category of exact solutions. As under uniform load the deflection surface of the plate will be symmetrical to the middle line perpendicular to the clamped edge, the lower index of a_m and E_m should be 1, 3, 5, ...

To satisfy the condition of the clamped edges *(b)*, we have to superpose the slopes given by equations (2.5), (2.11), (2.17), (2.21), (2.25) and equate their sum to zero. Then we obtain

$$(1-\mu)\frac{\pi}{4} \cdot \frac{\alpha_m}{\sinh\alpha_m}\left[\frac{1+\mu}{1-\mu} + \alpha_m \coth \alpha_m\right] + 2\frac{a}{b} \sum_{i=1} \frac{b_i}{i} \cdot \frac{\frac{m^2}{i^2} + (2-\mu)\frac{a^2}{b^2}}{\left(\frac{m^2}{i^2} + \frac{a^2}{b^2}\right)^2}$$

$$+ \frac{a^2}{4\pi D} \cdot \frac{E_m}{m^2}\left[\coth\alpha_m - \frac{\alpha_m}{\sinh^2\alpha_m}\right] + \frac{qa^4}{\pi^4 D} \cdot \frac{1}{m^5}\left[\tanh\frac{\alpha_m}{2} - \frac{\frac{\alpha_m}{2}}{\cosh^2\frac{\alpha_m}{2}}\right]$$

$$+\frac{2}{m^2}\cdot\frac{ka}{\pi}=0 \qquad m=1,\ 3,\ 5\cdots \tag{3.1}$$

To make the transverse forces along one of the edges $y = b$ vanish, the equations (2.2), (2.10), (2.14), (2.22) are superposed and equated to zero. Thus we have:

$$-\frac{qa^4}{D\pi^4}\cdot\frac{2}{m^5}\left[(3-\mu)\tanh\frac{\alpha_m}{2}-(1-\mu)\frac{\frac{\alpha_m}{2}}{\cosh^2\frac{\alpha_m}{2}}\right]$$

$$+4(1-\mu)^2\frac{a^3}{b^3}\sum_{i=1}\frac{b_i}{i}\frac{\cos i\pi}{\left(\frac{m^2}{i^2}+\frac{a^2}{b^2}\right)^2}+(1-\mu)^2\pi\frac{a_m}{2}\left[\frac{3+\mu}{1-\mu}\coth\alpha_m\right.$$

$$\left.+\frac{\alpha_m}{\sinh^2\alpha_m}\right]-(1+\mu)\frac{a^2}{2\pi D}\cdot\frac{E_m}{m^2\sinh\alpha_m}\left[1+\frac{1-\mu}{1+\mu}\alpha_m\coth\alpha_m\right]=0 \tag{3.2}$$

To make the transverse forces V_x along one of the edges $x = a$ vanish, superpose the equations of transverse forces by (2.3), (2.9), (2.15), (3.2) and equate their sum to zero. The third equation becomes:

$$-\frac{qb^4}{D\pi^4}\cdot\frac{1}{i^5}\left[(3-\mu)\tanh\frac{\beta_i}{2}-(1-\mu)\frac{\frac{\beta_i}{2}}{\cosh^2\frac{\beta_i}{2}}\right]$$

$$+(1-\mu)^2\frac{b^3}{a^3}\cos i\pi\cdot\sum_{m=1,3\cdots}\frac{a_m}{m\left(\frac{b^2}{a^2}+\frac{i^2}{m^2}\right)^2}$$

$$-\frac{1}{i^2\pi^2}\cdot\frac{b^3}{Da}\sum_{m=1,3\cdots}\frac{E_m\left[\frac{b^2}{a^2}+(2-\mu)\frac{i^2}{m^2}\right]}{m\left(\frac{b^2}{a^2}+\frac{i^2}{m^2}\right)^2}$$

$$+\frac{\pi}{4}(1-\mu)^2b_i\frac{\cosh\beta_i-1}{\sinh\beta_i}\left[\frac{3+\mu}{1-\mu}-\frac{\beta_i}{\sinh\beta_i}\right]=0 \tag{3.3}$$

In the above equation $i = 1, 2, 3, \ldots$, but for the first term when i is even, it is equal to zero. Owing to symmetry for the vanishing of transverse forces along one of the edges $x = 0$, we shall get an equation entirely the same as (3.3). As at the free corner (a, b) there is no concentrated force acting, we have to superpose the concentrated reactive forces given by equations (2.6), (2.12), (2.18), (2.23) and equate their sum to zero. Thus we obtain the following equation:

$$\frac{a^2}{b^2}\sum_{i=1}b_i i^2\cos i\pi\left[\frac{\cosh\beta_i-1}{\sinh\beta_i}\left(\beta_i\coth\beta_i+\frac{1+\mu}{1-\mu}\right)-\beta_i\right]$$

$$+\frac{1}{(1-\mu)\pi^2}\cdot\frac{a^2}{D}\sum_{m=1,3\cdots}\frac{E_m}{\sinh\alpha_m}(\alpha_m\coth\alpha_m-1)$$

$$-\sum_{m=1,3\cdots}m^2a_m\cdot\left[\frac{1+\mu}{1-\mu}\coth\alpha_m+\frac{\alpha_m}{\sinh^2\alpha_m}\right]$$

$$+\frac{4}{(1-\mu)\pi}\cdot\frac{qa^4}{D\pi^4}\sum_{m=1,3\cdots}\frac{1}{m^3}\left(\tanh\frac{\alpha_m}{2}-\frac{\frac{\alpha_m}{2}}{\cosh^2\frac{\alpha_m}{2}}\right)=0 \tag{3.4}$$

From another free corner we shall get an identical equation. In this manner, we have got three series of infinite simultaneous equations (3.1), (3.2), (3.3) and a single equation (3.4). Using them we can solve the unknown coefficients a_m, b_i, E_m, $\frac{ka}{\pi}$. As numerical examples, we shall solve two problems. One is for a uniformly loaded square cantilever plate and the other is for the edges $a : b$ equal to two.

(A) An Uniformly Loaded Square Cantilever Plate

As the coefficients E_m converge rather slowly, 24 terms are taken. However, the coefficients a_m and b_i converge very rapidly. Taking $\mu = 0.3$ from equations (3.1), (3.2), (3.3), (3.4) the computer gives:

$a_m = 0.16661 \frac{qa^4}{D\pi^4}$	0.23645×10^{-2}	0.20979×10^{-3}	0.45943×10^{-4}
0.15843×10^{-4}	0.71954×10^{-5}	0.39248×10^{-5}	0.24276×10^{-5}
0.18381×10^{-5}	0.11742×10^{-5}	0.87801×10^{-6}	0.67675×10^{-6}
0.53338×10^{-6}	0.42755×10^{-6}	0.34735×10^{-6}	0.28528×10^{-6}
0.23649×10^{-6}	0.19762×10^{-6}	0.16631×10^{-6}	0.14086×10^{-6}
0.12001×10^{-6}	0.10280×10^{-6}	0.88496×10^{-7}	0.73545×10^{-7}

$b_i = -2.1193 \frac{qa^4}{D\pi^4}$	-0.43681	-0.11579	-0.049370
-0.023495	-0.013385	-0.79574×10^{-2}	-0.52219×10^{-2}
-0.33926×10^{-2}	-0.24922×10^{-2}	-0.17934×10^{-2}	-0.12527×10^{-2}
-0.10234×10^{-2}	-0.80302×10^{-3}	-0.62996×10^{-3}	-0.50918×10^{-3}
-0.41063×10^{-3}	-0.33963×10^{-3}	-0.27992×10^{-3}	-0.23583×10^{-3}
-0.19881×10^{-3}	-0.16918×10^{-3}	-0.14379×10^{-3}	-0.12470×10^{-3}

$E_m = -65.860 \frac{qa^2}{\pi^4}$	-20.212	-10.780	-6.8540
-4.7718	-3.5163	-2.6969	-2.1325
-1.7282	-1.4297	-1.2039	-1.0298
-0.89311	-0.78423	-0.69634	-0.62455
-0.56526	-0.51583	-0.47424	-0.43896
-0.40881	-0.38285	-0.36036	-0.34075

$$\frac{ka}{\pi} = 4.0101 \frac{qa^4}{D\pi^4}$$

From the calculated results it can be seen that a_m and b_i converge very rapidly and that E_{47} is about 0.5% of E_1. Now let us calculate the deflections along the free edge $y = a$.

$$k = 4.0101 \frac{qa^3}{D\pi^3} = 0.12933 \frac{qa^3}{D}$$

The deflection curve of the free edge $y = a$ is

$$(W)_{y=a} = ka + \sum_{m=1,3\cdots} a_m \sin \frac{m\pi x}{a}$$

$$= 0.12933 \frac{qa^4}{D} + \frac{qa^4}{D\pi^4} \left\{ 0.16661 \sin \frac{\pi x}{a} + 0.0023645 \sin \frac{3\pi x}{a} \right.$$

$$+0.00020979\sin\frac{5\pi x}{a}+0.000045943\sin\frac{7\pi x}{a}+0.000015842\sin\frac{9\pi x}{a}\Big\}$$

The terms starting with 10^{-5} are all neglected.

The maximum deflection occurs at the mid-point of the edge and is equal to:

$$(W)=0.12933\frac{qa^4}{D}+\frac{qa^4}{D\pi^4}[0.16661-0.0023645+0.00020979$$

$$-0.000045943+0.000015842]=(0.12933+0.0016879)\frac{qa^4}{D}=0.13102\frac{qa^4}{D}$$

The earliest approximate result obtained by Kantorovich was $0.1192\frac{qa^4}{D}$. In the following table are given the deflections at several points along the free edge $y = a$. And for comparison, the results obtained by the method of finite elements are also listed*.

x	$0.5a$	$0.375a$	$0.25a$	$0.125a$	0
(W)	$0.13102\frac{qa^4}{D}$	0.13091	0.13056	0.12998	0.12933
F.E.M.	0.12905	0.12892	0.12851	0.12788	0.12708
Kantorovich	0.1192				0.1211

This deflected free edge is seen to be concave upwards. However, according to the solution of Kantorovich the deflections at the two corners are larger than that at the mid-point and the deflection curve of this free edge will become concave downwards.

The free edge $x = a$ will bend into a curve the equation of which is:

$$(W)_{x=a}=ky+\sum_{i=1}b_i\sin\frac{i\pi y}{a}=0.12933\frac{qa^3}{D}y-\frac{qa^4}{D\pi^4}\Big\{2.1193\sin\frac{\pi y}{a}$$

$$+0.43681\sin\frac{2\pi y}{a}+0.11580\sin\frac{3\pi y}{a}+0.04937\sin\frac{4\pi y}{a}$$

$$+0.023495\sin\frac{5\pi y}{a}+0.013385\sin\frac{6\pi y}{a}+0.0079574\sin\frac{7\pi y}{a}$$

$$+0.0052219\sin\frac{8\pi y}{a}+0.0033926\sin\frac{9\pi y}{a}+0.0024922\sin\frac{10\pi y}{a}$$

$$+0.0017935\sin\frac{11\pi y}{a}+0.0012528\sin\frac{12\pi y}{a}+0.0010234\sin\frac{13\pi y}{a}\Big\}$$

The terms of 10^{-3} are all neglected.

In the following table are given the deflections at several points along the free edge $x = a$. And for comparison the solution from the finite elements method is also tabulated.

y	0	$0.25a$	$0.5a$	$0.75a$	a
ky	0	$0.032333\frac{qa^4}{D}$	0.064666	0.096999	0.12933
$\sum_{i=1} b_i\sin\frac{i\pi y}{b}$	0	-0.020384	-0.020339	-0.011952	0
(W)	0	$0.011949\frac{qa^4}{D}$	0.044327	0.085046	0.12933

* This finite elements solution was communicated to the author by Mr. Wu Liang-tze of Peking University as a part of his research work.

F.E.M. (W)	0	0.01182	0.043221	0.083888	0.12708

The distribution of bending moments along the clamped edge is given by:

$$M(x)=\sum_{m=1,3\cdots} E_m \sin\frac{m\pi x}{a}=-\frac{qa^2}{\pi^4}\left\{65.859\sin\frac{\pi x}{a}+20.212\sin\frac{3\pi x}{a}\right.$$

$$+10.780\sin\frac{5\pi x}{a}+6.8540\sin\frac{7\pi x}{a}+4.7718\sin\frac{9\pi x}{a}+3.5163\sin\frac{11\pi x}{a}$$

$$+2.6969\sin\frac{13\pi x}{a}+2.1325\sin\frac{15\pi x}{a}+1.7282\sin\frac{17\pi x}{a}$$

$$+1.4298\sin\frac{19\pi x}{a}+1.2039\sin\frac{21\pi x}{a}+1.0297\sin\frac{23\pi x}{a}$$

$$+0.98311\sin\frac{25\pi x}{a}+0.78422\sin\frac{27\pi x}{a}+0.69633\sin\frac{29\pi x}{a}$$

$$+0.62454\sin\frac{31\pi x}{a}+0.56526\sin\frac{33\pi x}{a}+0.51583\sin\frac{35\pi x}{a}$$

$$+0.47425\sin\frac{37\pi x}{a}+0.43869\sin\frac{39\pi x}{a}+0.40881\sin\frac{41\pi x}{a}$$

$$\left.+0.38248\sin\frac{43\pi x}{a}+0.36035\sin\frac{45\pi x}{a}+0.34075\sin\frac{47\pi x}{a}\right\}$$

In the following table are given the bending moments at several points along the clamped edge.

X	$0.5a$	$0.375a$	$0.25a$	$0.125a$	$0.0625a$	$0.03125a$	0
M	$-0.53560qa^2$	-0.53550	-0.53353	-0.51270	-0.47314	-0.39115	0
F.E.M.	-0.53092	-0.53058	-0.52760	-0.50399			0.34571

As a check let us calculate the total bending moment along the clamped edge to see if it is in equilibrium statically.

$$\int_0^a M(x)dx=\int_0^a\sum_{m=1,3\cdots} E_m\sin\frac{m\pi x}{a}dx=\sum_{m=1,3\cdots}\frac{2a}{\pi}\cdot\frac{E_m}{m}=-\frac{2qa^3}{\pi^5}$$

$$\left\{65.859+\frac{1}{3}20.212+\frac{1}{5}10.780+\frac{1}{7}6.8540+\frac{1}{9}4.7718+\frac{1}{11}3.5163\right.$$

$$+\frac{1}{13}2.6960+\frac{1}{15}2.1325+\frac{1}{17}1.7282+\frac{1}{19}1.4298+\frac{1}{21}1.2039+\frac{1}{23}1.0297$$

$$+\frac{1}{25}0.89311+\frac{1}{27}0.78422+\frac{1}{29}0.69633+\frac{1}{31}0.62454+\frac{1}{33}0.56526$$

$$+\frac{1}{35}0.51583+\frac{1}{37}0.47425+\frac{1}{39}0.43896+\frac{1}{41}0.40881+\frac{1}{43}0.38284$$

$$\left.+\frac{1}{45}0.36035+\frac{1}{47}0.34075\right\}=0.50578qa^3$$

The error is $\frac{0.50578-0.5}{0.5}=1.16\%$

(B) Rectangular Cantilever Plate $\frac{a}{b} = 2$.

As the coefficients E_m converge slowly, 24 terms are taken for each coefficient. With $\mu = 0.3$, the computer gives:

$a_m = 0.019561\frac{qa^4}{D\pi^4}$	0.0016322	0.18039×10^{-3}	0.35757×10^{-4}
0.10885×10^{-4}	0.41559×10^{-5}	0.19308×10^{-5}	0.10196×10^{-5}
0.59338×10^{-6}	0.37283×10^{-6}	0.24922×10^{-6}	0.17549×10^{-6}
0.12860×10^{-6}	0.97725×10^{-7}	0.78471×10^{-7}	0.61322×10^{-7}
0.50186×10^{-7}	0.41778×10^{-7}	0.35277×10^{-7}	0.30144×10^{-7}
0.26018×10^{-7}	0.22647×10^{-7}	0.19856×10^{-7}	0.17516×10^{-7}

$b_i = -0.12897\frac{qa^4}{D\pi^4}$	-0.026367	-0.69674×10^{-2}	-0.29661×10^{-2}
-0.14090×10^{-2}	-0.80063×10^{-2}	-0.47476×10^{-3}	-0.31052×10^{-3}
-0.20701×10^{-3}	-0.14714×10^{-3}	-0.10547×10^{-3}	-0.79204×10^{-4}
-0.59652×10^{-4}	-0.46584×10^{-4}	-0.36369×10^{-4}	-0.29247×10^{-4}
-0.23465×10^{-4}	-0.19306×10^{-4}	-0.15827×10^{-4}	-0.13262×10^{-4}
-0.11064×10^{-4}	-0.94417×10^{-5}	-0.79656×10^{-5}	-0.68625×10^{-5}

$E_m = -15.980\frac{qa^2}{\pi^4}$	-5.287	-3.0173	-2.0395
-1.4958	-1.1595	-0.92960	-0.78506
-0.64253	-0.54849	-0.47458	-0.41636
-0.36717	-0.32742	-0.29426	-0.26636
-0.24262	-0.22232	-0.20484	-0.18968
-0.17647	-0.16490	-0.15472	-0.14572

$$k = 0.48604\frac{qa^3}{D\pi^3}$$

From the above coefficients it can be seen that E_{47} is less than 1% of E_1. Now let us calculate the deflections of the free edge $y = b$.

$$k = 0.48604\frac{qa^3}{D\pi^3} = 0.48604\times8\frac{qb^3}{D\pi^3} = 0.12540\frac{qb^3}{D}$$

The deflection curve of the free edge $y = b$ is:

$$(W)_{y=b} = kb + \sum_{m=1,3\cdots} a_m \sin\frac{m\pi x}{a} = 0.12540\frac{qb^4}{D} + 16\frac{qb^4}{D\pi^4}\Big\{0.019561\sin\frac{\pi x}{a}$$

$$+0.0016322\sin\frac{3\pi x}{a} + 0.00018039\sin\frac{5\pi x}{a} + 0.000035757\sin\frac{7\pi x}{a}$$

$$+0.000010885\sin\frac{9\pi x}{a}\Big\}$$

Terms starting with 10^{-5} are all neglected. The maximum deflection at the middle of the edge is equal to:

$$(W) = 0.12540\frac{qb^4}{D} + 16\frac{qb^4}{D\pi^4}\{0.019561 - 0.0016322 + 0.00018039$$

$$-0.000035757+0.000010885\}=(0.12540+0.0029704)\frac{qb^4}{D}=0.12837\frac{qb^4}{D}$$

This value is a little larger than $0.125\frac{qb^4}{D}$, when the plate with a very large width bends into a cylindrical surface. W. A. Nash obtained the corresponding value which is equal to $0.1585\frac{qb^4}{D}$ by using the method of finite difference. Later by using the method of collocation it was reduced to $0.141\frac{qb^4}{D}$. Both of them are too large. In the following table are given the deflections at several points along the free edge $y = b$, together with the results got by Nash.

x	$0.5a$	$0.375a$	$0.25a$	$0.125a$	0
(W)	$0.12837\frac{qb^4}{D}$	0.12825	0.12784	0.12691	0.12540
Nash	0.141		0.139		0.135

The distribution of bending moments along the clamped edge is:

$$M(x)=\sum_{m=1,3\cdots}E_m\sin\frac{m\pi x}{a}=-\frac{qa^2}{\pi^4}\Big\{15.980\sin\frac{\pi x}{a}+5.2857\sin\frac{3\pi x}{a}$$

$$+3.0173\sin\frac{5\pi x}{a}+2.0395\sin\frac{7\pi x}{a}+1.4985\sin\frac{9\pi x}{a}+1.1595\sin\frac{11\pi x}{a}$$

$$+0.92960\sin\frac{13\pi x}{a}+0.78506\sin\frac{15\pi x}{a}+0.64253\sin\frac{17\pi x}{a}$$

$$+0.54849\sin\frac{19\pi x}{a}+0.47458\sin\frac{21\pi x}{a}+0.41636\sin\frac{23\pi x}{a}$$

$$+0.36717\sin\frac{25\pi x}{a}+0.32742\sin\frac{27\pi x}{a}+0.29426\sin\frac{29\pi x}{a}$$

$$+0.26634\sin\frac{31\pi x}{a}+0.24262\sin\frac{33\pi x}{a}+0.22232\sin\frac{35\pi x}{a}$$

$$+0.20484\sin\frac{37\pi x}{a}+0.18968\sin\frac{39\pi x}{a}+0.17647\sin\frac{41\pi x}{a}$$

$$+0.16490\sin\frac{43\pi x}{a}+0.15472\sin\frac{45\pi x}{a}+0.14572\sin\frac{47\pi x}{a}\Big\}$$

In the following table are given the bending moments at several points along the clamped edge together with the Nash's results for comparison.

x	$0.5a$	$0.375a$	$0.25a$	$0.125a$	$0.0625a$	0
M	$0.51049qb^2$	0.51451	0.51386	0.51074	0.51472	0
Nash	0.5082		0.5047			0.4824

It can be seen that the distribution of bending moments along the clamped edge is almost uniform. As a check of the above calculation, the total bending moment will be found as follows.

$$\int_0^a M(x)dx=-\frac{2qa^3}{\pi^6}\Big\{15.980+\frac{1}{3}2.5827+\frac{1}{5}3.0173+\frac{1}{7}2.0395$$

$$+\frac{1}{9}1.4986+\frac{1}{11}1.1595+\frac{1}{13}0.9296+\frac{1}{15}0.76506+\frac{1}{17}0.64253$$

$$+\frac{1}{19}0.54849+\frac{1}{21}0.47458+\frac{1}{23}0.41536+\frac{1}{25}0.36717+\frac{1}{27}0.32742$$

$$+\frac{1}{29}0.29426+\frac{1}{31}0.26634+\frac{1}{33}0.24262+\frac{1}{35}0.22232+\frac{1}{37}0.20484$$

$$+\frac{1}{39}0.18968+\frac{1}{41}0.17647+\frac{1}{43}0.16490+\frac{1}{45}0.15472+\frac{1}{47}0.14572\Big\}$$

$$=-1.0049qb^3$$

The error is negligible.

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