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Embedding of Trees in Euclidean Spaces

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Abstract. It is proved that for any tree T the vertices of T can be placed on the surface of a sphere in R^3 in such a way that adjacent vertices have distance 1 and nonadjacent vertices have distances less than 1.

1. Introduction

The *sphericity* of a graph G, $\mathrm{sph}(G)$, is the smallest integer n such that the vertex set $V(G)$ of G can be embedded in Euclidean n-space R^n in such a way that $|u - v| < 1$ if and only if $uv \in E(G)$, where $|\ |$ is the Euclidean norm. In most cases, to determine $\mathrm{sph}(G)$ is difficult, and certain bounds on $\mathrm{sph}(G)$ for various types of graphs are considered e.g. in [1, 2, 3, 4, 5].

In [2] it was proved that $\mathrm{sph}(\bar{F}) \leq 8\lceil \log n \rceil$ for any forest F on n vertices, where $\bar{F}$ is the complement of F. This result was essentially improved in [6] to $\mathrm{sph}(\bar{F}) \leq 6$. Here we further improve this result to $\mathrm{sph}(\bar{F}) \leq 3$. We also give an example of a tree T whose complement has sphericity 3, which shows that the upper bound 3 is best possible.

2. Embeddings of Trees

Let T be a rooted tree with root u. The *level* of a vertex v of T is the number of edges in the path from the root u to v. Any vertex in that path is called an *ancestor* of v. A *proper* ancestor of v is any ancestor excluding v.

Theorem 1. *For any rooted tree T, we can embed the vertices of T in the plane R^2 in such a way that* (1) *and* (2) *hold.*
(1) *The Euclidean distances between adjacent vertices are equal to* 1.
(2) *The Euclidean distances between non-adjacent vertices of distinct levels are greater than* 1.

Proof. Let T be a rooted tree with $n + 1$ vertices. Label the vertices of T in the following way: Regarding T as a symmetric digraph, take an Euler tour starting

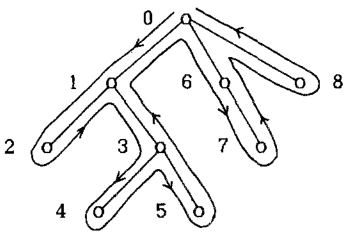

Fig. 1

from the root, and then label the vertices with 0, 1, 2, ..., n in the order of the first visit in that Euler tour, 0 is the root. See Fig. 1. Note the following fact:
(#) For two vertices b, c $(0 < b < c \le n)$, let a be their latest common ancestor, and let $ai_1 \dots i_s$ $(i_s = b)$, $aj_1 \dots j_t$ $(j_t = c)$ be the two paths from a to b, c. Then

$$a < i_1 < \cdots < i_s < j_1 < \cdots < j_t.$$

Now take n unit vectors

$$\mathbf{u}(i) = \left(\cos\frac{i\pi}{2n}, \sin\frac{i\pi}{2n}\right), \qquad i = 1, 2, \dots, n$$

in R^2. Note that all these vectors are in the positive quadrant. Define $f\colon V(T) \to R^2$ by

$$f(0) = (0,0), \qquad f(j) = \sum \mathbf{u}(i) \qquad \text{for } j > 0,$$

where the summation is taken over all the ancestors i of j other than 0. Hence, if i, j $(i < j)$ are adjacent in T, then $f(j) = f(i) + \mathbf{u}(j)$. Thus this embedding clearly satisfies the condition (1) of the theorem. To see that the condition (2) holds, let b, c $(b < c)$ be two non-adjacent vertices of distinct levels, a be their latest common ancestor, and $ai_1 \dots i_s$ $(i_s = b)$, $aj_1 \dots j_t$ $(j_t = c)$ be the two paths from a to b, c as in (#). Then it follows from (#) that for every $1 \le \eta \le s$, $1 \le \nu \le t$,

$$\mathbf{u}(i_1)\cdot\mathbf{u}(i_\eta) > \mathbf{u}(i_1)\cdot\mathbf{u}(j_\nu) > 0,$$
$$0 < \mathbf{u}(j_t)\cdot\mathbf{u}(i_\eta) < \mathbf{u}(j_t)\cdot\mathbf{u}(j_\nu).$$

Since the levels of b, c are different, we have $s \ne t$. If $s < t$ then

$$(f(j_t) - f(i_s))\cdot\mathbf{u}(j_t) = \mathbf{u}(j_t)\cdot\mathbf{u}(j_t) + \left(\sum_{\nu=1}^{t-1} \mathbf{u}(j_\nu) - \sum_{\eta=1}^{s} \mathbf{u}(i_\eta)\right)\cdot\mathbf{u}(j_t)$$
$$> 1,$$

whence $|f(j_t) - f(i_s)| > 1$. If $s > t$ then

$$(f(j_s) - f(i_t))\cdot\mathbf{u}(i_1) = \mathbf{u}(i_1)\cdot\mathbf{u}(i_1) + \left(\sum_{\eta=2}^{s} \mathbf{u}(i_\eta) - \sum_{\nu=1}^{t}\mathbf{u}(j_\nu)\right)\cdot\mathbf{u}(i_1)$$
$$> 1,$$

whence $|f(i_s) - f(j_t)| > 1$. Thus the condition (2) is also satisfied. □

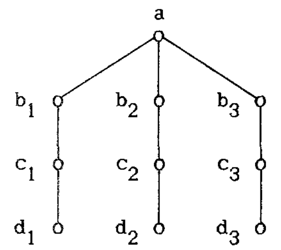

Fig. 2

Remark. Denote by S_r the sphere of radius r centered at the origin in R^3. Similarly to the above construction, we can embed the vertices of T on S_r so that the conditions (1) (2) of Theorem 1 hold, provided that r is sufficiently large.

Theorem 2. *For any tree T, there exists an embedding of the vertex set of T in R^3 such that the distances between adjacent vertices are equal to d, and the distances between non-adjacent vertices are less than d, with a fixed $d > 0$.*

Proof. Choose a vertex of T as a root. Then, as stated in the above remark, there is an embedding $f: V(T) \to S_r$ satisfying the conditions (1) (2) of Theorem 1, provided that r is sufficiently large. We may further assume that $f(V(T))$ is lying on a spherical cap of angular radius $< 30°$. Now define a new embedding $g: V(T) \to S_r$ by

$$g(v) = \begin{cases} f(v) & \text{if the level of } v \text{ is even} \\ f(v)^* & \text{if the level of v is odd,} \end{cases}$$

where $f(v)^*$ is the point antipodal to $f(v)$. If u, v are adjacent in T, then their levels are different by 1, so $|g(u) - g(y)| = ((2r)^2 - 1)^{1/2}$. Suppose now u, v are non-adjacent. If the difference of their levels is even, then $g(u)$, $g(v)$ are contained in a sperical cap of angular radius $< 30°$, and hence $|g(u) - g(v)| < r < ((2r)^2 - 1)^{1/2}$. If the difference of the levels of u and v is odd, then since $|f(u) - f(v)| > 1$, we have $|g(u) - g(v)| < ((2r)^2 - 1)^{1/2}$. Thus, letting $d = ((2r)^2 - 1)^{1/2}$, we have the theorem. □

Corollary. *For any forest F*, $\text{sph}(\bar{F}) \leq 3$.

Proof. Let F be a forest. Embed F in a tree T as an induced subgraph, and place the vertices of T in R^3 as in Theorem 2. Then, for $u, v \in V(F)$, $|u - v| = d$ if $uv \in E(F)$ and $|u - v| < d$ if $uv \notin E(F)$. This implies $\text{sph}(\bar{F}) \leq 3$ since may regard $d = 1$ by changing scale. □

3. An Example

For every tree T, let T' be the tree obtained from T by adding a new vertex v' and a new edge vv' for each vertex v of T.

Let T be the tree of Fig. 2.

Proposition. $\operatorname{sph}(\overline{T'}) = 3$.

Proof. Since $\operatorname{sph}(\overline{T'}) \leq 3$, we have only to prove $\operatorname{sph}(T') > 2$. Suppose $\operatorname{sph}(T') = 2$ and consider a placement of $V(T')$ on R^2 such that $|u - v| \geq 1$ iff $uv \in E(T')$.

Observation 1. Every vertex $v \in V(T)$ is exterior to the convex hull of $V(T) - \{v\}$.

Indeed, if $v \in V(T)$ is represented as $a_1 v_1 + \cdots + a_k v_k$ with $\sum a_i = 1$, $a_i \geq 0$ and $v \neq v_i \in V(T)$, then since $v'v_i$, $i = 1, \ldots, k$ are nonedges of T', it follows that $|v' - v_i| < 1$. Then also $|v' - v| < 1$, a contradiction for $v'v$ is an edge of T'.

Observation 2. If four points x_1, x_2, x_3, x_4 form a convex quadrilateral in R^2 such that edges x_1x_2, x_3x_4 have length ≥ 1 then at least one of the diagonals x_1x_3, x_2x_4 has length ≥ 1.

Indeed, letting y be the intersection of the two diagonals, we have

$$2 \leq |x_1 - x_2| + |x_3 - x_4| < |x_1 - y| + |x_2 - y| + |y - x_3| + |y - x_4|$$
$$= |x_1 - x_3| + |x_2 - x_4|.$$

As a corollary, we have:

(*) Suppose $v_1v_2, v_3v_4 \in E(T')$ and $v_1v_3, v_2v_4 \notin E(T')$. Then v_1, v_2, v_3, v_4 cannot form a convex quadrilateral with edges v_1v_2, v_3v_4 and diagonals v_1v_3, v_2v_4 (such a quadrilateral will be called a *forbidden quadrilateral*).

In what follows, when speaking about the string $\mathbb{S}$ of vertices of a convex polygon in R^2, $\mathbb{S}$ will be considered up to cyclic shift and inversion, and neighbors in the string will form the edges of the polygon.

Let $\mathbb{S}_6$ be the string of vertices of the convex polygon generated by b_i, c_i $(i = 1, 2, 3)$ (cf. Observation 1).

(1) $\mathbb{S}_6$ does not contain any substring of the form b_ic_i (or c_ib_i)

Indeed, otherwise choose $j \neq i$ and observe that either b_i, c_i, b_j, c_j or b_i, c_i, c_j, b_j form a forbidden quadrilateral.

(2) $\mathbb{S}_6$ does not contain any substring of the form $b_ic_kb_jb_k$ (or by symmetry, $c_kb_jc_ic_j$).

Indeed, otherwise, by (1), we have $k \neq i, j$ and c_k, b_k, c_i, b_i form a forbidden quadrilateral.

(3) If $\mathbb{S}_6$ contains a substring of the form $b_ic_kb_j$ (or by symmetry, $c_kb_jc_i$) then $\mathbb{S}_6$ is a cyclic shift of $b_ic_kb_jc_ib_kc_j$.

Indeed, by (1), $k \neq i, j$. Then $b_ic_kb_j$ cannot be extended to $b_ic_kb_jb_k$ by (2) or to $b_ic_kb_jc_j$ by (1). Thus it extends to $b_ic_kb_jc_i$ which cannot be extended to $b_ic_kb_jc_ic_j$ by (2), hence the only possibility is $b_ic_kb_jc_ib_kc_j$.

(4) If $\mathbb{S}_6$ contains a substring $b_ib_jb_k$ then it is a cyclic shift of $b_ib_jb_kc_ic_jc_k$.

In fact, $b_ib_jb_k$ cannot be extended to $b_ib_jb_kc_k$ by (1) and the extension $b_ib_jb_kc_j$ would lead to a forbidden quadrilateral $b_jc_jc_ib_i$. Thus $\mathbb{S}_6$ contains $b_ib_jb_kc_i$. Further $\mathbb{S}_6 \neq b_ib_jb_kc_ic_kc_j$ for the latter yields a forbidden quadrilateral $b_kc_kc_jb_j$. Thus $\mathbb{S}_6 = b_ib_jb_kc_ic_jc_k$.

(5) Without loss of generality, $\mathbb{S}_6$ is one of the strings

$$(\alpha)\ b_1c_2b_3c_1b_2c_3, \qquad (\beta)\ b_1b_2b_3c_1c_2c_3.$$

Consider the 7-gon $\mathbb{S}_7$ spanned by a, b_i, c_i $(i = 1, 2, 3)$.

(6) $\mathbb{S}_6$ is of the form (β).

Indeed, otherwise $\mathbb{S}_7$ contains a substring $c_k a b_i$ (or $b_1 a c_k$) and then b_j, c_j, b_i, a form a forbidden quadrilateral.

(7) $\mathbb{S}_7$ does not contain a substring $b_i a b_j$, for otherwise b_i, a, b_j, c_j form a forbidden quadrilateral.

As a corollary, we have:

(8) $\mathbb{S}_7$ is, without loss of generality, one of the strings

$$(\gamma)\ b_1 b_2 b_3 c_1 a c_2 c_3, \qquad (\delta)\ a b_1 b_2 b_3 c_1 c_2 c_3.$$

Let $\mathbb{S}_8$ be the 8-gon spanned by a, b_i, c_i, d_2 $(i = 1, 2, 3)$.

(9) Case (γ) is impossible.

In fact, if c_1, a are neighbors of d_2 in $\mathbb{S}_8$ then d_2, c_2, b_1, c_1 form a forbidden quadrilateral. If c_1, b_3 are neighbors of d_2 in $\mathbb{S}_8$ then d_2, c_2, c_3, b_3 is a forbidden quadrilateral. In all remaining cases, c_2, d_2, b_3, a form a forbidden quadrilateral.

(10) Case (δ) is impossible.

Indeed, if the neighbors of d_2 in $\mathbb{S}_8$ are a, b_1 then d_2, c_2, c_1, b_1 form a forbidden quadrilateral. In all remaining cases a, b_1, c_2, d_2 or a, b_1, d_2, c_2 form a forbidden quadrilateral.

The proof is finished. □

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