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Minimal Colorings for Properly Colored Subgraphs

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Abstract. We give conditions on the minimum number k of colors, sufficient for the existence of given types of properly edge-colored subgraphs in a k-edge-colored complete graph. The types of subgraphs we study include families of internally pairwise vertex-disjoint paths with common endpoints, hamiltonian paths and hamiltonian cycles, cycles with a given lower bound of their length, spanning trees, stars, and cliques. Throughout the paper, related conjectures are proposed.

1. Introduction

We study the problem of the existence of properly edge-colored subgraphs of various types in edge-colored complete graphs. The types of subgraphs we are interested in include families of internally pairwise vertex-disjoint paths with common endpoints, hamiltonian paths and hamiltonian cycles, cycles with a given lower bound of their length, spanning trees, stars, and cliques.

Various approaches have been proposed in the literature for this problem. The first one consists of giving algorithms which check the existence of a given properly edge-colored subgraph in a given edge-colored graph. Results in this direction are presented in [1], [2], [6], [12], [13]. In another approach, conditions of various types (such as color-degrees, structural conditions etc.) are established, sufficient for the existence of given edge-colored subgraphs [1, 3–6, 9, 11, 12, 16].

Here, we propose sufficient conditions on the *minimum number of colors* guaranteeing the existence of properly edge-colored subgraphs of various types in

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edge-colored complete graphs. Some results in this direction have been given in [18].

Notation

Formally, in what follows, unless ortherwise specified, we denote the vertex set, the edge set, the order (i.e., the number of vertices) and the number of edges of a graph G, by $V(G)$, $E(G)$ and $n(G)$, respectively. When just one graph is under discussion, we usually write V, E and n instead of $V(G)$, $E(G)$ and $n(G)$, respectively.

Let A, B be nonempty (not necessarily disjoint) subsets of V. The graph induced in G by A is denoted by $G[A]$. The set of all edges that have one endpoint in A and the other one in B is denoted by AB. If $A = \{x\}$, then for simplicity we write xB instead of $\{x\}B$.

A k-edge-coloring of G is a mapping c from E *onto* the set of "*colors*" $\{1, 2, \ldots, k\}$. For any $e \in E(G)$, $c(e)$ is the color of the edge e. Throughout this paper, "coloring" is understood to be *edge-coloring*. We let G^c denote a graph G colored by a k-coloring c. A subgraph of G^c is said to be properly colored, if any two adjacent edges of it are in different colors. A subgraph of G^c is said to be totally multicolored (for short, TMC), if any two edges of it are in different colors.

In the sequel, unless otherwise specified, the term "path" denotes *a simple path of length at least* 2.

Results

In Section 2, the types of subgraphs we study are families of internally pairwise vertex-disjoint, properly colored paths with common endpoints. We show that the minimum number k of colors ensuring the existence of at least $l + 1$ such paths between any two distinct vertices of a k-colored complete graph K_n^c is $(l + 1)n - \frac{1}{2}(l^2 + 3l)$ in case $n \geq \frac{5}{2}(l + 1)$, and it is $n + l(2l + 1)$ otherwise. We also show that for $n > 3l - 1$ the minimum number k of colors such that for any $2l$ distinct vertices $x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, y_l$ of a k-colored complete graph K_n^c there exists at least one properly colored path between x_i and y_i, for each i, $i = 1, 2, \ldots, l$, such that all these paths are pairwise internally vertex-disjoint is at least $f(n, l) = ln + \frac{1}{2}(3l^2 - 7l) + 2$.

In Section 3, the types of subgraphs we study are hamiltonian paths, hamiltonian cycles as well as cycles with a given lower bound of their length. We show that the minimum number k of colors such that every k-colored complete graph K_n^c contains a properly colored hamiltonian cycle is $\frac{1}{2}(n - 1)(n - 2) + 2$ whereas the minimum number k of colors such that every k-colored complete graph K_n^c contains a properly colored hamiltonian path is $\frac{1}{2}(n - 3)(n - 4) + 2$, provided that n is sufficiently large compared to k. We also show that for $n > l \geq 2$, if a complete graph K_n^c is colored with at least $\frac{1}{2}ln$ colors, then it contains a properly colored cycle of length at least $l + 1$.

In Section 4, the subgraph under study is the clique. We show that the minimum number k of colors such that there exists a properly colored clique on t

vertices, $t \geq 3$, in a k-colored complete graph K_n^c is $\frac{b-1}{2b}n^2 + o(n^2)$, where $b = \left\lfloor \frac{t-1}{2} \right\rfloor$.

Finally, in Section 5, the types of subgraphs we study are spanning trees as well as stars of an arbitrary order. We show that the minimum number k of colors such that there exists at least one properly colored spanning tree in a k-colored complete graph K_n^c is at most $\lceil \frac{1}{8}(n^2+1) \rceil$ and it is at least least $\frac{1}{8}(n^2 - 6n + 24)$ for n even and at least $\frac{1}{8}(n^2 - 4n + 19)$ for n odd. We also show that the minimum number k of colors such that there exists a totally multicolored spanning star in a k-colored complete graph K_n^c is $\frac{1}{2}n(n-3) + \left\lfloor \frac{n}{3} \right\rfloor + 1$. Next we show that the minimum number k of colors ensuring the existence of a totally multicolored star with r edges in a k-colored complete graph K_n^c is at least $\lfloor \frac{1}{2}(n(r-2)+4) \rfloor$ and at most $\lfloor \frac{1}{2}(n(r-2)+r+2) \rfloor$.

2. Properly Colored Paths

The main result of this section is a formula for the minimum number of colors ensuring a fixed number of internally pairwise vertex-disjoint, properly colored paths joining any two vertices.

Theorem 2.1. *The minimum number k of colors such that there exist at least $l+1$ internally pairwise vertex-disjoint properly colored paths between any two distinct vertices of a k-colored complete graph K_n^c is*

$$k = f(n,l) = \begin{cases} (l+1)n - \frac{1}{2}(l^2+3l) & \text{if } n \geq \frac{5}{2}(l+1) \\ n + l(2l+1) & \text{if } n \leq \frac{5}{2}(l+1). \end{cases}$$

Proof. First, we describe two constructions showing that k has to be at least as large as given in the above formula. For the first case, $n \geq \frac{5}{2}(l+1)$, let $V_1 = \{z_i | 1 \leq i \leq l\}$, $V_2 = \{t_i | 1 \leq i \leq n-l-2\}$, $V_3 = \{x,y\}$ be three pairwise disjoint sets of vertices and let V denote their union. Consider the complete graph K_n on V and color its edges as follows:

$$c(t_i x) = c(t_i y) = i, \qquad i = 1, 2, \ldots, n-l-2,$$
$$c(t_i t_j) = min(\{i,j\}), \qquad i,j = 1, 2, \ldots, n-l-2, \qquad i \neq j.$$

The colors of all other edges are taken to be pairwise distinct and greater than $n-l-2$. The number of colors in this coloring of K_n is

$$g_1(n,l) = (n-l-2) + \frac{1}{2}(l+2)(l+1) + (n-l-2)l = (l+1)n - \frac{1}{2}l(l+3) - 1.$$

Clearly, any properly colored path between x and y must contain at least one vertex in V_1 and therefore K_n does not contain $l+1$ internally pairwise vertex-disjoint properly colored paths between x and y.

For the second case, $n \leq \frac{5}{2}(l+1)$, let $V_1' = \{v_i | 1 \leq i \leq 2l+1\}$, $V_2' = \{v_i | 2l + 2 \leq i \leq n-2\}$, $V_3' = \{x, y\}$ be three pairwise disjoint sets of vertices and let V denote their union. Consider the complete graph K_n' on V and color its edges as follows:

$$c(v_i x) = c(v_i y) = i, \qquad i = 1, 2, \ldots, n-2,$$
$$c(xy) = n-1.$$

The colors of the edges which have both endpoints in V_1' are taken to be pairwise distinct and greater than $n-1$. For any edge $v_i v_j$, $v_i \neq v_j$, having one endpoint in V_2' and the other one in $V_2' \cup V_1'$ we set $c(v_i v_j) = min(\{i, j\})$.

The number of colors in K_n' is $g_2(n, l) = n - 1 + (2l+1)l$. As any properly colored path between x and y must contain at least two vertices in V_1', we conclude that K_n' does not contain $l+1$ such paths.

Notice that $g_1(n,l) \geq g_2(n,l)$ if, and only if, $n \geq \frac{5}{2}(l+1)$. The lower bounds follow because in order to ensure $l+1$ internally pairwise vertex-disjoint properly colored paths between x and y, the value of k can not be less than $\max\{g_1(n,l), g_2(n,l)\} + 1$.

Consider now a complete graph K_n^c on a vertex set V, with an arbitrary coloring c with at least $k = f(n,l)$ colors. Let x, y be any two distinct vertices in V. For later use, we first show that the assertion is valid for $l = 0$ and any n. Let $V = \{x, y, t_1, t_2, \ldots, t_{n-2}\}$ be the vertex set of K_n^c. Suppose there is no properly colored path between x and y. Then $c(xt_i) = c(yt_i)$ and $c(t_i t_j)$ is identical either to $c(xt_i)$ or to $c(xt_j)$, for all $i, j \in \{1, 2, \ldots, n-2\}$, $i \neq j$. It follows that the number of colors in K_n^c is $|\{c(xt_i) | i = 1, 2, \ldots, n-2\} \cup \{c(xy)\}| \leq n-1$ a contradiction to the fact that the edges of K_n^c are colored by $f(n, 0) = n$ colors.

Now we distinguish between two cases:

Case 1. There exists $z \in V$ such that xzy is a properly colored path.

We proceed by induction on l. Suppose that any K_{n-1}^c colored with at least $f(n-1, l-1)$ colors contains at least l internally pairwise vertex-disjoint alternating paths between any two distinct vertices. Since $f(n-1, l-1) = f(n,l) - (n-1)$, if the first formula applies, and $f(n-1, l-1) = f(n,l) - 4l \leq f(n,l) - (n-1)$, in the second case (assuming that $n \leq \frac{5}{2}(l+1)$ and $l \geq 1$), the induction hypothesis yields that $K_n^c - z$ contains at least l internally pairwise vertex-disjoint properly colored paths between x and y. Considering also the path xzy we conclude that K_n^c contains at least $l+1$ internally pairwise vertex-disjoint properly colored paths between x and y.

Case 2. $c(zx) = c(zy)$ for all $z \in V \backslash \{x, y\}$.

We are going to show that K_n^c contains at least $l+1$ internally pairwise vertex-disjoint properly colored paths between x and y each of which has length 3. Let r denote the number of colors in the edge set $\{xz | z \in V \backslash \{x, y\}\}$. Clearly,

$$1 \leq r \leq n-2. \tag{1}$$

In order to simplify notation we denote these colors by 1, 2, ..., r. We partition

$V\setminus\{x,y\}$ into blocks $V_1, V_2, \ldots, V_r$ according to the color in which its vertices "view" the vertex x (or y):

$$V_i = \{z \in V\setminus\{x,y\} | c(zx) = c(zy) = i\}, \qquad i = 1, 2, \ldots, r.$$

Let G denote the subgraph of K_n^c obtained by deleting the edge xy and all edges with color less than or equal to r. Clearly, the number of colors of G is at least $f(n,l) - r - 1$. Taking into account (1) we obtain that G has at least $f(n,l) - (n-1)$ colors and of course, at least so many edges. Thus, by the Erdös–Gallai theorem [8], G admits a matching of size $l + 1$. The edges of such a matching yield $l + 1$ internally pairwise vertex-disjoint properly colored paths between x and y. □

Conjecture 2.2. *Let $n > 3l - 1$ and $x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, y_l$ be any $2l$ distinct vertices of a k-colored complete graph K_n^c. The minimum number k of colors such that there exist at least one properly colored path between x_i and y_i, $i = 1, 2, \ldots, l$, such that all these paths are pairwise internally vertex disjoint is $f(n,l) = ln + \frac{1}{2}(3l^2 - 7l) + 2$.*

We describe a construction showing that k has to be at least as large as given in the above formula. Consider 4 pairwise disjoint sets:

$$V_x = \{x_i | i = 1,2,\ldots,l\},$$
$$V_y = \{y_i | i = 1,2,\ldots,l\},$$
$$V_1 = \{v_i | i = 1,2,\ldots,l-1\},$$
$$V_2 = \{w_i | i = 1,2,\ldots,n-3l+1\}.$$

Let $V = V_x \cup V_y \cup V_1 \cup V_2$ and consider a complete graph K_n with vertex set V. We define a coloring c on K_n as follows:

$$c(w_i w_j) = min(\{i,j\}), \qquad i,j = 1, 2, \ldots, n-3l+1, \qquad i \neq j.$$
$$c(w_i V_x) = c(w_i V_y) = \{i\}, \qquad i = 1, 2, \ldots, n-3l+1.$$

All other edges of K_n are colored in a totally multicolored way using new colors. The number of colors in K_n^c is

$$n - 3l + 1 + \tfrac{1}{2}(3l-1)(3l-2) + (l-1)(n-3l+1) = ln + \tfrac{1}{2}(3l^2 - 7l) + 1.$$

Any properly colored path with endpoints x_i and y_i must contain at least one vertex of V_1. Our assertion follows, as K_n^c does not contain one properly colored path between x_i and y_i for every $i = 1, 2, \ldots, l$, such that they all form a set of pairwise internally vertex-disjoint paths.

3. Properly Colored Hamiltonian Paths and Cycles

In this section we concentrate on properly colored paths and cycles of various lengths. The most interesting case is where this length is equal to the order of the graph in question, i.e., where a hamiltonian path or cycle has to be found.

Theorem 3.1. *The minimum number k of colors such that every k-colored complete graph K_n^c contains a properly colored hamiltonian cycle is* $\frac{1}{2}(n-1)(n-2)+2$.

Proof. Assigning color 1 to all edges incident to a fixed vertex $x \in V(K_n)$, and a distinct color $\neq 1$ to each edge not incident to x, we see that $\frac{1}{2}(n-1)(n-2)+1$ colors in K_n^c are not enough to ensure the existence of a properly colored hamiltonian cycle.

Suppose now that K_n is a complete graph colored with at least $\frac{1}{2}(n-1)(n-2)+2$ colors. Clearly, it has a totally multicolored spanning subgraph G with $\frac{1}{2}(n-1)(n-2)+2$ edges. Then, by Ore's theorem [14], G contains a hamiltonian cycle. This cycle is clearly totally multicolored and thus properly colored. □

Theorem 3.2. *The minimum number k of colors such that every k-colored complete graph K_n^c contains a properly colored hamiltonian path is* $\frac{1}{2}(n-3)(n-4)+2$, *provided that n is sufficently large compared to k.*

Proof. In order to show that k cannot be smaller, consider a complete graph K_{n-3} whose edges are colored with the colors 1, 2, ..., $\frac{1}{2}(n-3)(n-4)$. Extend K_{n-3} to K_n by adding three new vertices x_1, x_2, x_3 and assigning the color $\frac{1}{2}(n-3)(n-4)+1$ to all new edges. Since no new vertex can be an internal vertex of a properly colored path, K_n contains no properly colored hamiltonian path and therefore $k \geq \frac{1}{2}(n-3)(n-4)+2$.

In order to prove the converse inequality, suppose n is large and assume that at least $m = \frac{1}{2}(n-3)(n-4)+2$ colors appear in K_n^c. From each color class i we arbitrarily select an edge e_i and denote by G the spanning subgraph of K_n^c whose edge set is $\{e_i | i = 1, 2, \ldots, m\}$. Let $d_1, d_2, \ldots, d_n$ denote the degree sequence of G arranged in a non-decreasing order: $d_1 \leq d_2 \leq \cdots \leq d_n$. There are many ways to select G. We shall assume that the triple (d_1, d_2, d_3) is lexicographically maximum. Since G is totally multicolored, if it is also hamiltonian, then K_n^c will contain a properly colored hamiltonian path.

Suppose G is not hamiltonian. Then, by a theorem of Pósa [15], there exists an integer t, $1 \leq t < \frac{n}{2}$, such that G contains at least t vertices of degree at most t. Therefore $d_t \leq t$. Notice that the complement of G contains as few as $\frac{1}{2}n(n-1) - \frac{1}{2}(n-3)(n-4) - 2 = 3n - 8$ edges, therefore, $t \leq 3$ follows. Moreover, $d_4 \geq \frac{n}{3} - c$ holds for some constant c. $\left(\text{For } d_2 \leq 2 \text{ or } d_3 \leq 3, \text{ also } d_4 \geq \frac{n}{2} - c \text{ or } d_4 \geq \frac{1}{3}n - c\right.$ $\left.\text{can be shown, respectively.}\right)$ In order to simplify the notation, we assume that the vertices of K_n^c are relabelled in such a way that the vertex x_i has degree d_i in G.

Case 1. $d_2 \geq 3$ *and* $d_1 \geq 2$.

Now $t = 3$ and $d_3 = 3$. Let x_i be an arbitrary vertex not adjacent to x_3 in G. Adding the edge x_3x_i to G, we obtain by Pósa's theorem that the new graph contains a hamiltonian cycle C. Removing the edge x_3x_i from C we obtain a properly colored hamiltonian path of K_n^c.

Case 2. $d_2 \geq 3$ and $d_1 \leq 1$.

Now $G - x_1$ contains a hamiltonian cycle C. Consider the graph $H = C + x_1x_n$. The two edges of C incident to x_n are in distinct colors, and at most one of them can be in color $c(x_1x_n)$. Removing it from H we obtain a properly colored hamiltonian path of K_n^c.

Case 3. $d_2 \leq 2$, $d_3 \geq 4$ and all edges meeting $\{x_1, x_2\}$ are in the same color.

By the maximality of (d_1, d_2, d_3), the color $c(x_1x_2)$ is represented by the edge x_1x_2 in G, i.e., it does not occur on the edges of $G' = G - \{x_1, x_2\}$. Moreover, G' is hamiltonian: let C be one of its hamiltonian cycles, and choose two consecutive vertices x_i, x_j on C. Clearly $C - x_ix_j + \{x_1x_i, x_2x_j\}$ is a properly colored hamiltonian path of K_n^c.

Case 4. $d_2 \leq 2$, $d_3 \geq 4$ and at least two color classes meet $\{x_1, x_2\}$.

Choose an edge e joining $\{x_1, x_2\}$ with a hamiltonian cycle C of $G - \{x_1, x_2\}$ such that $c(e) \neq c(x_1x_2)$. As in Case 2, an edge can be removed from $C + e$ to obtain a properly colored path with one endpoint in $\{x_1, x_2\}$. This path can be completed to a properly colored hamiltonian path of K_n^c by adding the edge x_1x_2 to it.

From now on, we can assume $d_2 \leq 2$ and $d_3 \leq 3$. It follows that at least two colors meet $T = \{x_1, x_2, x_3\}$, and the graph $G' = G - T$ contains a hamiltonian cycle C (as G' has minimum degree $n - c$ for some constant c). Moreover, the colors appearing on the edges of the triangle T are represented inside T, and the color classes meeting T are not represented by the edges of G'.

Case 5. T is monochromatic.

Let x_3x_i be an edge with $c(x_3x_i) \neq c(x_1x_2)$, and x_ix_j an edge of C. Since the color $c(x_ix_j)$ is not assigned to any edge meeting $\{x_1, x_3\}$, $C - x_ix_j + \{x_1x_j, x_2x_3, x_3x_i\}$ is a properly colored hamiltonian path of K_n^c.

Case 6. T is not monochromatic.

There exists an edge in T, say x_1x_2, whose color is not repeated inside T. If all edges joining $\{x_1, x_2\}$ with C are in color $c(x_1x_2)$, then we can find a properly colored hamiltonian path P with endpoints x_1, x_2 in $G - x_3$ in the same way as described in Case 3, and $P + x_1x_3$ is a properly colored hamiltonian path of K_n^c. On the other hand, if some edge, say, x_1x_i, $i \geq 4$, has $c(x_1x_i) \neq c(x_1x_2)$, we can traverse the vertices of $C + x_1x_i$ by a properly colored path P, and $P + \{x_1x_2, x_2x_3\}$ is a properly colored hamiltonian path of K_n^c. □

It remains an open problem to determine the smallest integer n_0 such that the conclusion of Theorem 3.2 is valid for all $n \geq n_0$. To see that for $n = 4$ and $n = 5$, $\frac{1}{2}(n-3)(n-4) + 2$ colors do not suffice, one can take a triangle $x_1x_2x_3$ in color 1, joining it to a vertex x_4 with three edges in color 2, and (for $n = 5$) color all edges x_5x_i, $1 \leq i \leq 4$, in color 3.

It is interesting to see what happens if, instead of hamiltonian cycles and paths, we consider cycles and paths of a given length l, $3 \le l \le n-1$. For the case of paths, Simonovits and Sós [18] have proved that if K_n^c is colored with at least $\frac{nl}{2}$ colors and $n \ge n_0(l)$, then K_n^c admits a totally multicolored path of length $l+3$.

For the case of cycles we prove the following:

Theorem 3.3. *Let $n > l \ge 2$. If K_n^c is colored with at least $\frac{ln}{2}$ colors, then it contains a properly colored cycle of length at least $l+1$.*

Proof. For $n = l+1$, K_n^c is totally multicolored, therefore each of its hamiltonian cycles is properly colored and the assertion is obviously valid. Hence, if $n \ge l+2$ and there exists a vertex whose removal decreases the number of colors by at most $\frac{1}{2}l$, then we can apply induction on n.

Suppose, on the contrary, that for every $v \in V(K_n^c)$ there are more than $\frac{1}{2}l$ colors having the property that for each of them, say, color i, all edges of color i are incident to v. Choose a subgraph G of K_n^c such that each color occurs precisely once in G, and the largest connected component of G has as many vertices as possible. By our assumption on the vertices, G has minimum degree $\delta(G)$ at least $\frac{1}{2}(l+1)$.

Let G_1 be the largest component of G. Suppose first that $|V(G_1)| \ge l+2$. If G_1 is 2-connected, then it contains a cycle of length at least $2\delta(G_1) \ge l+1$ (see [7]), and the proof is done. Hence, assume that v is a cut-vertex of G_1. Let $P(v,u)$ and $P(v,w)$ be two nonextendible paths starting at v, where u and w belong to distinct components of $G_1 - v$. Since both u and w have at least $\frac{1}{2}(l-1)$ neighbors on the path $P = P(u,v) \cup P(u,w)$, different from v, P contains more than $l+1$ vertices. Denote by u_1, u_2 and w_1, w_2 the neighbors of u and w which are closest and second closest to u and w, respectively, on P. If $c(uu_1)$, $c(uw)$, $c(ww_1)$ are pairwise distinct then $P + uw$ is a properly colored cycle of length at least $l+2$. Otherwise, if, say, $c(uu_1) = c(uw)$, then the $u_2 - w$ segment of P together with the edges uu_2 and uw form a properly colored cycle of length at least $l+1$.

Suppose next that $|V(G_1)| \le l+1$. In this case G contains more than one component, and each of them is hamiltonian. Let H_1 and H_2 be hamiltonian cycles in G_1 and G_2 respectively, and choose two edges u_1v_1 in H_1 and u_2v_2 in H_2. If the color of u_1u_2 were present on some edge e of $H_1 \cup H_2$, then $G - e + u_1u_2$ would have a component larger than G, a contradiction. Thus, $H_1 \cup H_2 + \{u_1u_2, v_1v_2\} - \{u_1v_1, u_2v_2\}$ is a properly colored cycle of length $|V(G_1)| + |V(G_2)| \ge 2\delta(G) + 2 \ge l+3$. □

Notice that the above proof does not ensure the existence of a cycle of length exactly $l+1$. It would be interesting to see under what conditions such a cycle exists. Notice also that the lower bound of the above theorem does not seem to be optimal. In particular, in the sequel we define on a set V of n vertices two k-colored complete graphs $K_n^{c_1}$ and $K_n^{c_2}$ none of which admits a properly colored cycle of length $l+1$ whereas they both admit a cycle of length l.

Definition of $K_n^{c_1}$. Partition V into two blocks, $V_1 = \{v_i | i = 1, 2, \ldots, n-l\}$ and $V_2 = \{v_i | i = n-l+1, n-l+2, \ldots, n\}$ and color the edges as follows:

Let $c(v_iv_j) = min(\{i,j\})$, for any two distinct vertices v_i, v_j of V_1, and $c(v_iv_j) = i$, for any $v_i \in V_1$ and $v_j \in V_2$. Finally, color the edges with both endpoints in V_2 in a totally multicolored way using colors greater than $n - l$. Thus, $K_n^{c_1}$ is colored with $\frac{1}{2}l(l + 1) + n - l$ colors.

Definition of $K_n^{c_2}$. Partition V in two blocks V_1 and V_2 so that V_1 contains $n - \frac{1}{3}l$ vertices and V_2 contains $\frac{l}{3}$ vertices. All edges with both endpoints in V_1 are colored 1 and all the other edges are colored in a totally multicolored way with colors greater than 1. Thus, $K_n^{c_2}$ is colored with $\frac{1}{3}ln - \frac{1}{18}l(l + 3) + 1$ colors.

Based on the above graphs, we propose the following:

Conjecture 3.4. *Let $n > k \geq 2$ and $l \geq 4$. Assume that K_n^c is colored with at least k colors, where*

$$k = \begin{cases} \dfrac{1}{2}l(l + 1) + n - l + 1 & \text{if } n \leq \dfrac{10l^2 - 6l - 18}{6(l - 3)} \\[2ex] \dfrac{1}{3}ln - \dfrac{1}{18}l(l + 3) + 2 & \text{if } n \geq \dfrac{10l^2 - 6l - 18}{6(l - 3)}. \end{cases}$$

Then, K_n^c admits a properly colored cycle of length $l + 1$.

4. Cliques

First we state a known result [18] concering totally multicolored triangles, for which we give a new proof.

Lemma 4.1. *The minimum number k of colors such that there exists a totally multicolored triangle in a k-colored complete K_n^c is n.*

Proof. First we prove that there exists an $(n - 1)$-coloring of K_n yielding no totally multicolored triangle. Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of K_n. For $i = 1, 2, \ldots, n - 1$, color i all edges incident to v_i which are not colored so far. It is easy to see that in this coloring no totally multicolored triangle occurs.

Consider now a complete colored graph K_n^c having the property that the number of its colors is at least equal to its order. Let G be a subgraph of minimum order m having the above property. Clearly, $3 \leq m \leq n$. From the minimality of G, it follows that for any $v \in V(G)$, there exist $x, y \in V(G)$ with $c(vx) \neq c(vy)$ such that neither $c(vx)$ nor $c(vy)$ appears in $G - v$. Hence $\{v, x, y\}$ induces on G a TMC triangle (which is in fact G itself). □

Theorem 4.2. *Let $t \geq 3$ be a fixed integer and $n \to \infty$. The minimum number k of colors such that there exists a properly colored clique of order t in a k-colored complete graph K_n^c is $\dfrac{b - 1}{2b}n^2 + o(n^2)$, where $b = \lfloor \frac{1}{2}(t - 1) \rfloor$.*

Proof. Let $b = \lfloor \frac{1}{2}(t-1) \rfloor$. Color a complete graph K_n as follows: Partition its vertex set V into b blocks $V_0, V_1, \ldots, V_{b-1}$, each containing $\left\lfloor \frac{n}{b} \right\rfloor$ or $\left\lceil \frac{n}{b} \right\rceil$ vertices. For $i = 0, 1, 2, \ldots, b-1$, we color all edges with both endpoints in V_i using $|V_i| - 1$ new colors, so that no MTC triangle occurs. Further, to each edge joining two blocks V_i, V_j, $i \neq j$, we assign a new distinct color. The largest properly colored complete subgraphs in this coloring have $2b \leq t-1$ vertices, showing that fewer than $\frac{b-1}{2b} n^2 + o(n^2)$ colors are not enough to ensure a properly colored K_t.

Next, we prove that for every fixed $\varepsilon > 0$, and for n large enough with respect to t and ε, $\left(\frac{b-1}{2b} + \varepsilon\right) n^2$ colors in K_n^c are sufficient for the existence of a properly colored K_t in K_n^c. Take an edge from color class of K_n^c. In the graph of $\left(\frac{b-1}{2b} + \varepsilon\right) n^2$ edges obtained in this way, for an arbitrarily fixed s and for a large n, by the Erdös–Stone theorem [10], there exists a complete $(b+1)$-partite subgraph $K_{s,s,\ldots,s}$ with s vertices in each class. We take $s = 2^{b+1}$.

Denote by V the vertex set of $K_{s,s,\ldots,s}$ and by $V_1, V_2, \ldots, V_{b+1}$ its vertex classes. We apply the procedure that follows.

For each $i = 1, 2, \ldots, b+1$ do sequentially the following:

(1) Select arbitrarily 2^{b+1-i} pairwise disjoint pairs (u_{ij}, v_{ij}) in V_i, $j = 1, 2, \ldots, 2^{b+1-i}$.
(2) For $j = 1, 2, \ldots, 2^{b+1-i}$ delete from $K_{s,s,\ldots,s}$ the (at most one) vertex $z \in V \backslash V_i$ for which either $c(zu_{ij}) = c(u_{ij}v_{ij})$ or $c(zv_{ij}) = c(u_{ij}v_{ij})$, and if z has already been selected in a previous pair $(u_{i'j'}, v_{i'j'})$, (for some $i' < i$), then also delete the other member of its pair.

The fact that $\sum_{0 \leq p \leq q} 2^p = 2^{q+1} - 1$ ensures that it is possible to execute the above procedure and that at the end of the execution, in each V_i, at least one pair, say, (u_i, v_i), remains undeleted.

Indeed, in the beginning, V_i contains 2^{b+1} vertices, $i = 1, 2, \ldots, b+1$. In the first iteration, $i = 1$, we can easily execute instructions (1) and (2). Suppose we have executed up to the $(i-1)$-th iteration. Before executing the i-th iteration observe that at most $\sum_{b-i+2 \leq p \leq b} 2^p = 2^{b+1} - 2^{b+2-i}$ vertices have been deleted from V_i. Thus, V_i contains at least 2^{b+2-i} vertices, enough to execute instruction (1) in the i-th iteration.

On the other hand, for any $i = 1, 2, \ldots, b$, from the $(i+1)$-th iteration up to the end, due to instructions of type (2), at most $\sum_{0 \leq p \leq b-i} 2^p = 2^{b+1-i} - 1$ pairs of V_i have been deleted and thus at least one pair of V_i, say, (u_i, v_i) there remains undeleted. Note also that V_{b+1} contains one pair of vertices and no deletion of pair occurs there.

Since $K_{s,s,\ldots,s}$ was totally multicolored, the vertex set $\{u_i | 1 \leq i \leq b+1\} \cup \{v_i | 1 \leq i \leq b+1\}$ induces a properly colored complete subgraph of $2b + 2 \geq t$ vertices in K_n^c. □

We do not know whether the coloring constructed in the above proof contains the largest number of colors without a properly colored K_t. It shows, however, that the number of colors needed for a properly colored K_t is strictly larger than that for a totally multicolored $K_{\lceil (t-1)/2 \rceil}$, though the two numbers are asymptotically the same as $n \to \infty$ (cf. [9]). We should also note that if t is close to n, more precisely if $t \geq \frac{2}{3}n + 1$, then the minimum number of colors required for a properly colored K_t is the same as that for a TMC K_t. In both cases, the value is $\binom{n}{2} - (n - t)$ and the extremal construction consists of $n - t + 1$ monochromatic pairwise vertex-disjoint paths in distinct colors each of length 2 and a distinct new color for every other edge.

5. Trees and Stars

In this last section we consider properly colored spanning trees and stars. The main result is the following:

Theorem 5.1. *The minimum number k of colors such that there exists at least one properly colored spanning tree in a k-colored complete graph K_n^c is at most $\lceil \frac{1}{8}(n^2 + 1) \rceil$ and it is at least $\frac{1}{8}(n^2 - 6n + 24)$ for n even and at least $\frac{1}{8}(n^2 - 4n + 19)$ for n odd.*

Proof. We first prove the lower bound. Consider a complete graph K_n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Partition V into two blocks V_1 and V_2 with respective cardinalities $\frac{n}{2} - 1, \frac{n}{2} + 1$ if n is even, or $\frac{n-1}{2}, \frac{n+1}{2}$ if n is odd. We color K_n^c as follows: All edges with both endpoints in V_1 are colored in a totally multicolored way whereas all other edges are colored with the same new color. The number of colors used is $\frac{1}{2}\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right) + 1$ if n is even and it is $\frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) + 1$ if n is odd. Clearly, in any properly colored spanning tree, all vertices in V_2 must be matched with distinct vertices of V_1, but this is impossible because $|V_1| < |V_2|$.

We prove next the upper bound by induction on n. The assertion is obvious for $n = 3$, and it is also easy to verify for $n = 4$ (take any two disjoint edges and join them by an edge colored distinctly from each of them).

From now on we assume $n \geq 5$ and that the theorem holds for every coloring of K_{n-1} and K_{n-2}, and we will show that it holds for every coloring of K_n^c. Call a vertex v *critical* for a color i if all edges of color i are incident to v, and denote by $r(v)$ the number of colors for which v is a critical vertex. If $r(v) > 0$, then v is called a critical vertex. Moreover, a color is said to be critical if it has at least one critical vertex.

We distinguish between four cases:

Case 1. K_n^c contains a critical vertex v with $r(v) \leq \frac{1}{4}(n - 1)$.

The assertion follows by induction on n. We have at least $\lceil \frac{1}{8}(n^2+1) \rceil - \lfloor \frac{1}{4}(n-1) \rfloor \geq \lceil \frac{1}{8}((n-1)^2+1) \rceil$ colors in $K_n^c - v$; thus, by the induction hypothesis, it contains a properly colored spanning tree (of order $n-1$) which can be extended to a properly colored spanning tree of K_n^c, by adding an edge incident to v colored with a color for which v is critical.

Case 2. There are critical vertices in K_n^c, all of them satisfy $r(v) \geq \frac{1}{4}n$, but not all vertices are critical.

Let w be a non critical vertex. If $K_n^c - w$ contains a vertex v which is critical for at least $\lceil \frac{1}{2}n \rceil$ colors, then we can construct explicitly a properly colored spanning tree of K_n^c as follows: Choose a subset $V_1 = \{u_i | i = 1, 2, \ldots, t\}$ of $V(K_n^c)$ with $t \geq r(v) - 1 \geq \lceil \frac{1}{2}n \rceil - 1$ such that the edge set $\{vu_i | i = 1, 2, \ldots, t\}$ contains one edge of each color ($\neq c(vw)$) for which v is a critical vertex in $K_n^c - w$. Clearly, $|V_1 \cup \{v, w\}| \geq \frac{1}{2}n + 1$. Let $\{z_i | i = 1, 2, \ldots, s\}$ denote the set $V(K_n^c) \backslash (V_1 \cup \{v, w\})$. Clearly, $s \leq \frac{1}{2}n - 1$ and thus, $s \leq t$. Then the set of edges $\{vw\} \cup \{vu_i | i = 1, 2, \ldots, t\} \cup \{u_i z_i | i = 1, 2, \ldots, s\}$ forms a properly colored spanning tree of K_n^c.

On the other hand, if no vertex of $K_n^c - w$ is critical for more than $\lceil \frac{1}{2}n \rceil - 1$ colors, we can apply induction on n. Let v be a critical vertex. We have at least

$$\left\lceil \frac{1}{8}(n^2+1) \right\rceil - \left(\left\lceil \frac{1}{2}n \right\rceil - 1 \right) \geq \left\lceil \frac{1}{8}((n-2)^2+1) \right\rceil$$

colors in $K_n^c - \{v, w\}$ and thus, by the induction hypothesis, it contains a properly colored spanning tree (of order $n-2$). Add the edge vw and another edge incident to v colored with a color different from $c(vw)$ for which v is critical, to obtain a properly colored spanning tree of K_n^c.

Case 3. For every vertex v in K_n^c, $r(v) \geq \frac{1}{4}n$.

We construct a spanning subgraph G by selecting exactly one edge from each critical color in such a way that G has as few connected components as possible and the smallest among the orders of the components is as large as possible. Clearly, G has at most three components because the degree of every vertex in G is at least $\frac{1}{4}n$ and therefore each component contains at least $\frac{1}{4}n + 1$ vertices.

If G is connected, then obviously any one of its spanning trees is a properly colored (in fact, a totally multicolored) spanning tree also in K_n^c. Assume next G has three components G_1, G_2, G_3. Since any two components contain together at least $2(\frac{1}{4}n + 1)$ vertices, the order of each component is at most $\frac{1}{2}n - 2$. Notice that the degree of each vertex is at least half the order of its corresponding component, thus by Dirac's theorem [7], every component has a hamiltonian cycle. In particular, each G_i is 2-connected.

Now pick arbitrarily u_1 in $V(G_1)$, u_2 and u_2' in $V(G_2)$ and u_3 in $V(G_3)$, such that $u_2 \neq u_2'$. We claim that neither G_1 nor G_2 contains the color $c(u_1 u_2)$ and also neither G_2 nor G_3 contains the color $c(u_2' u_3)$. To show this, suppose that an edge xy of G_1 has color $c(u_1 u_2)$. Since G_1 has a hamiltonian cycle, $G_1 - xy$ is a connected graph and so is $(G_1 - xy + u_1 u_2) \cup G_2$, contradicting the assumption that G has the smallest possible number of components. Thus G_1 does not contain the color $c(u_1u_2)$. We can verify the other two cases in a similar way. It follows that $(G_1 + u_1u_2) \cup (G_2 + u_2'u_3) \cup G_3$ is a properly colored connected spanning subgraph from which we can easily construct a properly colored spanning tree of K_n^c.

Let us suppose therefore that G consists of two components G_1, G_2. The order n_1 of at least one component, say, G_1, is not greater than $\frac{1}{2}n$. If d is the degree of an arbitrary vertex of G_1 we have $d \geq \frac{1}{4}n$. It follows that $d \geq \frac{1}{2}n_1$. Thus, by Dirac's theorem, G_1 contains a hamiltonian cycle. Add an arbitrary edge u_1u_2 with $u_i \in V(G_i)$, $i = 1, 2$. There exists at most one edge xy of G with the same color as u_1u_2. Define $G' = G + u_1u_2$ if there is no such edge xy. If xy is in G_1, then define $G' = G + u_1u_2 - xy$. If xy is in G_2 but it is not adjacent to u_1u_2 then define $G' = G + u_1u_2$. Finally, suppose xy is in G_2 and it is adjacent to u_1u_2, say, $x = u_2$. Now consider the subgraph $H = G_2 - xy$. If H is connected, then define $G' = (G_1 + u_1u_2) \cup H$, otherwise denote the components of H by G_2' (containing x) and G_2'' (containing y). By the choice of G, G_2'' has at most as many vertices as G_1 (thus, it has at most $\frac{1}{2}n$ vertices) and therefore it contains a hamiltonian cycle. Now consider an edge $u_1'u_2'$ with $u_1' \in V(G_1)$, $u_1' \neq u_1$ and $u_2' \in V(G_2'')$. If there exists an edge e' adjacent to $u_1'u_2'$ and having the same color with it, then define $G' = (G_1 + u_1u_2) \cup G_2' \cup (G_2'' + u_1'u_2') - e'$ otherwise define $G' = (G_1 + u_1u_2) \cup G_2' \cup (G_2'' + u_1'u_2')$.

In every case, G' is a properly colored connected spanning subgraph from which we can easily construct a properly colored spanning tree of K_n^c.

Case 4. K_n^c contains no critical vertex.

First, we suppose that K_n^c contains a vertex v such that for some color i the i-degree of v is equal to one. Let w be the unique vertex of K_n^c for which $c(vw) = i$. Then, we remove w from K_n^c and the assertion follows easily by induction on n. Indeed, the number of colors in $K_n^c - w$ is $\lceil \frac{1}{8}(n^2 + 1) \rceil > \lceil \frac{1}{8}((n-1)^2 + 1) \rceil$ and thus, by the induction hypothesis, $K_n^c - w$ has a properly colored spanning tree (of order $n - 1$) which by the addition of the edge vw can be extended to a properly colored spanning tree of K_n^c.

Next, assume that for every color i the i-degree of any vertex of K_n^c is either 0 or at least 2. Then, the number of colors in any star of K_n^c cannot exceed $\lfloor \frac{1}{2}(n-1) \rfloor$. Notice that the current case can only appear for $n \geq 7$. If $n = 5$ or $n = 6$, K_n^c must contain a color class with a vertex of degree 1. In K_5^c the 10 edges are split into 4 color classes, one of which should contain at most 2 edges. In K_6^c, if each color class has size at least 3 with minimum degree 2, then the 15 edges are partitionned into 5 color classes, each of which is a monochromatic triangle. This is impossible because each vertex of K_6^c has degree 5.

Now we apply Lemma 4.1. Since $\lceil \frac{1}{8}(n^2 + 1) \rceil \geq n$ for $n \geq 7$, K_n^c contains a totally multicolored triangle with vertices, say, u, v, w. Let $c(uv) = i$, $c(vw) = j$ and $c(wu) = m$. First remove vertex u. We do not lose any color because no vertex is critical. Then remove vertex v. By the assumption, there exists a vertex t in K_n^c such that $c(wt) = j$. Therefore, we lose at most $\lfloor \frac{1}{2}(n-3) \rfloor$ colors. Now, in $K_n^c - \{u, v\}$ change the color of every edge of color j to color m. We have thus lost at most $\lfloor \frac{1}{2}(n-1) \rfloor$ colors. Thus, the number of colors in $K_n^c - \{u, v\}$ is at least $\lceil \frac{1}{8}(n^2 + 1) \rceil - \lfloor \frac{1}{2}(n-1) \rfloor \geq \lceil \frac{1}{8}((n-2)^2 + 1) \rceil$. By the induction hypothesis, $K_n^c -$

$\{u, v\}$ has a properly colored spanning tree (of order $n-2$), with respect to the new coloring. Now, in the initial coloring of this tree, the vertex w has degree 0 in at least one of the colors j, m. If the j-degree of w is 0 then add the edges wv and vu, otherwise add the edges wu and uv to obtain a properly colored spanning tree of K_n^c. □

Theorem 5.2. *The minimum number k of colors such that there exists a totally multicolored spanning star in a k-colored complete graph K_n^c is $\frac{1}{2}n(n-3) + \lfloor \frac{1}{3}n \rfloor + 1$.*

Proof. First we construct a coloring of K_n with $\frac{1}{2}n(n-3) + \lfloor \frac{1}{3}n \rfloor$ colors, containing no totally multicolored spanning star. Consider a complete graph K_n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Set $q = \left\lfloor \frac{n}{r} \right\rfloor$ and $r = n\ (mod\,3)$, $r < 3$. Let C_i denote the triangle on the vertices v_{3i-2}, v_{3i-1}, v_{3i}, $i = 1, 2, \ldots, q$. We color the edges by the coloring c: $E(K_n) \to \{1, 2, \ldots, \frac{1}{2}n(n-3) + q\}$ defined as follows: $c(C_i) = i$, $i = 1, 2, \ldots, q$. Further, if $r = 1$ then we put $c(v_n v_1) = c(v_n v_2) = q + 1$ and if $r = 2$ then we put $c(v_n v_{n-1}) = c(v_{n-1} v_1) = c(v_n v_1) = q + 1$. Next, we color all the other edges of K_n in a totally multicolored way, using new colors. There are $2q + r$ edges with a duplicate color, so the number of colors used is $\frac{1}{2}n(n-1) - (2q + r) = \frac{1}{2}n(n-3) + q$ and thus c is indeed a $\frac{1}{2}n(n-3) + \lfloor \frac{1}{3}n \rfloor$-coloring of K_n^c. Clearly, K_n^c contains no totally multicolored spanning star.

Now, let K_n be colored with $\frac{1}{2}n(n-3) + \lfloor \frac{1}{3}n \rfloor + 1$ colors. We define a totally multicolored subgraph G with $\frac{1}{2}n(n-3) + \lfloor \frac{1}{3}n \rfloor + 1$ edges choosing one edge of each color class. Then the sum of the degrees of the vertices of G is equal to $n(n-3) + 2\lfloor \frac{1}{3}n \rfloor + 2$.

If there exists a vertex v in G with degree $n-1$, then we can immediately have a totally multicolored spanning star in v.

Hence, suppose that not all vertices in G have degrees smaller than $n-1$. Denote by V_1 the set of vertices with degree $n-2$ in G. It follows that $|V_1| \geq 2\lfloor \frac{1}{3}n \rfloor + 2$, otherwise the degree sum would be too small.

If there are two vertices v and u in V_1 which are not adjacent in G, then we add the edge uv to G. Since G is a totally multicolored graph, we have a totally multicolored spanning star either in v or in u. If this is not the case, then the subgraph of G which is induced by V_1 is complete. Now we consider the set $E(\overline{G})$ of the edges of the complement of G. Each vertex of V_1 has degree 1 in $\overline{G}$ but there is no edge in $E(\overline{G})$ with both endpoints in V_1. It follows that $|E(\overline{G})| \geq 2\left\lfloor \frac{n}{3} \right\rfloor + 2$. Consequently, the number of edges of K_n is

$$|E(K_n)| = |E(G)| + |E(\overline{G})| \geq \frac{1}{2}n(n-3) + \left\lfloor \frac{1}{3}n \right\rfloor + 1 + 2\left\lfloor \frac{1}{3}n \right\rfloor + 2$$

$$\geq \frac{1}{2}n(n-1) + 1,$$

a contradiction. □

Theorem 5.3. *The minimum number k of colors ensuring the existence of a totally multicolored star with r edges in every k-colored complete graph K_n^c is at least $\lfloor \frac{1}{2}(n(r-2)+4) \rfloor$ and at most $\lfloor \frac{1}{2}(n(r-2)+r+2) \rfloor$.*

Proof. To see that $\lfloor \frac{1}{2}n(r-2) \rfloor + 1$ colors are not always sufficient, consider an $(r-2)$-regular graph G on n vertices if either n or r is even, and a graph G with $n-1$ vertices of degree $r-2$ and one vertex of degree $r-3$ respectively. (For n even, this G can be obtained as the union of $r-2$ perfect matchings in a 1-factorization of K_n. If n is odd and r is even, G can be the union of $\frac{r}{2}-1$ hamiltonian cycles in a hamiltonian decomposition of K_n. Finally, if both n and r are odd, G can be the union of $\frac{1}{2}(r-3)$ hamiltonian cycles and a maximum matching of another hamiltonian cycle). Coloring all edges in the complement of G with the same (new) color, the assertion follows.

Suppose now that K_n^c contains no totally multicolored star with r edges. Let v be an arbitrarily chosen vertex, and for each color not incident to v, choose an edge in $K_n^c - v$. The graph obtained in $K_n^c - v$ has maximum degree at most $r-2$, therefore the number of colors in K_n^c cannot exceed $\lfloor \frac{1}{2}(n-1)(r-2) \rfloor + r - 1 = \lfloor \frac{1}{2}(n(r-2)+r) \rfloor$. □

We conjecture that the lower bound is the best possible.

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