Three-Dimensional Nonlinear Vibrations of Composite Beams – I. Equations of Motion

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Abstract. Newton's second law is used to develop the nonlinear equations describing the extensional-flexural-flexural-torsional vibrations of slewing or rotating metallic and composite beams. Three consecutive Euler angles are used to relate the deformed and undeformed states. Because the twisting-related Euler angle ϕ is not an independent Lagrangian coordinate, twisting curvature is used to define the twist angle, and the resulting equations of motion are symmetric and independent of the rotation sequence of the Euler angles. The equations of motion are valid for extensional, inextensional, uniform and nonuniform, metallic and composite beams. The equations contain structural coupling terms and quadratic and cubic nonlinearities due to curvature and inertia. Some comparisons with other derivations are made, and the characteristics of the modeling are addressed. The second part of the paper will present a nonlinear analysis of a symmetric angle-ply graphite-epoxy beam exhibiting bending-twisting coupling and a two-to-one internal resonance.

Key words: Composite beams, flexural-flexural-torsional-extensional vibrations, nonlinear equations of motion.

1. Introduction

The dynamics and control of flexible beams are of interest in connection with helicopter rotor blades, prop-fan blades, aviation propeller blades, wind-turbine blades, lengthy robot manipulators, large-space structures, and other systems that perform large and/or complex motions.

In the development of nonlinear equations of motion for flexible beams, three successive Euler-like rotations are commonly used to relate the deformed and undeformed states [1–14]. But different sequences of Euler rotations will result in different equations of motion [1]. Moreover, the equations are not symmetric because the twist-related Euler angle ϕ is not an independent Lagrangian coordinate [2, 3]. Hodges and Dowell [2] developed a comprehensive set of equations with quadratic nonlinearities to describe the dynamics of beams. They also identified several nonlinear terms. To evaluate the theory of Hodges and Dowell [2], Dowell, Traybar, and Hodges [4] carried out an experiment on a simple, non-rotating beam with a tip weight. The results show that there are systematic differences between theory and experiment when tip deflections become large because the equations of motion include only quadratic nonlinearities and are expanded about the undeformed rather than the deformed position [5]. Hodges *et al.* [6] stated that the nonlinear equations governing flexural deformations are inconsistent unless they include all nonlinear terms to the same order. Rosen *et al.* [3, 7] derived a set of equations that is more accurate than those of Hodges and Dowell [2] by including some nonlinear terms with order higher than two. Their numerical results are in good agreement with the experimental data of [4]. Crespo da Silva and Glynn [8] and Crespo da Silva [9] used the extended Hamilton's principle to derive mathematically consistent, nonlinear third-order equations of motion for inextensional and extensional isotropic beams, respectively.

Minguet and Dugundji [13] used Newton's second law and Euler rotations to derive the equations of motion for composite beams, but they solved the nonlinear trigonometric equations

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directly for arbitrary large static deflections. Sato, Saito, and Otomi [15] treated parametric vibrations of a simply supported horizontal beam carrying a concentrated mass under the influence of gravity. The nonlinear response of a slender cantilever beam carrying a lumped mass to a principal parametric base excitation was investigated theoretically and experimentally by Zavodney and Nayfeh [16]. Hinnant and Hodges [17] combined multibody and finite-element technology to analyze the nonlinear vibrations of a cantilever beam. Their results show good agreement with the experimental data of [4].

Because of their high strength-to-weight ratio, long fatigue life, resistance to corrosion, high damping, structural simplicity, and possible use for aeroelastic tailoring, advanced laminated structures made of fiber-reinforced composite materials, such as boron-epoxy, graphite-epoxy, and boron-aluminum, have emerged as primary materials for rotor blades of a new generation of helicopters and for other advanced aerospace vehicles and are showing great promise for improved performance. Adams *et al.* [18] stated that shear deformation can be neglected for isotropic beams if the aspect ratio L/h (i.e., length/thickness) > 20; but for unidirectional carbon-fiber-reinforced plastics, an aspect ratio the order of 100 is necessary if shear effects are to be ignored. Rao *et al.* [19] stated that when $L/r > 100$, where r denotes the radius of gyration, the shear effect and rotary inertia are negligible. Kapania and Raciti [20] indicated that, for thin composite plates whose length/thickness is greater than 50, transverse shear effects can be neglected in the large deflection theory. One important elastic behavior of composite structures is the elastic coupling among extensional, bending, and torsional stiffnesses, which is desirable in some cases. For example, the extension-twisting coupling produces a different twist distribution when a two-speed rotor of a helicopter is rotating at different speeds. Moreover, the bending-twisting coupling produces a pitch-flap stability of helicopter rotor blades. But, for a composite structural element, elastic couplings may result in complicated vibrations, especially in flexible structures. Hence, developing a precise nonlinear modeling of composite beams has become an important goal of research in recent years.

The purpose of the present paper is to generalize the equations derived by Crespo da Silva and Glynn [8] for the case of isotropic beams to the case of composite beams. We use Newton's second law to develop the nonlinear equations governing extensional-flexural-flexural-torsional vibrations of a slewing or rotating beam. Three consecutive Euler angles are used to relate the deformed and undeformed states. Moreover, the definition of twisting curvature is used to define the twist angle. The resulting equations of motion are symmetric and independent of the rotation sequence of the Euler angles.

2. Derivation of the Equations of Motion

For a long slender beam with small strains and moderate deflections and rotations, the following assumptions are usually true: (a) warping, shear deformation, and Poisson effect are negligible; and (b) bending and twisting moments at any arbitrary position along the beam are proportional, respectively, to the nonlinear expressions of the local bending curvature and twist rate. Although warping of the cross-section is neglected, its influence on the torsional rigidity can be accounted for by using the theory of elasticity [21]. To simplify the analysis, we assume that the beam is initially straight and has closed cross-sections.

Following Crespo da Silva and Glynn [8], we introduce two coordinate systems, as shown in Figure 1. The x-y-z coordinate system is fixed to the hub, and we will call it the reference

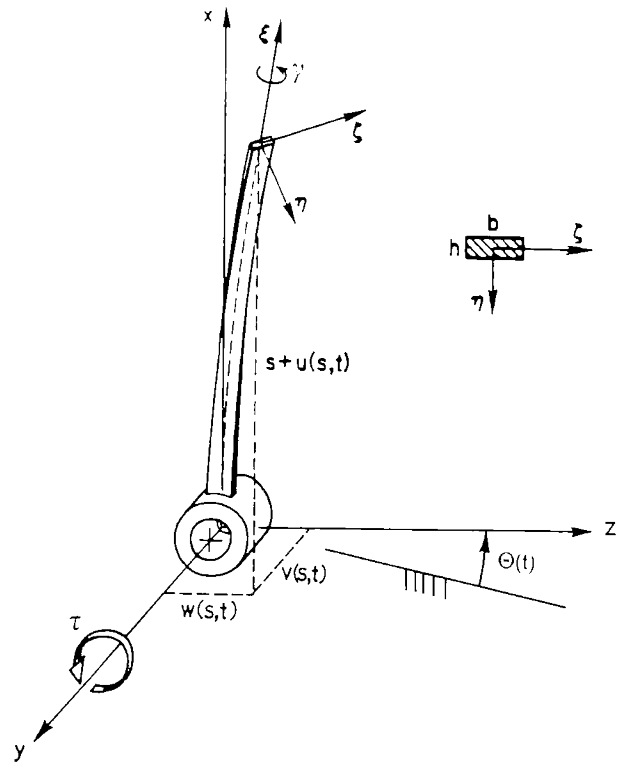

Fig. 1. Coordinate systems: x-y-z = reference frame, which is fixed on the hub; $\xi - \eta - \zeta$ coincides with the principal axes of the cross-section, which is a local frame fixed on the cross-section.

coordinate system. The $\xi - \eta - \zeta$ system, which will be called the local coordinate system, coincides with the principal axes of the observed differential element. Here we use three consecutive Euler angles ψ, θ, and ϕ to describe the rotation from the undeformed position to the deformed one. The rotations are executed in the order shown in Figure 2: first, ψ around the z-axis; second, θ around the y_1-axis; and last, ϕ around the ξ-axis. In other words, we assume deformations in the following sequence: u, v, w, and ϕ. This sequence is only tied to the mathematical modeling.

It follows from Figure 2 that the transformation that relates the undeformed coordinate system x-y-z to the deformed coordinate system $\xi - \eta - \zeta$ is

$$\begin{Bmatrix} \mathbf{i}_\xi \\ \mathbf{i}_\eta \\ \mathbf{i}_\zeta \end{Bmatrix} = [T] \begin{Bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{Bmatrix}, \qquad (1)$$

where $\mathbf{i}_\xi$, $\mathbf{i}_\eta$, $\mathbf{i}_\zeta$ are base vectors of the local coordinate system, $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ are base vectors of the reference coordinate system, and the transformation matrix $[T]$ is given by

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (2)$$

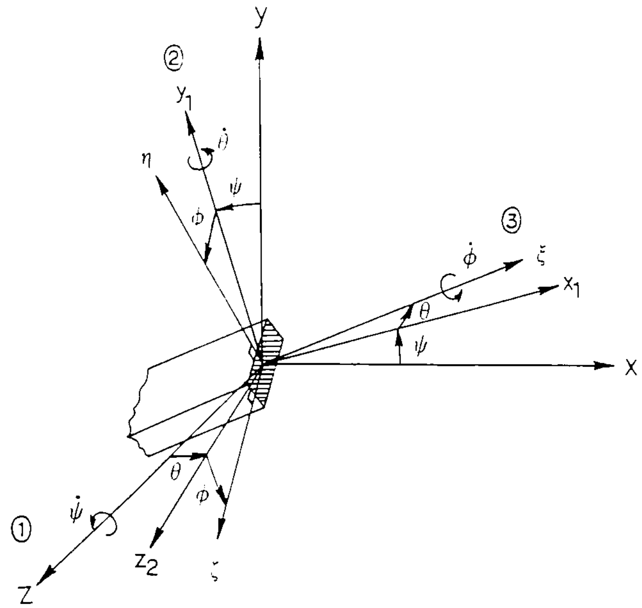

Fig. 2. Three successive Euler angle rotations of a differential beam element.

According to transformation theory,

$$[T]^{-1} = [T]^{t} . \tag{3}$$

To avoid the calculation of comoving derivatives and troubles in evaluating Euler strains in the derivation, we use the Lagrangian coordinate s, which is the distance from the origin of the x-y-z coordinate to the undeformed position of the observed element.

To obtain the relation between the curvatures and the Euler angles, we use Kirchhoff's kinetic analogy [22]. It follows from equation (1) and Figures 1 and 2 that the angular velocity of the element is

$$\boldsymbol{\omega}(s, t) = \dot{\psi}\mathbf{i}_z + \dot{\theta}\mathbf{i}_{y_1} + \dot{\phi}\mathbf{i}_\xi + \dot{\Theta}\mathbf{i}_x = \omega_\xi \mathbf{i}_\xi + \omega_\eta \mathbf{i}_\eta + \omega_\zeta \mathbf{i}_\zeta , \tag{4}$$

where

$$\omega_\xi = \dot{\phi} - \dot{\psi} \sin\theta + \dot{\Theta} \sin\psi \cos\theta , \tag{5}$$

$$\omega_\eta = \dot{\psi} \cos\theta \sin\phi + \dot{\theta} \cos\phi + \dot{\Theta} \cos\psi \cos\phi + \dot{\Theta} \sin\psi \sin\theta \sin\phi , \tag{6}$$

$$\omega_\zeta = \dot{\psi} \cos\theta \cos\phi - \dot{\theta} \sin\phi - \dot{\Theta} \cos\psi \sin\phi + \dot{\Theta} \sin\psi \sin\theta \cos\phi , \tag{7}$$

and the dot stands for $\partial/\partial t$. According to Kirchhoff's kinetic analogy [22], the curvatures can be obtained from equations (5)–(7) by replacing $\partial/\partial t$ with $\partial/\partial \bar{s}$, where $\bar{s}$ is the deformed arc length measured from the origin of the x-y-z coordinate system to the deformed position of the observed element, and putting $\Theta = 0$ (because $\Theta \neq 0$ corresponds to a rigid-body motion). Hence the twisting curvature $\tilde{\rho}_\xi$ and bending curvatures $\tilde{\rho}_\eta$ and $\tilde{\rho}_\zeta$ are given by

$$\tilde{\rho}_\xi = \phi^{+} - \psi^{+} \sin\theta , \tag{8}$$

$$\tilde{\rho}_\eta = \psi^{+} \cos\theta \sin\phi + \theta^{+} \cos\phi , \tag{9}$$

$$\tilde{\rho}_\zeta = \psi^{+} \cos\theta \cos\phi - \theta^{+} \sin\phi , \tag{10}$$

where the plus stands for $\partial/\partial\tilde{s}$. If e denotes the normal strain along the ξ-axis, we have

$$\mathrm{d}\tilde{s} = (1 + e)\,\mathrm{d}s . \tag{11}$$

Moreover, letting

$$\rho_i = \tilde{\rho}_i(1 + e) \tag{12}$$

and substituting equations (11) and (12) into equations (8)–(10), we obtain

$$\rho_\xi = \phi' - \psi' \sin\theta , \tag{13}$$

$$\rho_\eta = \psi' \cos\theta \sin\phi + \theta' \cos\phi , \tag{14}$$

$$\rho_\zeta = \psi' \cos\theta \cos\phi - \theta' \sin\phi , \tag{15}$$

where the prime denotes $\partial/\partial s$. We note that the ρ_i are 'normalized' and not real curvatures unless the beam is inextensional.

Using the transport theorem, Kirchhoff's kinetic analogy, equation (1), and the fact that the base vectors along the coordinates x, y and z are independent of $\tilde{s}$, we obtain

$$\frac{\partial[T]}{\partial\tilde{s}} = [\tilde{K}][T] , \quad \frac{\partial[T]}{\partial s} = [K][T] , \tag{16}$$

where the curvature matrix $[\tilde{K}]$ is given by

$$[\tilde{K}] \equiv \begin{bmatrix} 0 & \tilde{\rho}_\zeta & -\tilde{\rho}_\eta \\ -\tilde{\rho}_\zeta & 0 & \tilde{\rho}_\xi \\ \tilde{\rho}_\eta & -\tilde{\rho}_\xi & 0 \end{bmatrix} . \tag{17}$$

In the derivation that follows, the following nomenclature will be used:

- $\{A\}$: rotary inertia per unit of undeformed length, defined in local frame, $\{A\} = \{A_\xi A_\eta A_\zeta\}^T$.
- $\{a\}$: accelerations defined in reference frame, $\{a\} = \{a_x a_y a_z\}^T$.
- $[C_1]$: translational damping matrix of a unit of undeformed length, defined in reference frame.
- $[C_2]$: shear damping matrix of a unit of undeformed length, defined in local frame (i.e., twisting and shear dampings).
- $\{d\}$: displacements defined in reference frame, $\{d\} = \{uvw\}^T$.
- $\{F\}$: internal resultant forces defined in local frame, $\{F\} = \{F_1 F_2 F_3\}^T$.

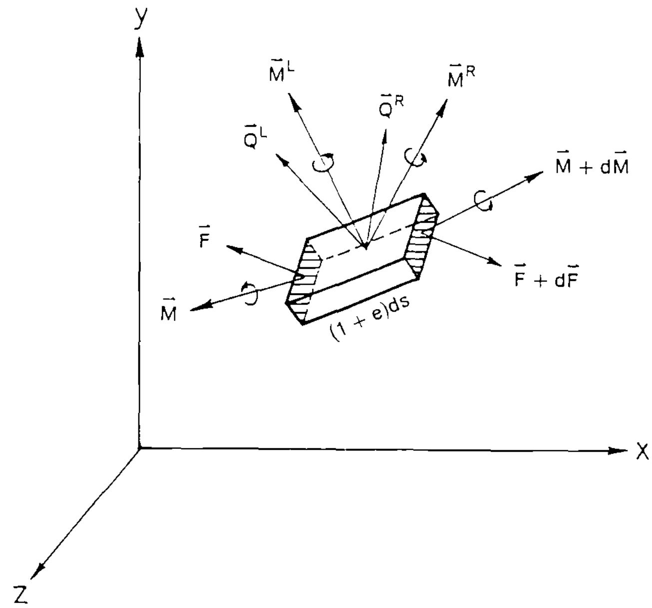

Fig. 3. Free-body diagram of a differential beam element.

$\{M\}$: internal resultant moments defined in local frame, $\{M\} = \{M_1 M_2 M_3\} T$.
$\{M^L\}$: load per unit of undeformed length, measured in local frame.
$\{M^R\}$: load per unit of undeformed length, measured in reference frame.
m : mass per unit of undeformed length.
$\{Q^L\}$: load per unit of undeformed length, measured in local frame (e.g., aerodynamic loads).
$\{Q^R\}$: load per unit of undeformed length, measured in reference frame (e.g., gravity).
$\{\omega\}$: angular velocity defined in local frame.

It follows from the free-body diagram in Figure 3, Newton's second law, and equations (11), (12), and (16) that, in the reference frame, the force-equilibrium equation is

$$\frac{\partial}{\partial s}\{[T]^T\{F\}\} + [T]^T\{Q^L\} + \{Q^R\} = m\{a\} + [C_1]\{\dot{d}\} , \tag{18}$$

and the moment-equilibrium equation, in the local frame, is

$$\frac{\partial\{M\}}{\partial s} + [K]^T\{M\} + \{(1+e)\boldsymbol{i}_\xi \times \mathbf{F}\} + \{M^L\} + [T]\{M^R\} = \{A\} + [C_2]\{\omega\} . \tag{19}$$

To complete the formulation, we need to express $\{a\}$, $\{A\}$, $\{F\}$, and $\{M\}$ in terms of u, v, w, ψ, θ, and ϕ.

Because the hub can only rotate, the acceleration of the observed element is given by the formula

$$\mathbf{a} = a_x\mathbf{i}_x + a_y\mathbf{i}_y + a_z\mathbf{i}_z , \tag{20}$$

where

$$a_x = \ddot{\Theta} w - \dot{\Theta}^2 (s + u) + 2\dot{\Theta}\dot{w} + \ddot{u} , \tag{21}$$

$$a_y = \ddot{v} , \tag{22}$$

$$a_z = -\ddot{\Theta}(s + u) - \dot{\Theta}^2 w - 2\dot{u}\dot{\Theta} + \ddot{w} . \tag{23}$$

To calculate the rotary inertias A_ξ, A_η, and A_ζ, we use the Euler equations

$$A_\xi = j_\xi \dot{\omega}_\xi - (j_\eta - j_\zeta)\omega_\eta \omega_\zeta , \tag{24}$$

$$A_\eta = j_\eta \dot{\omega}_\eta - (j_\zeta - j_\xi)\omega_\zeta \omega_\xi , \tag{25}$$

$$A_\zeta = j_\zeta \dot{\omega}_\zeta - (j_\xi - j_\eta)\omega_\xi \omega_\eta , \tag{26}$$

where j_ξ, j_η, and j_ζ are the rotary intertias per unit length of the beam, and $\boldsymbol{\omega}$ is expressed in terms of the Euler angles as shown in equations (5)–(7). Hence, $\{\mathbf{A}\}$ can be represented in terms of ψ, θ, and ϕ and their derivatives.

The last step to obtain a set of complete equations of motion is to relate $\{F\}$ and $\{M\}$ to the three displacements and three rotations. In other words, we need constitutive equations and strain-displacement relations. The constitutive equations relate the three resultant forces and moments to the six strains – one axial strain, two shear strains, one twisting curvature, and two bending curvatures. And, the strain-displacement relations relate the six strains to the three displacements and three rotations.

For the constitutive equations, we include all possible structural couplings. Hence, we express the constitutive equations in the most general form

$$\{F_1 F_2 F_3 M_1 M_2 M_3\}^T = [S]\{e \gamma_{\xi\eta} \gamma_{\xi\zeta} \tilde{\rho}_\xi \rho_\eta \rho_\zeta\}^T , \tag{27}$$

where the stiffness matrix $[S]$ is given by

$$[S] \equiv \begin{bmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ A_{12} & A_{22} & A_{23} & B_{21} & B_{22} & B_{23} \\ A_{13} & A_{23} & A_{33} & B_{31} & B_{32} & B_{33} \\ B_{11} & B_{21} & B_{31} & D_{11} & D_{12} & D_{13} \\ B_{12} & B_{22} & B_{32} & D_{12} & D_{22} & D_{23} \\ B_{13} & B_{23} & B_{33} & D_{13} & D_{23} & D_{33} \end{bmatrix} = \begin{bmatrix} [A] & [B] \\ [B]^T & [D] \end{bmatrix} , \tag{28}$$

and $\gamma_{\xi\eta}$ and $\gamma_{\xi\zeta}$ are the averaged shear strains, which are the same as those used in Timoshenko's beam theory. We note that the submatrices $[A]$, $[B]$, and $[D]$ are not the same as those used in the classical laminated theory (CLT) of plates [23]. In CLT, ρ_η and $\gamma_{\xi\eta}$ are assumed to be zeros, and the bending curvature in the $\eta - \zeta$ plane is not zero, which is assumed to be zero for the beam theory we develop here. Berdichevskii [24] shows that the geometrically nonlinear problem of beams can be decoupled into a nonlinear one-dimensional vibration problem and a two-dimensional section problem. To obtain analytically the stiffness matrices $[A]$, $[B]$, and $[D]$ by solving the two-dimensional section problem, one can use theories [13, 25] similar to the classical laminated plate theory when the cross section is simple, or finite-element methods [26, 27] when the cross-section is complicated.

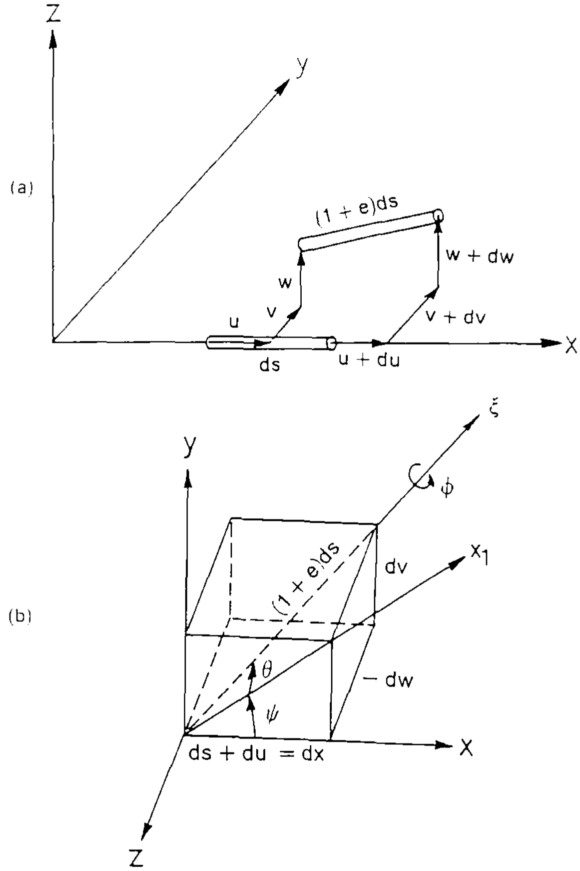

Fig. 4. (a) Transformation geometry, (b) relationship between displacements and Euler angles.

The strain-displacement relations relate the curvatures to the Euler angles as in equations (8)–(10) and (12)–(15). It follows from Figure 4(a) and transformation theory that the relationship between the axial strain e and the displacements is

$$e = \sqrt{(1 + u')^2 + v'^2 + w'^2} - 1 . \tag{29}$$

The ξ-axis is chosen as the neutral axis of the beam and perpendicular to the cross-section. Hence ψ and θ can be represented (see Figure 4(b)) in terms of the derivatives of the displacements as

$$\cos\psi = \frac{1 + u'}{\sqrt{(1 + u')^2 + v'^2}} , \quad \sin\psi = \frac{v'}{\sqrt{(1 + u')^2 + v'^2}} , \tag{30}$$

$$\cos\theta = \sqrt{\frac{(1 + u')^2 + v'^2}{(1 + u')^2 + v'^2 + w'^2}} , \quad \sin\theta = \frac{-w'}{\sqrt{(1 + u')^2 + v'^2 + w'^2}} . \tag{31}$$

With these equations, the unknown variables are reduced to four; namely, u, v, w, and ϕ, which means that only four of equations (18) and (19) are independent. The other two give expressions for the internal shear forces in terms of these four displacements, as shown later.

Because the shear strains are functions of the local coordinates (η, ζ) on the cross-section, the shear effect can only be included in the one-dimensional theory developed here in an averaged sense, as in Timoshenko's beam theory. When the shear strains $\gamma_{\xi\eta}$ and $\gamma_{\xi\zeta}$ are included, the ξ-axis can be chosen as the neutral axis, and a shear coefficient [28] must be introudeced, and the shear strains must be included in the expressions of $\{\omega\}$, $\{A\}$, and ρ_i. Then, we need to solve six second-order partial-differential equations or four highly coupled fourth-order partial-differential equations. However, for a long beam with a solid cross-section, the shear effects are negligible as indicated earlier.

Although general damping can be added to the equations of motion through the damping matrices $[C_1]$ and $[C_2]$, in the following derivation we assume that the dissipation of energy due to internal friction and relative motion between the beam and its support system can be modeled by small, uncoupled, viscous dampers. If we neglect the shear strains and assume that any possible external damping (e.g., air damping) can be treated as an external load, we do not need to consider the shear deformation related damping in the matrix $[C_2]$. Consequently, we put

$$[C_1] = \begin{bmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{bmatrix} \quad \text{and} \; [C_2] = \begin{bmatrix} \mu_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

Here, μ_0, μ_1, μ_2, and μ_3 denote the damping coefficients with respect to u, v, w, and ϕ, respectively.

To implement the assumption that $\gamma_{\xi\eta} = \gamma_{\xi\zeta} = 0$, two artificial constraint forces λ_2 (along the η-direction) and λ_3 (along the ζ-direction) are needed for the prevention of shear deformations. The constraint forces can be looked as applied local loads and their values indicate the magnitude of shear effect. Any other external load can be added to the equations of motion whenever the need arises, but we drop them in the following expressions to simplify the equations. Expanding equations (18) and (19) yields the following complete equations of motion:

$$[T_{11}F_1 + T_{21}(F_2 + \lambda_2) + T_{31}(F_3 + \lambda_3)]' = ma_x + \mu_0\dot{u}\,, \tag{33}$$

$$[T_{12}F_1 + T_{22}(F_2 + \lambda_2) + T_{32}(F_3 + \lambda_3)]' = ma_y + \mu_1\dot{v}\,, \tag{34}$$

$$[T_{13}F_1 + T_{23}(F_2 + \lambda_2) + T_{33}(F_3 + \lambda_3)]' = ma_z + \mu_2\dot{w}\,, \tag{35}$$

$$M_1' - \rho_\zeta M_2 + \rho_\eta M_3 = A_\xi + \mu_3\omega_\xi\,, \tag{36}$$

$$M_2' + \rho_\zeta M_1 - \rho_\xi M_3 - (1 + e)(F_3 + \lambda_3) = A_\eta\,, \tag{37}$$

$$M_3' - \rho_\eta M_1 + \rho_\xi M_2 + (1 + e)(F_2 + \lambda_2) = A_\zeta\,. \tag{38}$$

These are six equations governing the four unknowns u, v, w, and ϕ. Equations (37) and (38) can be used to eliminate the shear forces $F_2 + \lambda_2$ and $F_3 + \lambda_3$ from equations (33)–(35), yielding four fully nonlinear partial-differential equations for u, v, w, and ϕ. Because u, v, w, and ϕ are the generalized displacements corresponding to the coordinates x, y, z, and ξ, respectively, the natural boundary conditions are

$$F_x \equiv T_{11}F_1 + T_{21}(F_2 + \lambda_2) + T_{31}(F_3 + \lambda_3) = 0\,, \tag{39}$$

$$F_y \equiv T_{12}F_1 + T_{22}(F_2 + \lambda_2) + T_{32}(F_3 + \lambda_3) = 0\,, \tag{40}$$

$$F_z \equiv T_{13}F_1 + T_{23}(F_2 + \lambda_2) + T_{33}(F_3 + \lambda_3) = 0\,, \tag{41}$$

$$M_1 = 0\,, \tag{42}$$

$$M_y \equiv T_{22}M_2 + T_{32}M_3 = 0\,, \tag{43}$$

$$M_z \equiv T_{23}M_2 + T_{33}M_3 = 0\,, \tag{44}$$

at $s = L$. The kinematic boundary conditions are

$$u = v = \frac{\partial v}{\partial x} = w = \frac{\partial w}{\partial x} = \phi = 0 \tag{45}$$

at $s = 0$.

3. Expanded Form of Equations of Motion

Because the complete nonlinear equations of motion (i.e., equations (33)–(38) are transcendental, closed-form solutions of this nonlinear, two-point-boundary-value-problem are not available. Moreover, direct numerical procedures suffer from instability and convergence problems. Alternatively, one can use a combination of numerical and perturbation methods to solve them. The first step in such an approach is to expand the nonlinear transcendental terms into polynomials about the static equilibrium position. In what follows, we assume that the equilibrium position is close to the undeformed position.

To obtain the governing equations in polynomial form for small but finite oscillations about the undeformed position, we assume that u, v, w, ϕ, and their derivatives are $O(\varepsilon)$, where ε ($\ll 1$) is a small dimensionless parameter that is used as a bookkeeping device. In this section, we expand all terms in equations (33)–(45) in Taylor series by using MACSYMA [29] and keep nonlinear terms up to $O(\varepsilon^3)$, thereby obtaining third-order nonlinear equations of motion. We note that u can be $O(\varepsilon)$ or $O(\varepsilon^2)$, which depends on the boundary and loading conditions. When u is $O(\varepsilon^2)$, the obtained equations of motion contain extra higher-order terms. Normalization results (shown later) show that the rotary inertias are $O(\varepsilon^2)$, and this fact is used to simplify the equations.

Substituting for ψ and θ from equations (30) and (31) into equation (13) and expanding the result, we obtain

$$\rho_\xi = \phi' + v''w' - 2u'v''w' - u''v'w' + \dots, \tag{46}$$

Owing to the use of finite rotations and the contributions from nonlinear bendings, ϕ does not represent the real twist angle with respect to the ξ axis [1, 3]. It can be shown that any twist variable ϕ, defined by using a sequence of three Euler-like rotations or even two sequential rotations, is not a real twisting angle because the deformations u, v, w, and ϕ do not occur in sequence as assumed in the mathematical model that uses Euler angles. Consequently, the twisting curvature and kinematic boundary conditions must be used to define a real twist angle.

Hence, we define the elastic twist angle γ as

$$\gamma \equiv \phi + \int_0^s (v''w' - 2u'v''w' - u''v'w')\,\mathrm{d}s\,, \tag{47}$$

which satisfies the boundary conditions $\gamma(0, t) = \phi(0, t) = 0$. Hence,

$$\rho_\xi = \gamma'\,. \tag{48}$$

Here we keep all nonlinar terms due to the motion of the hub because Θ can be of $O(\varepsilon)$ (e.g., free vibration of a hinged-free beam) or of $O(1)$ (e.g., slewing beam with large $\dot{\Theta}$). Because the linear inertia term $j_\xi \ddot{\gamma}$ in the equation for γ is $O(\varepsilon^3)$, we keep quadratic and cubic inertia terms in the expansion of A_ξ. The term $\mu_3 \omega_\xi$ is expanded as

$$\mu_3 \omega_\xi = \mu_3 \dot{\gamma}\,. \tag{49}$$

Using equations (27), (29), (12), (48), (14), and (15), and neglecting shear strains, one can obtain the Taylor-series expansions of the internal force F_1 and the internal moments M_1, M_2, and M_3. Then, substituting the expanded form of e, A_η, and A_ζ and the expressions of the internal moments into equations (37) and (38), one can obtain the expanded forms of the shear forces $F_2 + \lambda_2$ and $F_3 + \lambda_3$. Finally, using the expanded forms of F_1, $F_2 + \lambda_2$, $F_3 + \lambda_3$, M_1, M_2, M_3, T_u, and A_ξ, and equations (21)–(23) and (49), one can obtain the equations of motion as

$$m\ddot{u} + \mu_0 \dot{u} = G_u' + F_u\,, \tag{50}$$

$$m\ddot{v} + \mu_1 \dot{v} = G_v' + F_v\,, \tag{51}$$

$$m\ddot{w} + \mu_2 \dot{w} = G_w' + F_w\,, \tag{52}$$

$$j_\xi \ddot{\gamma} + \mu_3 \dot{\gamma} = G_\gamma + F_\gamma\,, \tag{53}$$

where

$$\begin{aligned}
G_u \equiv{}& A_{11}\Big(u' + \frac{1}{2} v'^2 + \frac{1}{2} w'^2 - u'v'^2 - u'w'^2\Big) \\
&+ B_{11}(\gamma' - \gamma'u' + u'v'w'' - \gamma'w'^2 - u'v''w' - \gamma'v'^2 + \gamma'u'^2) \\
&- B_{12}\Big(w'' - u'w'' - \gamma v'' + v''\int_0^s v''w'\,\mathrm{d}s - \frac{1}{2} w'^2 w'' - v'^2 w'' + u'^2 w'' - \frac{1}{2}\gamma^2 w'' \\
&\quad + \gamma u'v'' - \gamma'u'v'\Big) \\
&- B_{13}\Big(-v'' + u'v'' - \gamma w'' - w''\int_0^s w''v'\,\mathrm{d}s + \frac{1}{2} v'^2 v'' + w'^2 v'' - u'^2 v'' \\
&\quad + \frac{1}{2}\gamma^2 v'' + \gamma u'w'' - \gamma'u'w'\Big) \\
&+ D_{11}(\gamma'v'w'' - \gamma'v''w')
\end{aligned}$$

$$
\begin{aligned}
&+ D_{12}(-\gamma'' w' - v' w''^2 + v'' w' w'' + \gamma' u'' w' + 3\gamma'' u' w' + \gamma\gamma'' v' + \gamma'^2 v') \\
&+ D_{13}(\gamma'' v' - w' v''^2 + w'' v' v'' - \gamma' u'' v' - 3\gamma'' u' v' + \gamma\gamma'' w' + \gamma'^2 w') \\
&+ D_{22}(w' w''' - 3u' w' w''' - \gamma v' w''' - 2u'' w' w'' - \gamma' v' w'' - u''' w'^2 - \gamma v''' w' - \gamma' v'' w') \\
&+ D_{23}(-v' w''' - v''' w' - 2\gamma w' w''' + 2\gamma v' v''' + 2u'' v' w'' \\
&\quad + 2u'' v'' w' + 3u' v''' w' + 3u' w''' v' + 2u''' v' w' + 2\gamma' v' v'' - 2\gamma' w' w'') \\
&+ D_{33}(v' v''' - 3u' v' v''' + \gamma w' v''' - 2u'' v' v'' + \gamma' w' v'' - u''' v'^2 + \gamma w''' v' + \gamma' w'' v') \\
&+ B'_{12}\left(-u' w' - \frac{1}{2} w'^3 - \frac{1}{2} v'^2 w' + 2u'^2 w' + \gamma u' v'\right) \\
&+ B'_{13}\left(u' v' + \frac{1}{2} v'^3 + \frac{1}{2} w'^2 v' - 2u'^2 v' + \gamma u' w'\right) \\
&+ D'_{12}(-\gamma' w' + \gamma\gamma' v' + 3\gamma' u' w') + D'_{13}(\gamma' v' + \gamma\gamma' w' - 3\gamma' u' v') \\
&+ D'_{22}(w' w'' - 3u' w' w'' - \gamma v' w'' - u'' w'^2 - \gamma v'' w') \\
&+ D'_{23}(-v' w'' - v'' w' - 2\gamma w' w'' + 2\gamma v' v'' + 3u' v' w'' + 3u' w' v'' + 2u'' v' w') \\
&+ D'_{33}(v' v'' - 3u' v' v'' + \gamma w' v'' - u'' v'^2 + \gamma w'' v') \qquad (54)
\end{aligned}
$$

$$
\begin{aligned}
G_v \equiv\; & A_{11}\left(u' v' + \frac{1}{2} v' w'^2 + \frac{1}{2} v'^3 - u'^2 v'\right) \\
&- B_{11}\left(u' w'' - \gamma' v' + \frac{1}{2} w'^2 w'' + \frac{1}{2} v'^2 w'' - 2u'^2 w'' - u' u'' w' + 2\gamma' u' v'\right) \\
&+ B_{12}\Big(-v' w'' - \gamma u'' - \gamma' u' + u'' \int_0^s v'' w' \, ds \\
&\quad - \gamma w' w'' + 2u' v' w'' - \frac{1}{2} \gamma' w'^2 - \frac{1}{2} \gamma' v'^2 + \gamma u' u'' + \gamma' u'^2\Big) \\
&+ B_{13}\Big(-u'' + u' u'' - w' w'' + 2u' w' w'' + \gamma v' w'' + u'' w'^2 + \frac{1}{2} u'' v'^2 - u'^2 u'' \\
&\quad + \frac{1}{2} \gamma^2 u'' + \gamma\gamma' u'\Big) \\
&+ D_{11}(-\gamma' w'' + 3\gamma' u' w'' + \gamma' u'' w') \\
&+ D_{12}\Big(w''^2 - \gamma\gamma'' - \gamma'^2 - \gamma v'' w'' - \gamma'' \int_0^s w'' v' \, ds \\
&\quad - 3u' w''^2 - 2u'' w' w'' + \gamma\gamma' u'' + 2\gamma\gamma'' u' + 2\gamma'^2 u'\Big) \\
&- D_{13}\Big(\gamma'' + v'' w'' - \gamma' u'' - 2\gamma'' u' + \gamma w''^2 - \frac{3}{2} \gamma'' v'^2 - \frac{1}{2} \gamma^2 \gamma'' - \gamma\gamma'^2 \\
&\quad - \gamma' w' w'' - 3u' v'' w'' - u'' v' w'' - \gamma'' w'^2 - u'' v'' w' - \gamma' v' v'' + 3\gamma' u' u'' + 3\gamma'' u'^2\Big) \\
&+ D_{22}\Big(\gamma w''' + \gamma' w'' - \gamma^2 v''' - 2\gamma\gamma' v'' + w''' \int_0^s w'' v' \, ds
\end{aligned}
$$

$$- 2\gamma u'w''' - 2\gamma u''w'' - 2\gamma' u'w'' - \gamma u'''w' - \gamma' u''w'\Big)$$

$$+ D_{23}\Big(w''' - 2\gamma v''' - 2\gamma' v'' - 2u'w''' - 2u''w'' - u'''w' - 2v''' \int_0^s w''v' \, ds - \frac{3}{2} w'^2 w'''$$

$$- \frac{3}{2} v'^2 w''' - 2\gamma^2 w''' - 2w'w''^2 - 4\gamma\gamma' w'' + 3w'''u'^2 - 2v'v''w'' + 6u'u''w''$$

$$+ 3u'u'''w' + 2u''^2 w' + 4\gamma u'v''' + 4\gamma u''v'' + 4\gamma' u'v'' + 2\gamma u'''v' + 2\gamma' u''v'\Big)$$

$$+ D_{33}\Big(-v''' - \gamma w''' - w''\gamma' + 2u''v'' + v'u''' + 2u'v''' + w''' \int_0^s v''w' \, ds + 2v'^2 v'''$$

$$+ \gamma^2 v''' + 2v'v''^2 + 2\gamma\gamma' v'' + 2\gamma u'w''' + 2v''w'w'' + 2\gamma u''w'' + 2\gamma' u'w''$$

$$+ v'''w'^2 + \gamma u'''w' + \gamma' u''w' - 3u'^2 v''' - 6u'u''v'' - 3u'u'''v' - 2u''^2 v'\Big)$$

$$+ B'_{12}\Big(-\gamma u' - u' \int_0^s w''v' \, ds - \frac{1}{2} \gamma w'^2 - \frac{1}{2} \gamma v'^2 + \gamma u'^2\Big)$$

$$+ B'_{13}\Big(-u' + u'^2 - \frac{1}{2} w'^2 - \frac{1}{2} v'^2 + \frac{3}{2} u'w'^2 + 2u'v'^2 - u'^3 + \frac{1}{2} \gamma^2 u'\Big)$$

$$+ D'_{12}\Big(-\gamma\gamma' - \gamma' \int_0^s w''v' \, ds + 2\gamma\gamma' u'\Big)$$

$$+ D'_{13}\Big(-\gamma' + 2\gamma' u' + \frac{3}{2} \gamma' v'^2 + \frac{1}{2} \gamma^2 \gamma' + \gamma' w'^2 - 3\gamma' u'^2\Big)$$

$$+ D'_{22}\Big(\gamma w'' - \gamma^2 v'' + w'' \int_0^s w''v' \, ds - 2\gamma u'w'' - \gamma u''w'\Big)$$

$$+ D'_{23}\Big(w'' - 2\gamma v'' - 2u'w'' - u''w' - \frac{3}{2} w''w'^2 - \frac{3}{2} v'^2 w''$$

$$- 2\gamma^2 w'' - 2v'' \int_0^s w''v' \, ds + 3u'^2 w'' + 3u'u''w' + 4\gamma u'v'' + 2\gamma u''v'\Big)$$

$$+ D'_{33}\Big(-v'' - \gamma w'' + u''v' + 2u'v'' + 2v'^2 v'' + \gamma^2 v''$$

$$+ w'' \int_0^s v''w' \, ds + 2\gamma u'w'' + v''w'^2 + \gamma u''w' - 3u'^2 v'' - 3u'u''v'\Big) + j_\zeta \ddot{v}' \tag{55}$$

$$G_w \equiv A_{11}\Big(u'w' + \frac{1}{2} w'v'^2 + \frac{1}{2} w'^3 - u'^2 w'\Big)$$

$$+ B_{11}\Big(u'v'' + \gamma' w' + \frac{1}{2} v'^2 v'' + \frac{1}{2} w'^2 v'' - 2u'^2 v'' - u'u''v' - 2\gamma' u'w'\Big)$$

$$+ B_{13}\Big(w'v'' - \gamma u'' - \gamma' u' - u'' \int_0^s w''v' \, ds - \gamma v'v'' - 2u'w'v'' - \frac{1}{2} \gamma' v'^2 - \frac{1}{2} \gamma' w'^2$$

$$+ \gamma u'u'' + \gamma' u'^2\Big)$$

$$+ B_{12}\Big(u'' - u'u'' + v'v'' - 2u'v'v'' + \gamma w'v''$$

$$
\begin{aligned}
&- u''v'^2 - \frac{1}{2} u''w'^2 + u'^2u'' - \frac{1}{2} \gamma^2 u'' - \gamma\gamma' u' \Big) \\
&+ D_{11}(\gamma' v'' - 3\gamma' u' v'' - \gamma' u'' v') \\
&+ D_{13}\Big(v''^2 - \gamma\gamma'' - \gamma'^2 + \gamma w'' v'' + \gamma'' \int_0^s v'' w' \, ds \\
&- 3u'v''^2 - 2u''v'v'' + \gamma\gamma' u'' + 2\gamma\gamma'' u' + 2\gamma'^2 u' \Big) \\
&- D_{12}\Big(-\gamma'' + w''v'' + \gamma' u'' + 2\gamma'' u' - \gamma v''^2 + \frac{3}{2} \gamma'' w'^2 + \frac{1}{2} \gamma^2 \gamma'' + \gamma\gamma'^2 \\
&+ \gamma' v' v'' - 3u' w'' v'' - u'' w' v'' + \gamma'' v'^2 - u'' w'' v' + \gamma' w' w'' - 3\gamma' u' u'' - 3\gamma'' u'^2 \Big) \\
&+ D_{33}\Big(-\gamma v''' - \gamma' v'' - \gamma^2 w''' - 2\gamma\gamma' w'' + v''' \int_0^s v'' w' \, ds \\
&+ 2\gamma u' v''' + 2\gamma u'' v'' + 2\gamma' u' v'' + \gamma u''' v' + \gamma' u'' v' \Big) \\
&+ D_{23}\Big(v''' + 2\gamma w''' + 2\gamma' w'' - 2u' v''' - 2u'' v'' - u''' v' - 2w''' \int_0^s v'' w' \, ds - \frac{3}{2} v'^2 v''' \\
&- \frac{3}{2} w'^2 v''' - 2\gamma^2 v''' - 2v' v''^2 - 4\gamma\gamma' v'' + 3u'^2 v''' - 2w' w'' v'' + 6u' u'' v'' \\
&+ 3u' u''' v' + 2u''^2 v' - 4\gamma u' w''' - 4\gamma u'' w'' - 4\gamma' u' w'' - 2\gamma u''' w' - 2\gamma' u'' w' \Big) \\
&- D_{22}\Big(w''' - \gamma v''' - v'' \gamma' - 2u'' w'' - u''' w' - 2u' w''' - v''' \int_0^s w'' v' \, ds \\
&- 2w'^2 w''' - \gamma^2 w''' - 2w' w''^2 - 2\gamma\gamma' w'' + 2\gamma u' v''' - 2w'' v' v'' + 2\gamma u'' v'' \\
&+ 2\gamma' u' v'' - w''' v'^2 + \gamma u''' v' + \gamma' u'' v' + 3u'^2 w''' + 6u' u'' w'' + 3u' u''' w' + 2u''^2 w' \Big) \\
&+ B'_{13}\Big(-\gamma u' + u' \int_0^s v'' w' \, ds - \frac{1}{2} \gamma v'^2 - \frac{1}{2} \gamma w'^2 + \gamma u'^2 \Big) \\
&- B'_{12}\Big(-u' + u'^2 - \frac{1}{2} v'^2 - \frac{1}{2} w'^2 + \frac{3}{2} u' v'^2 + 2u' w'^2 - u'^3 + \frac{1}{2} \gamma^2 u' \Big) \\
&+ D'_{13}\Big(-\gamma\gamma' + \gamma' \int_0^s v'' w' \, ds + 2\gamma\gamma' u' \Big) \\
&- D'_{12}\Big(-\gamma' + 2\gamma' u' + \frac{3}{2} \gamma' w'^2 + \frac{1}{2} \gamma^2 \gamma' + \gamma' v'^2 - 3\gamma' u'^2 \Big) \\
&+ D'_{33}\Big(-\gamma v'' - \gamma^2 w'' + v'' \int_0^s v'' w' \, ds + 2\gamma u' v'' + \gamma u'' v' \Big) \\
&+ D'_{23}\Big(v'' + 2\gamma w'' - 2u' v'' - u'' v' - \frac{3}{2} v'' v'^2 - \frac{3}{2} w'^2 v'' - 2\gamma^2 v'' \\
&- 2w'' \int_0^s v'' w' \, ds + 3u'^2 v'' + 3u' u'' v' - 4\gamma u' w'' - 2\gamma u'' w' \Big)
\end{aligned}
$$

$$- D'_{22}\Big(w'' - \gamma v'' - u''w' - 2u'w'' - 2w'^2 w'' - \gamma^2 w''$$

$$- v'' \int_0^s w''v' \, \mathrm{d}s + 2\gamma u'v'' - w''v'^2 + \gamma u''v' + 3u'^2 w'' + 3u'u''w' \Big) + j_\eta \ddot{w}' \qquad (56)$$

$$G_\gamma \equiv B_{11}\Big(u'' + w'w'' + v'v'' - u'w'w'' - u'v'v'' - \frac{1}{2} u''w'^2 - \frac{1}{2} u''v'^2 \Big)$$

$$+ B_{12}\Big(-u'v'' - \gamma u'w'' - \frac{1}{2} v''w'^2 - \frac{1}{2} v'^2 v'' + u'^2 v'' + u'u''v' \Big)$$

$$+ B_{13}\Big(-u'w'' + \gamma u'v'' - \frac{1}{2} w''v'^2 - \frac{1}{2} w'^2 w'' + u'^2 w'' + u'u''w' \Big)$$

$$- D_{11}\Big(-\gamma'' + \gamma'u'' + \gamma''u' + \gamma'w'w'' + \gamma'v'v'' + \frac{1}{2} \gamma''w'^2 + \frac{1}{2} \gamma''v'^2 - 2\gamma'u'u'' - \gamma''u'^2 \Big)$$

$$- D_{12}\Big(w''' - \gamma v''' - u'w''' + 4u'u''w'' - 2w''u'' - u'''w' - v''' \int_0^s w''v' \, \mathrm{d}s$$

$$- \frac{1}{2} \gamma^2 w''' - w'^2 w''' - 2w'w''^2 - \frac{1}{2} v'^2 w''' + u'^2 w'''$$

$$- 2v'v''w'' + 2u'u'''w' + 2u''^2 w' + \gamma u'v''' + 2\gamma u''v'' + \gamma u'''v' - \gamma'u'v'' \Big)$$

$$- D_{13}\Big(-v''' - \gamma w''' + u'v''' - 4u'u''v'' + 2u''v'' + u'''v' + w''' \int_0^s v''w' \, \mathrm{d}s$$

$$+ \frac{1}{2} \gamma^2 v''' + v'^2 v''' + 2v'v''^2 + \frac{1}{2} w'^2 v''' - u'^2 v'''$$

$$+ 2w'w''v'' - 2u'u'''v' - 2u''^2 v' + \gamma u'w''' + 2\gamma u''w'' + \gamma u'''w' - \gamma'u'w'' \Big)$$

$$+ D_{22}(v''w'' + \gamma w''^2 - \gamma v''^2 - 2u'v''w'' - u''v'w'' - u''v''w')$$

$$+ D_{23}(w''^2 - v''^2 - 4\gamma v''w'' - 2u'w''^2 + 2u'v''^2 - 2u''w'w'' + 2u''v'v'')$$

$$+ D_{33}(-v''w'' + \gamma v''^2 - \gamma w''^2 + 2u'v''w'' + u''w'v'' + u''w''v')$$

$$+ B'_{11}\Big(u' + \frac{1}{2} v'^2 + \frac{1}{2} w'^2 - \frac{1}{2} u'w'^2 - \frac{1}{2} u'v'^2 \Big)$$

$$+ D'_{11}\Big(\gamma' - \gamma'u' - \frac{1}{2} \gamma'w'^2 - \frac{1}{2} \gamma'v'^2 + \gamma'u'^2 \Big)$$

$$- D'_{12}\Big(w'' - \gamma v'' - u'w'' - u''w' - v'' \int_0^s w''v' \, \mathrm{d}s - w'^2 w''$$

$$- \frac{1}{2} \gamma^2 w'' - \frac{1}{2} v'^2 w'' + u'^2 w'' + 2u'u''w' + \gamma u'v'' + \gamma u''v' \Big)$$

$$- D'_{13}\Big(-v'' - \gamma w'' + u'v'' + u''v' + w'' \int_0^s v''w' \, \mathrm{d}s + v'^2 v''$$

$$+ \frac{1}{2} \gamma^2 v'' + \frac{1}{2} w'^2 v'' - u'^2 v'' - 2u'u''v' + \gamma u'w'' + \gamma u''w' \Big)$$

$$
-j_\xi\Big[\int_0^s(-w'v''+2u'v''w'+u''v'w')^{\cdot\cdot}\,\mathrm{d}s+\ddot v'w'
$$
$$
-2\ddot v'u'w'-3\dot v'\dot u'w'-\ddot u'v'w'+\dot v'\dot w'-2u'\dot v'\dot w'-\dot u'v'\dot w'\Big]
$$
$$
+(j_\eta-j_\zeta)(\gamma\dot v'^2+\dot u'\dot v'w'-\dot v'\dot w'+2u'\dot v'\dot w'+\dot u'v'\dot w'-\gamma\dot w'^2) \tag{57}
$$

The forcing functions F_u, V_v, F_w, and F_γ are due to the motion of the hub, which are given by

$$
F_u=m[-\ddot\Theta w+\dot\Theta^2(s+u)-2\dot\Theta\dot w]+(j_\eta\ddot\Theta w')', \tag{58}
$$

$$
F_v=\{(j_\eta-j_\xi)(\dot\Theta\dot\gamma+\dot\Theta^2v')+j_\eta\ddot\Theta\gamma-j_\zeta(\dot\Theta\gamma)^{\cdot}\}', \tag{59}
$$

$$
F_w=m[\dot\Theta^2w+\ddot\Theta(s+u)+2\dot u\dot\Theta]+\ddot\Theta(u'j_\eta-j_\eta)', \tag{60}
$$

$$
F_\gamma=-j_\xi\Big[\ddot\Theta\Big(v'-u'v'+u'^2v'-\frac{1}{2}v'^3-\frac{1}{2}v'w'^2\Big)+
$$
$$
\dot\Theta\Big(\dot v'-u'\dot v'-\dot u'v'+u'^2\dot v'+2\dot u'u'v'-\frac{3}{2}\dot v'v'^2-\frac{1}{2}\dot v'w'^2-v'w'\dot w'\Big)\Big]
$$
$$
+(j_\eta-j_\zeta)\Big[\dot\Theta^2\Big(-\gamma+\int_0^s(v''w'+2u'v'w''+u''v'w')\,\mathrm{d}s+\gamma v'^2+\frac{2}{3}\gamma^3-v'w'\Big)+
$$
$$
\dot\Theta\Big(\dot v'-u'\dot v'-\dot u'v'+u'^2\dot v'+2\dot u'u'v'-\frac{3}{2}\dot v'v'^2
$$
$$
-\frac{1}{2}\dot v'w'^2-2\gamma^2\dot v'+2\gamma\dot w'-2\gamma u'\dot w'-2\gamma\dot u'w'+v'w'\dot w'\Big)\Big]. \tag{61}
$$

The forcing functions can also be obtained by using D'Alembert principle [25].

The boundary conditions (39)–(45) become

$$
G_u+j_\eta\ddot\Theta w'=0 \tag{62}
$$

$$
G_v+(j_\eta-j_\xi)(\dot\Theta\dot\gamma+\dot\Theta^2v')+j_\eta\ddot\Theta\gamma-j_\zeta(\dot\Theta\gamma)^{\cdot}=0 \tag{63}
$$

$$
G_w+j_\eta\ddot\Theta(u'-1)=0 \tag{64}
$$

$$
B_{11}\Big(u'+\frac{1}{2}w'^2+\frac{1}{2}v'^2-\frac{1}{2}u'w'^2-\frac{1}{2}u'v'^2\Big)
$$
$$
+D_{11}\Big(\gamma'-\gamma'u'-\frac{1}{2}\gamma'w'^2-\frac{1}{2}\gamma'v'^2+\gamma'u'^2\Big)
$$
$$
+D_{12}\Big(-w''+u'w''+u''w'+\gamma v''+v''\int_0^s w''v'\,\mathrm{d}s+w'^2w''
$$
$$
+\frac{1}{2}v'^2w''-u'^2w''+\frac{1}{2}\gamma^2w''-2u'u''w'-\gamma u'v''-\gamma u''v'\Big)
$$
$$
+D_{13}\Big(v''-u'v''-u''v'+\gamma w''-w''\int_0^s v''w'\,\mathrm{d}s-v'^2v''
$$
$$
-\frac{1}{2}w'^2v''+u'^2v''-\frac{1}{2}\gamma^2v''+2u'u''v'-\gamma u'w''-\gamma u''w'\Big)=0 \tag{65}
$$

$$B_{12}\Big(u' + \frac{1}{2}w'^2 + \frac{1}{2}v'^2 - \frac{1}{2}u'w'^2 - u'v'^2 - \frac{1}{2}\gamma^2 u'\Big)$$

$$+ B_{13}\Big(-\gamma u' - u'\int_0^s w''v'\,\mathrm{d}s - \frac{1}{2}\gamma w'^2 - \frac{1}{2}\gamma v'^2\Big)$$

$$+ D_{12}\Big(\gamma' - \gamma' u' - \frac{1}{2}\gamma' w'^2 - \gamma' v'^2 + \gamma' u'^2 - \frac{1}{2}\gamma^2\gamma'\Big)$$

$$+ D_{13}\Big(-\gamma\gamma' - \gamma'\int_0^s w''v'\,\mathrm{d}s + \gamma\gamma' u'\Big)$$

$$+ D_{22}\Big(-w'' + u'w'' + u''w' + \gamma v'' + v''\int_0^s w''v'\,\mathrm{d}s + w'^2 w''$$

$$+ v'^2 w'' - u'^2 w'' + \gamma^2 w'' - 2u'u''w' - \gamma u'v'' - \gamma u''v'\Big)$$

$$+ D_{23}\Big(v'' + 2\gamma w'' - u'v'' - u''v' - 2w''\int_0^s v''w'\,\mathrm{d}s + v'w'w''$$

$$- 2\gamma u'w'' - \frac{1}{2}v''w'^2 - 2\gamma u''w' - \frac{3}{2}v'^2 v'' + u'^2 v'' - 2\gamma^2 v'' + 2u'u''v'\Big)$$

$$+ D_{33}\Big(-\gamma v'' - v''\int_0^s w''v'\,\mathrm{d}s - \gamma^2 w'' + \gamma u'v'' + \gamma u''v'\Big) = 0 \qquad (66)$$

$$B_{13}\Big(u' + \frac{1}{2}v'^2 + \frac{1}{2}w'^2 - \frac{1}{2}u'v'^2 - u'w'^2 - \frac{1}{2}\gamma^2 u'\Big)$$

$$+ B_{12}\Big(\gamma u' - u'\int_0^s v''w'\,\mathrm{d}s + \frac{1}{2}\gamma v'^2 + \frac{1}{2}\gamma w'^2\Big)$$

$$+ D_{13}\Big(\gamma' - \gamma' u' - \frac{1}{2}\gamma' v'^2 - \gamma' w'^2 + \gamma' u'^2 - \frac{1}{2}\gamma^2\gamma'\Big)$$

$$+ D_{12}\Big(\gamma\gamma' - \gamma'\int_0^s v''w'\,\mathrm{d}s - \gamma\gamma' u'\Big)$$

$$+ D_{33}\Big(v'' - u'v'' - u''v' + \gamma w'' - w''\int_0^s v''w'\,\mathrm{d}s - v'^2 v''$$

$$- w'^2 v'' + u'^2 v'' - \gamma^2 v'' + 2u'u''v' - \gamma u'w'' - \gamma u''w'\Big)$$

$$+ D_{23}\Big(-w'' + 2\gamma v'' + u'w'' + u''w' + 2v''\int_0^s w''v'\,\mathrm{d}s - w'v'v''$$

$$- 2\gamma u'v'' + \frac{1}{2}w''v'^2 - 2\gamma u''v' + \frac{3}{2}w'^2 w'' - u'^2 w'' + 2\gamma^2 w'' - 2u'u''w'\Big)$$

$$+ D_{22}\Big(-\gamma w'' + w''\int_0^s v''w'\,\mathrm{d}s + \gamma^2 v'' + \gamma u'w'' + \gamma u''w'\Big) = 0 \qquad (67)$$

$s = L$ and

$$u = v = v' = w = w' = \gamma = 0\,, \qquad (68)$$

$$\begin{aligned}\tau &= I_{11}\ddot{\Theta} - M_2 \\ &= I_{11}\ddot{\Theta} - B_{12}u' - D_{12}(\gamma' - \gamma'u' + \gamma'u'^2) - D_{22}\left(-w'' + u'w'' - u'^2w'' - v''\int_0^s v''w'\,\mathrm{d}s\right) \\ &\quad - D_{23}\left(v'' - u'v'' + u'^2v'' - w''\int_0^s v''w'\,\mathrm{d}s\right)\end{aligned} \tag{69}$$

at $s = 0$.

Using the following transformation, which corresponds to the case in which the reference coordinate system x-y-z is rotated by 90° with respect to the x-axis,

$$\begin{aligned}&\gamma \rightarrow \gamma\,, \quad u \rightarrow u\,, \quad v \rightarrow -w\,, \quad w \rightarrow v\,, \\ &j_\xi \rightarrow j_\xi\,, \quad j_\eta \rightarrow j_\zeta\,, \quad j_\zeta \rightarrow j_\eta\,, \quad \mu_1 \rightarrow \mu_1\,, \quad \mu_2 \rightarrow \mu_3\,, \quad \mu_3 \rightarrow \mu_2\,, \\ &B_{11} \rightarrow B_{11}\,, \quad B_{12} \rightarrow -B_{13}\,, \quad B_{13} \rightarrow B_{12}\,, \\ &D_{11} \rightarrow D_{11}\,, \quad D_{22} \rightarrow D_{33}\,, \quad D_{33} \rightarrow D_{22}\,, \quad D_{12} \rightarrow -D_{13}\,, \quad D_{13} \rightarrow D_{12}\,, \\ &\quad D_{23} \rightarrow -D_{32}\ (= -D_{23})\end{aligned}$$

in equations (50)–(53), we can interchange the equations of motion and boundary conditions governing v and w with the equations and boundary conditions governing γ and u being unchanged. So, the equations of motion are symmetric, unique, and independent of the rotation sequence of the three Euler angles used in the mathematical modeling. And u, v, w, and γ are independent coordinates.

Because a different ordering scheme is used in the derivation, the intrinsic axial strain e [30] is used instead of the Green strain measure, and the influence of stretching on the twisting curvature is taken into consideration in the constitutive equation (27), equations (50)–(53) do not reduce to those of Crespo da Silva [9] when $F_u = F_v = F_w = F_\gamma = 0$, $A_{mn} = B_{mn} = 0$, and $D_{12} = D_{13} = D_{23} = 0$.

4. Inextensional Beams

For an inextensional beam, the number of equations can be reduced from four to three if an artificial constraint force is added into the equations. Next, we develop three third-order nonlinear partial-differential equations, which describe the slewing motion of an inextensional composite beam. For an inextensional beam,

$$s = \tilde{s}\,, \quad \rho_i = \tilde{\rho}_i\,, \quad \text{and } e = 0\,. \tag{70}$$

Putting $e = 0$ in equation (29), solving for u', and neglecting terms of order higher than three, we obtain

$$u' = -\frac{1}{2}(v'^2 + w'^2)\,, \tag{71}$$

Integrating equation (71) and using the geometric boundary condition $u(0, t) = 0$, we obtain

$$u = -\frac{1}{2}\int_0^s (v'^2 + w'^2)\,ds, \tag{72}$$

which shows that u is a second-order quantity. Following the ordering scheme, substituting equation (72) and its derivatives into equations (13)–(15) and (2), we obtain the third-order expansions of the curvatures and $[T]$ as

$$\rho_\xi = \gamma', \tag{73}$$

$$\rho_\eta = -w'' + v''\gamma - v''\int_0^s v''w'\,ds + \frac{1}{2}w''(\gamma^2 - w'^2), \tag{74}$$

$$\rho_\zeta = v'' + w''\gamma + w''\int_0^s v'w''\,ds + \frac{1}{2}v''(v'^2 - \gamma^2), \tag{75}$$

$$[T] = \begin{bmatrix} 1 - \frac{1}{2}(v'^2 + w'^2) & v' & w' \\ \frac{1}{2}v'(\gamma^2 + w'^2 - 2) - w'(\gamma + l_1) & 1 - \frac{1}{2}(v'^2 + \gamma^2 + 2\gamma l_1) & \gamma - l_2 - \frac{1}{2}\gamma\left(w'^2 + \frac{1}{3}\gamma^2\right) \\ \frac{1}{2}w'(\gamma^2 + v'^2 - 2) + v'(\gamma - l_2) & \frac{1}{2}\gamma\left(v'^2 + \frac{1}{3}\gamma^2\right) - \gamma - l_1 & 1 - \frac{1}{2}(w'^2 + \gamma^2 - 2\gamma l_2) \end{bmatrix} \tag{76}$$

where $l_1 \equiv \int_0^s v'w''\,ds$ and $l_2 \equiv \int_0^s v''w'\,ds$. Comparing equations (74) and (75) with the equations (A13) and (A14) used by Rosen *et al.* [7] for the nonlinear curvatures, we find that each of the latter is missing a third-order term.

Because the beam is not a rigid body, the axial strain can never be zero. One way for implementing the assumption of inextensionality is the addition of an artificial, axial constraint force λ_1 in the equations. Hence, F_1 should be replaced by $F_1 + \lambda_1$ in equations (33)–(35) and (39)–(41). The constraint force λ_1 indicates the magnitude of extension effects, which is of second order if $\Theta(t) = 0$ and $B_{ij} = 0$ [25]. The assumption of inextensionality is valid only when Θ is small. If $\Theta(t)$ is large, λ_1 can be large due to centrifugal forces. For asymmetric composite beams, the assumption of inextensionality is invalid because of the existence of extension-twisting (B_{11}) and extension-bending (B_{12} and B_{13}) couplings. Because the axial strain is assumed to be zero, the axial damping μ_0 vanishes. Using equations (33) and (39) and $\mu_0 = 0$, one can obtain the expanded form of $F_1 + \lambda_1$. Then, the equations of motion are obtained from equations (34)–(36) as

$$m\ddot{v} + \mu_1\dot{v} = G_v' + F_v, \tag{77}$$

$$m\ddot{w} + \mu_2\dot{w} = G_w' + F_w, \tag{78}$$

$$j_\xi\ddot{\gamma} + \mu_3\dot{\gamma} = G_\gamma + F_\gamma, \tag{79}$$

where

$$
\begin{aligned}
G_v \equiv & -(D_{33}v'')' - D_{11}(\gamma' w'') - D_{33}v'(v'v'' + w'w'')' \\
& + (D_{22} - D_{33})\left[(w''\gamma - v''\gamma^2)' - w''' \int_0^s v''w' \, \mathrm{d}s \right] \\
& + D'_{22}\left(w''\gamma - w'' \int_0^s v''w' \, \mathrm{d}s - v''\gamma^2 \right) - D'_{33}\left(w''\gamma + w'' \int_0^s v'w'' \, \mathrm{d}s - v''\gamma^2 + v''v'^2 \right) \\
& + D_{12}\left(w''^2 - \gamma'^2 - \gamma\gamma'' + \gamma'' \int_0^s v''w' \, \mathrm{d}s - \gamma v''w'' \right) \\
& + D_{13}\left(-\gamma'' - v''w'' - \frac{1}{2} v'^2\gamma'' - w''^2\gamma + \frac{1}{2} \gamma^2\gamma'' + \gamma\gamma'^2 \right) \\
& + D_{23}\left(w''' - 2v''\gamma' + v''^2 w' - 2v'''\gamma + 2v''' \int_0^s v''w' \, \mathrm{d}s + \frac{1}{2} v'^2 w''' \right. \\
& \quad \left. + \frac{1}{2} w'''w'^2 - 4w''\gamma\gamma' + w''^2 w' - 2w'''\gamma^2 \right) \\
& + D'_{12}\left(-\gamma\gamma' + \gamma' \int_0^s v''w' \, \mathrm{d}s \right) + D'_{13}\left(-\gamma' - \frac{1}{2} v'^2\gamma' + \frac{1}{2} \gamma^2\gamma' \right) \\
& + D'_{23}\left(w'' + \frac{1}{2} v'^2 w'' + \frac{1}{2} w''w'^2 - 2\gamma^2 w'' - 2v''\gamma + 2v'' \int_0^s v''w' \, \mathrm{d}s \right) \\
& + j_\zeta \ddot{v}' - \frac{1}{2} v' \int_L^s m\left[\int_0^s (v'^2 + w'^2) \, \mathrm{d}s \right]^{\cdot\cdot} \mathrm{d}s \,,
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
G_w \equiv & -(D_{22}w'')' + D_{11}(\gamma' v'') - D_{22}w'(w'w'' + v'v'')' \\
& + (D_{22} - D_{33})\left[(v''\gamma + w''\gamma^2)' + v''' \int_0^s w''v' \, \mathrm{d}s \right] \\
& - D'_{33}\left(v''\gamma + v'' \int_0^s w''v' \, \mathrm{d}s + w''\gamma^2 \right) + D'_{22}\left(v''\gamma - v'' \int_0^s w'v'' \, \mathrm{d}s + w''\gamma^2 - w''w'^2 \right) \\
& + D_{13}\left(v''^2 - \gamma'^2 - \gamma\gamma'' - \gamma'' \int_0^s w''v' \, \mathrm{d}s + \gamma w''v'' \right) \\
& + D_{12}\left(\gamma'' - v''w'' + \frac{1}{2} w'^2\gamma'' + v''^2\gamma - \frac{1}{2} \gamma^2\gamma'' - \gamma\gamma'^2 \right) \\
& + D_{23}\left(v''' + 2w''\gamma' + w''^2 v' + 2w'''\gamma + 2w''' \int_0^s w''v' \, \mathrm{d}s + \frac{1}{2} w'^2 v''' \right. \\
& \quad \left. + \frac{1}{2} v'''v'^2 - 4v''\gamma\gamma' + v''^2 v' - 2v'''\gamma^2 \right) \\
& + D'_{13}\left(-\gamma\gamma' - \gamma' \int_0^s w''v' \, \mathrm{d}s \right) + D'_{12}\left(\gamma' + \frac{1}{2} w'^2\gamma' - \frac{1}{2} \gamma^2\gamma' \right) \\
& + D'_{23}\left(v'' + \frac{1}{2} w'^2 v'' + \frac{1}{2} v''v'^2 - 2\gamma^2 v'' + 2w''\gamma + 2w'' \int_0^s w''v' \, \mathrm{d}s \right) \\
& + j_\eta \ddot{w}' - \frac{1}{2} w' \int_L^s m\left[\int_0^s (v'^2 + w'^2) \, \mathrm{d}s \right]^{\cdot\cdot} \mathrm{d}s \,.
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
G_\gamma \equiv & (D_{11}\gamma')' + (D_{33} - D_{22})[(v''^2 - w''^2)\gamma - v''w''] \\
& + \left[D_{12}\left(-w'' + v''\gamma - v'' \int_0^s v''w' \,\mathrm{d}s + \frac{1}{2} w''\gamma^2 - \frac{1}{2} w''w'^2 \right) \right]' \\
& + \left[D_{13}\left(v'' + w''\gamma + w'' \int_0^s w''v' \,\mathrm{d}s - \frac{1}{2} v''\gamma^2 + \frac{1}{2} v''v'^2 \right) \right]' \\
& - D_{12}(v''\gamma' + w''\gamma\gamma') + D_{13}(-w''\gamma' + v''\gamma\gamma') + D_{23}(w''^2 - 4\gamma v''w'' - v''^2) \\
& + j_\xi \left(\int_0^s v''w' \,\mathrm{d}s \right)^{\cdot\cdot} - j_\xi(\dot v'w')^{\cdot} + (j_\eta - j_\zeta)[(\dot v'^2 - \dot w'^2)\gamma - \dot w'\dot v'] \,.
\end{aligned} \tag{82}
$$

The forcing functions become

$$
\begin{aligned}
F_v = & \Big\{ (j_\eta - j_\xi)(\dot\Theta\dot\gamma + \dot\Theta^2 v') + j_\eta\ddot\Theta\gamma - j_\zeta(\dot\Theta\gamma)^{\cdot} - \frac{1}{2}\dot\Theta^2 v'(v'^2 + w'^2) \int_L^s ms \,\mathrm{d}s \\
& + v' \int_L^s m\left[\ddot\Theta w + 2\dot\Theta\dot w - \dot\Theta^2 s + \frac{1}{2}\dot\Theta^2 \int_0^s (v'^2 + w'^2)\,\mathrm{d}s \right] \mathrm{d}s \Big\}' \,,
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
F_w = & m\dot\Theta^2 w - m\ddot\Theta\left(-s + \frac{1}{2}\int_0^s (v'^2 + w'^2)\,\mathrm{d}s \right) - m\dot\Theta\left[\int_0^s (v'^2 + w'^2)\,\mathrm{d}s \right]^{\cdot} + \Big\{ -j_\eta\ddot\Theta - \frac{1}{2}\dot\Theta^2 w' \\
& \times (v'^2 + w'^2) \int_L^s ms \,\mathrm{d}s + w' \int_L^s m\left[\ddot\Theta w + 2\dot\Theta\dot w - \dot\Theta^2 s + \frac{1}{2}\dot\Theta^2 \int_0^s (v'^2 + w'^2)\,\mathrm{d}s \right] \mathrm{d}s \Big\}' \,,
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
F_\gamma = & -j_\xi(\dot\Theta v')^{\cdot} + (j_\eta - j_\zeta)\Big[\dot\Theta\Big(\dot v' - 2\dot v'\gamma^2 + 2\dot w'\gamma + 2\dot w' \int_0^s v'w''\,\mathrm{d}s \Big) \\
& + \dot\Theta^2\Big(-\gamma - \int_0^s v'w''\,\mathrm{d}s + \gamma v'^2 + \frac{2}{3}\gamma^3 \Big) \Big] \,.
\end{aligned} \tag{85}
$$

Equations (77)–(82) reduce to those of Crespo da Silva and Glynn [8] when $D_{12} = D_{13} = D_{23} = 0$ and $F_v = F_w = F_\gamma = 0$. The boundary conditions become

$$
\begin{aligned}
& (D_{22} - D_{33})\left(w'''\gamma - v'''\gamma^2 - w''' \int_0^L v''w' \,\mathrm{d}s \right) - D_{33}v''' - D_{33}v'(v'v''' + w'w''') + j_\zeta \ddot v' \\
& + D_{12}\left(-\gamma\gamma'' + \gamma'' \int_0^L v''w' \,\mathrm{d}s \right) + D_{13}\left(-\gamma'' - \frac{1}{2} v'^2\gamma'' + \frac{1}{2}\gamma^2\gamma'' \right) \\
& + D_{23}\left(w''' - 2v'''\gamma + 2v''' \int_0^L v''w' \,\mathrm{d}s + \frac{1}{2} v'^2 w''' + \frac{1}{2} w'''w'^2 - 2w'''\gamma^2 \right) \\
& + (j_\eta - j_\xi)(\dot\Theta\dot\gamma + \dot\Theta^2 v') + j_\eta\ddot\Theta\gamma - j_\zeta(\dot\Theta\gamma)^{\cdot} = 0 \,,
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
& (D_{22} - D_{33})\left(v'''\gamma + w'''\gamma^2 + v''' \int_0^L w''v' \,\mathrm{d}s \right) - D_{22}w''' - D_{22}w'(w'w''' + v'v''') + j_\eta \ddot w' \\
& + D_{13}\left(-\gamma\gamma'' - \gamma'' \int_0^L w''v' \,\mathrm{d}s \right) + D_{12}\left(\gamma'' + \frac{1}{2} w'^2\gamma'' - \frac{1}{2}\gamma^2\gamma'' \right) \\
& + D_{23}\left(v''' + 2w'''\gamma + 2w''' \int_0^L w''v' \,\mathrm{d}s + \frac{1}{2} w'^2 v''' + \frac{1}{2} v'''v'^2 - 2v'''\gamma^2 \right) - j_\eta\ddot\Theta = 0 \,.
\end{aligned} \tag{87}
$$

$$\gamma' = 0 , \tag{88}$$

$$w'' = 0 , \tag{89}$$

$$v'' = 0 \tag{90}$$

at $s = L$ and

$$v = v' = w = w' = \gamma = 0 , \tag{91}$$

$$\tau = I_{11}\ddot{\Theta} - D_{12}\gamma' + D_{22}w'' - D_{23}v'' \tag{92}$$

at $s = 0$.

For an isotropic beam, $D_{12} = D_{13} = D_{23} = 0$. If the cross-section dimensions and material properties of the beam are uniform (i.e., m, D_{11}, D_{22}, D_{33}, j_ξ, j_η, and j_ζ are constant), the equations of motion can be considerably simplified and rewritten in nondimensional form by normalizing the variables using the characteristic length L and the characteristic time $L^2\sqrt{m/D_{22}}$ as

$$s^* = s/L , \quad v^* = v/L , \quad w^* = w/L , \quad t^* = t\sqrt{D_{22}/mL^4} ,$$

$$\mu_1^* = \mu_1 L^2/\sqrt{mD_{22}} , \quad \mu_2^* = \mu_2 L^2/\sqrt{mD_{22}} , \quad \mu_3^* = \mu_3/\sqrt{mD_{22}}\, j_1^* ,$$

$$j_1^* = j_\xi/mL^2 , \quad j_2^* = j_\eta/mL^2 , \quad j_3^* = j_\zeta/mL^2 ,$$

For ease of notation, we drop the superscript * and use primes and dots to denote $\partial/\partial s^*$ and $\partial/\partial t^*$, respectively. Because $j_1 = j_2 + j_3 = 1/12(b/L)^2 + 1/12(h/L)^2$ for a rectangular cross-section, where b and h denote the width and height of the cross-section, we assume that j_1, j_2 and j_3 are second-order quantities. Letting $\beta_x \equiv D_{33}/D_{22}$ and $\beta_y \equiv D_{11}/D_{22}$, we rewrite equations (77)–(79) in the nondimensional form

$$\begin{aligned}\ddot{v} + \mu_1\dot{v} + \beta_x v^{iv} = {} & -\beta_y(\gamma'w'')' - \beta_x[v'(v'v'' + w'w'')']' \\ & - (1-\beta_y)\left[(v''\gamma^2 - w''\gamma)' + w'''\int_0^s v''w'\,\mathrm{d}s\right]' + j_3\ddot{v}'' \\ & - \frac{1}{2}\left\{v'\int_1^s\left[\int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right]^{\cdot\cdot}\mathrm{d}s\right\}' + F_v ,\end{aligned} \tag{93}$$

$$\begin{aligned}\ddot{w} + \mu_2\dot{w} + w^{iv} = {} & \beta_y(v''\gamma')' - [w'(v'v'' + w'w'')']' \\ & + (1-\beta_y)\left[(v''\gamma + w''\gamma^2)' + v'''\int_0^s v'w''\,\mathrm{d}s\right]' + j_2\ddot{w}'' \\ & - \frac{1}{2}\left\{w'\int_1^s\left[\int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right]^{\cdot\cdot}\mathrm{d}s\right\}' + F_w ,\end{aligned} \tag{94}$$

$$\ddot{\gamma} + \mu_3\dot{\gamma} - \frac{\beta_\gamma}{j_1}\gamma'' = \frac{\beta_y - 1}{j_1}(v''^2\gamma - w''^2\gamma - v''w'') + \left(\int_0^s v''w'\,ds\right)^{\cdot\cdot} - (\dot{v}'w')^{\cdot}$$
$$+ \frac{j_2 - j_3}{j_1}(\dot{v}'^2\gamma - \dot{w}'^2\gamma - \dot{v}'\dot{w}') + F_\gamma . \quad (95)$$

Because the rotary inertias are small, the boundary conditions can be simplified. Thus, the boundary conditions become

$$v = w = v' = w' = \gamma = 0 \quad \text{at } s = 0 , \quad (96)$$

$$I_h\ddot{\Theta}(t) + w''(0, t) = \tau_0(t) , \quad (97)$$

$$\beta_y v''' - j_3\ddot{v}' = w''' - j_2(\ddot{w}' - \ddot{\Theta}) = v'' = w'' = \gamma' = 0 \quad \text{at } s = 1 , \quad (98)$$

where $I_h \equiv I_H/mL^3$ and $\tau_0 \equiv \tau L/D_{22}$. Equations (93)–(95) reduce to those derived by Crespo da Silva and Glynn [8] when $F_v = F_w = F_\gamma = 0$.

If the torsional rigidity is relatively high (e.g., a beam with a square cross-section and long span), the torsional motion cannot be excited by low-frequency input forces because the twisting natural frequency is relatively high. Moreover, in the absence of a torque applied along the beam, the twist is only induced by bending deflections. The induced twist angle can be found by setting all terms with time derivatives equal to zero in equation (95). The result is

$$\frac{\beta_\gamma}{j_1}\gamma'' + \frac{1 - \beta_y}{j_1}(v''w'' + w''^2\gamma - v''^2\gamma) = 0 . \quad (99)$$

Using the boundary conditions $\gamma(0, t) = \gamma'(1, t) = 0$ in equation (125), we find that the twist angle due to bending is

$$\gamma = \frac{\beta_y - 1}{\beta_\gamma}\int_0^s \int_1^s v''w''\,ds\,ds + \text{fourth-order quantities} . \quad (100)$$

Equation (100) shows that the bending induced twisting is a nonlinear phenomenon. For a lengthy isotropic beam, the frequencies of twist oscillations are linear combinations of the bending frequencies in the x and z directions. If any pretwist angle γ_p exists, it must be be considered before using the transformation [1]. Any torque applied to the beam produces a forced torsional motion γ_f. Furthermore, for a composite beam, bending-twisting couplings may induce a torsional motion γ_c. Hence, the total twist angle consists of four parts: γ_p, γ_f, γ_c, and γ. The frequency of γ_c is the same as that of v if the bending motion in the y-direction is coupled with the torsional motion, and the forced torsional motion γ_f may have a different frequency. Hence, we expect the nonlinear torsional motion to be rather complicated.

Using equation (100) and neglecting terms of order higher than three, we reduce equations (93) and (94) to

$$\ddot{v} + \mu_1\dot{v} + \beta_y v^{iv} = (1 - \beta_y)\left[w''\int_1^s v''w''\,ds - w'''\int_0^s v''w'\,ds\right]' + j_3\ddot{v}''$$

$$-\frac{(1-\beta_y)^2}{\beta_y}\left[w''\int_0^s\int_1^s v''w''\,\mathrm{d}s\,\mathrm{d}s\right]'' - \beta_y[v'(v'v''+w'w'')']'$$

$$-\frac{1}{2}\left\{v'\int_1^s\left[\int_0^s (v'^2+w'^2)\,\mathrm{d}s\right]^{\cdot\cdot}\mathrm{d}s\right\}' + F_v\,, \tag{101}$$

$$\ddot{w} + \mu_2\dot{w} + w'''' = -(1-\beta_y)\left[v''\int_1^s v''w''\,\mathrm{d}s - v'''\int_0^s w''v'\,\mathrm{d}s\right]' + j_2\ddot{w}''$$

$$-\frac{(1-\beta_y)^2}{\beta_y}\left[v''\int_0^s\int_1^s v''w''\,\mathrm{d}s\,\mathrm{d}s\right]'' - [w'(w'w''+v'v'')']'$$

$$-\frac{1}{2}\left\{w'\int_1^s\left[\int_0^s (v'^2+w'^2)\,\mathrm{d}s\right]^{\cdot\cdot}\mathrm{d}s\right\}' + F_w\,. \tag{102}$$

The boundary conditions become

$$v = w = v' = w' = 0 \quad \text{at } s = 0\,, \tag{103}$$

$$I_h\ddot{\Theta}(t) + w''(0,t) = \tau_0(t)\,, \tag{104}$$

$$\beta_y v''' - j_3\ddot{v}' = w''' - j_2(\ddot{w}' - \ddot{\Theta}) = v'' = w'' = 0 \quad \text{at } s = 1\,, \tag{105}$$

Equations (101) and (102) are the same as those derived by Crespo da Silva and Glynn [8], except for the presence of F_v and F_w.

For the case of an in-plane vibration of an inextensional cantilever beam, we substitute $w = \gamma = F_v = 0$ into equation (77) and obtain

$$m\ddot{v} + \mu_1\dot{v} = -D_{33}(v'''' + v''''v'^2 + 4v'v''v''' + v''^3)$$

$$+ j_\xi\ddot{v}'' - \frac{1}{2}\left[v'\int_L^s m\left(\int_0^s v'^2\,\mathrm{d}s\right)^{\cdot\cdot}\mathrm{d}s\right]'\,. \tag{106}$$

The nonlinear curvature and inertia terms in equation (106) are the same as those derived by Sato *et al.* [15].

5. Concluding Remarks

Newton's second law is used to develop the nonlinear equations describing the motions of extension, flexures along two principal directions, and torsion of a slewing composite beam exhibiting structural couplings. Three consecutive Euler angles are used to relate the deformed and undeformed states, but twisting curvature is used to define the twist angle and the resulting equations of motion are symmetric and independent of the rotation sequence of the Euler angles. The equations contain bending-twisting, bending-bending, extension-twisting, and extension-bending coupling terms and quadratic and cubic nonlinearities due to curvature, inertia, and midplane stretching. The equations of motion are valid for extensional, inextensional, uniform and nonuniform, and metallic and composite beams. Some comparisons with other derivations are made, and the characteristics of the modeling are addressed.

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