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The complex geometry of the Kowalewski-Painlevé analysis

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Algebraic complete integrable systems are integrable systems whose trajectories are straight line motions on complex algebraic tori, themselves completions of the level manifolds of the system; here space and time must be thought of as complex. Such systems can then be solved by quadratures, that is to say their solutions can be expressed in terms of Abelian integrals. In fact most integrable systems, old and new ones, enjoy this remarkable property.

Exactly one hundred years ago, S. Kowalewski [15] discovered a necessary condition for an n-dimensional system to be algebraic complete integrable: namely, it must possess Laurent solutions depending on $n-1$ parameters. This criterion enabled her to classify all integrable solid body motions about a fixed point; among them she discovered her celebrated top. This criterion, used in a heuristic way by her, was only proven and fully exploited recently.

The purpose of this paper is to show how the Kowalewski criterion and the theory of Abelian varieties are intimately related. Does the Kowalewski criterion guarantee algebraic complete integrability? This outstanding question, often called the Painlevé problem, consists of two parts: (i) given a Hamiltonian system in R^n having the required number (approximately $n/2$) of polynomial constants of motion and possessing families of Laurent solutions having $n-1$ free parameters, is it algebraic complete integrable? (ii) given a Hamiltonian system in R^n, having families of Laurent solutions with $n-1$ free parameters, does the system possess the right number of polynomial constants of motion? In Sect. 1, we answer question (i) and we show how the existence of a coherent set of Laurent solutions depending on $n-1$ free parameters is necessary and sufficient for a Hamiltonian system with the right number of constants of motion to be algebraic complete integrable; question (ii) will be addressed elsewhere.

The main observation is that, if the system possesses several families of $n-1$-dimensional Laurent solutions (principal Laurent solutions), they must fit together in a coherent way; this means the system must, in addition, possess $n-2$, $n-3$, … dimensional Laurent solutions which are the glueing agents of the principal families. The glueing occurs by means of a rational change of

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coordinates, in which both the generic and lower parameter Laurent solutions are holomorphic functions of time and initial data; in terms of these functions, the lower parameter solutions are bona fide limits of the generic Laurent solutions. The generic Laurent solutions correspond to trajectories on the tori passing through the divisors (Painlevé divisors) where the phase variables blow up; the lower-dimensional solutions correspond to the intersection of these divisors, their singularities and the points where the flow is tangent to these divisors. In fact, in most instances, the generic Laurent solutions have the virtue of blowing up the invariant manifold at infinity, at least embedded into some projective space, whereas the rational change of coordinates alluded to above often blows down the embedded variety along a variety at infinity. This will be illustrated in the example of Sect. 4.

Section 1 also shows the following striking fact: given an algebraic complete integrable system of differential equations, it is possible to replace the original set of variables by a new or extended set having the property of forming a closed system of quadratic differential equations. Not only is the derivative of any new variable expressible quadratically in terms of the new variables, but also the derivative of the ratio by an arbitrary variable is expressible quadratically in terms of such ratios! It is indeed these ratios which provide the rational change of coordinates which does the gluing discussed above. This procedure will be illustrated in the example of Sect. 4.

In Sect. 2 and 3 we study the nature of Laurent solutions of weight-homogeneous systems; these sections will play an important role in the effective implementation of the Kowalewski criterion. At first we show that "formal" Laurent solutions, consistent with the weighting, must be convergent. Secondly, finding these Laurent solutions is done by an inductive procedure, involving the so-called Kowalewski matrix $\mathscr{L}$. Let $\mathscr{A}$ be the invariant (or level) manifolds for the flow embedded into an appropriate weighted projective space. In Sect. 3, we establish the precise relationship between

- the spectrum of the Kowalewski matrix $\mathscr{L}$
- the free parameters
- the weighted degrees of the invariants
- the singularity of the projective closure $\mathscr{A}$ at infinity
- the description of the Painlevé divisor.

Indeed, we show that confining each family of Laurent solutions to the invariant manifolds leads to a natural variety, called the Painlevé variety associated with that family; it parametrizes the Laurent solutions on $\mathscr{A}$. These Painlevé varieties will play an important role in this work.

In Sect. 4, the techniques of the previous chapters are applied and illustrated on a geodesic flow on SO(4), for a certain family of metrics; this flow turns out to be equivalent to the motion of a rigid body in fluids, as investigated last century by Lyapounov and Steklov.

Classically, Painlevé has classified the differential equations of the first and second order having "uniform integrals". For complete and definitive results, see Bureau [22]. In 1982, we applied the Kowalewski criterion to classify the integrable geodesic flows on SO(4) [2, 4]. In 1985 we have shown that the

Laurent solutions contain encoded a great deal of information about the invariant tori, their periods and other geometrical features; these ideas were then applied to many integrable systems [5]. In these systems it was shown that the Laurent solutions of different dimensions provide the variables which blow up and blow down the invariant manifolds at infinity (embedded into projective space). Also, among systems of particles with non-nearest neighbor exponential interactions (generalized periodic Toda lattices), we found that the only algebraic complete integrable systems are those whose interaction is governed by the Cartan matrices of Kac-Moody Lie algebras [3]. There we show that the natural phase variables $x_1, x_2, \ldots, x_l$, namely the exponentials of the interactions for the Toda lattices, have the following divisor structure in terms of l divisors on an Abelian variety T^{l-1}:

$$(\text{divisor of } x_i) = -\sum_j a_{ij}\theta_j,$$

where the integers a_{ij} are the entries of the Kac-Moody Cartan matrix. Flaschka and Cheng [9] have further studied Laurent solutions of different dimensions and tied them in with Lie algebras and Sato's theory. Recently Ercolani and Siggia [8] have studied the Laurent solutions going with separable systems. There is an extensive literature on applying the Painlevé test to various systems, by Ablowitz, Ramani and Segur [1], Bountis [7], Dorizzi and Grammaticos [10], Haine [12], Weiss, Tabor and Carnevale [20], and Steeb [19], just to name a few; for excellent review articles, see Kruskal and Clarkson [16] and Hietarinta [13]. Also H. Yoshida [21] has investigated the relation between the spectrum of the Kowalewski matrix and the degrees of the invariants. Finally we thank J. Harris, T. Matsusaka and A. Mayer for valuable help with this project.

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§ 1. Algebraic complete integrable systems

A Hamiltonian vector field X_1,

$$\dot{z} = f(z) = J\frac{\partial H}{\partial z}, \qquad z \in \mathbb{R}^n, \tag{1.1}$$

$$J = J(z) = \left(\begin{array}{l}\text{skew-symmetric matrix with polynomial} \\ \text{entries in } z \text{, for which the corresponding} \\ \text{Poisson bracket } \{H_i, H_j\} \equiv \langle \partial H_i/\partial z, J\,\partial H_j/\partial z\rangle \\ \text{satisfies the Jacobi identities}\end{array}\right)$$

with polynomial right hand side will be called *algebraic complete integrable* (a.c.i.) when:

1. the system is completely integrable with polynomial invariants, i.e., besides the (polynomial) Casimir functions $H_1, \ldots, H_k$ (functions whose gradients are null vectors of J), the system possesses $m=(n-k)/2$ polynomial constants of motion $H_{k+1}=H, \ldots, H_{k+m}$ in involution (i.e., $\{H_i, H_j\}=0$), which give rise to m commuting vector fields X_i generated by (1.1) applied to H_{k+i}, $1 \leqq i \leqq m$; for generic A_i, the invariant manifolds

$$\bigcap_1^{k+m} \{H_i=A_i, z \in \mathbb{R}^n\} \tag{1.2}$$

are assumed compact and connected and therefore real tori according to the classical Arnold-Liouville theorem (see [6]).

2. The invariant manifolds, thought of as living in $\mathbb{C}^n$

$$\mathscr{A}=\bigcap_1^{k+m} \{H_i=A_i, z \in \mathbb{C}^n\}$$

are related, for generic A_i, to an Abelian variety T^m as follows

$$\mathscr{A}=T^m \backslash D$$

where D is a divisor in T^m. In the natural coordinates $(t_1, \ldots, t_m)$ of $T^m=\mathbb{C}^m/L$ coming from $\mathbb{C}^m$, the coordinates $z_i=z_i(t_1, \ldots, t_m)$ are meromorphic and D is the minimal divisor on T^m where the variables z_i blow up. Moreover, the Hamiltonian flows (run with complex time) $\dot{z}=J\,\partial H_{k+i}/\partial z$ $(i=1, \ldots, m)$ are straight-line motions on T^m.

An a.c.i. system will be called *irreducible* when the generic invariant tori do not contain Abelian subvarieties. *Poincaré's reducibility theorem* states that if an Abelian variety T^m contains an Abelian subvariety T^k, there exists another Abelian subvariety T^l and an isomorphism

$$T^k \oplus T^l \to T^m,$$

modulo a finite group $T^k \cap T^l$.

For a divisor D, define

$$L(D)=\{f \text{ meromorphic on } T^m \text{ such that } (f) \geqq -D\}, \tag{1.3}$$

i.e., for $D=\sum k_i D_i$ a function $f \in L(D)$ has at worst a k_i-fold pole along D_i. The divisor D will be called *ample* when a basis $f_0, \ldots, f_N$ of $L(kD)$ embeds T^m smoothly into $\mathbb{P}^N$ for some k, via the map

$$p \in T^m \hookrightarrow (f_0(p), f_1(p), \ldots, f_N(p)) \in \mathbb{P}^N;$$

then kD is called very ample. It is known that every positive divisor D on an irreducible Abelian variety is ample and thus some multiple of D embeds T^m into $\mathbb{P}^N$. By a theorem of Lefschetz (see [11]), any $k \geqq 3$ will work.

We now state three theorems, containing some concepts to be made precise immediately after the statement.

Theorem 1 *(necessary and sufficient conditions for algebraic complete integrability).*

I. *a.c.i. system $\dot{z}=f(z)$, $z \in \mathbb{C}^n$ with invariant tori not containing elliptic curves* $\Rightarrow$ *the system has a coherent tree of Laurent solutions*

II. *a regular Hamiltonian system having $k+m$ polynomial invariants in involution with a coherent tree of Laurent solutions* $\Rightarrow$ *the system is a.c.i.*

Theorem 2 *(closed systems of quadratic differential equations). If an irreducible system $\dot{z}=f(z)$, $z \in \mathbb{C}^n$, is algebraic complete integrable, there exist polynomials $y_0=1, y_1, \ldots, y_N$ of z having the following property: for any choice of holomorphic vector field and for arbitrary $0 \leqq \alpha \leqq N$, we have*

$$\frac{d}{dt}\left(\frac{y_i}{y_\alpha}\right)=\textit{quadratic polynomial}\left(\frac{y_0}{y_\alpha}, \ldots, \frac{y_N}{y_\alpha}\right), \qquad i=0, \ldots, N;$$

in particular, setting $\alpha=0$, the y_i form a closed quadratic system. Moreover the invariant tori of the system are smoothly embedded into $\mathbb{P}^N$ by means of the map:

$$p \in T^m \curvearrowright (y_0, \ldots, y_N)(p) \in \mathbb{P}^N.$$

Theorem 3 *(the complex Arnold-Liouville theorem). Suppose M is an m-dimensional complex compact manifold with m independent meromorphic functions. In addition, assume for some divisor $\bar{D}$, the affine variety $M \backslash \bar{D}$ supports m everywhere independent, commuting, nonvanishing holomorphic vector fields $X_1, \ldots, X_m$, with flows g^{t_i}. Assume that one vector field, say X_1, extends to a holomorphic vector field on M having the property that all of its orbits through $\bar{D}$ go immediately into the affine $M \backslash \bar{D}$, i.e.,*

$$\{g^{t_1}(p) \mid 0<|t_1|<\varepsilon(p)\} \subset M \backslash \bar{D}, \qquad \forall p \in \bar{D}.$$

Then M is an Abelian variety and the X_i extend to holomorphic vector fields on M.

In many problems it is natural to embed the invariant manifolds $\mathscr{A}$ into weighted projective spaces $\mathbb{P}_\nu^n$ for the weights $\nu=(\nu_0, \nu_1, \ldots, \nu_n)$, $\nu_i \in \mathbb{Z}$, $\geqq 1$. These spaces are defined by identifying all points on the curve

$$(z_0 t^{\nu_0}, z_1 t^{\nu_1}, \ldots, z_n t^{\nu_n}), \qquad t \in \mathbb{C}^*$$

running through the origin; whenever $\nu_0=\nu_1=\ldots=\nu_n$, we have the isomorphism $\mathbb{P}_\nu^n=\mathbb{P}^n$. In fact it is always possible to embed $\mathscr{A}$ into some weighted projective space. Indeed, if a system $\dot{z}=f(z)$, $z\in\mathbb{C}^n$, has invariants $H_i(z)$, $i=1,\ldots,N$, of weighted degrees d_i, we embed $\mathscr{A}$ into $\mathbb{P}^n_v$ in the following way:

$$\mathscr{A}=\bigcap_1^N\{H_i(z)=A_i\}\subset\bar{\mathscr{A}}\equiv\{H_i(z)=A_i z_0^{d_i}\}\subset\mathbb{P}^n_v; \tag{1.4}$$

let $\mathscr{A}_\infty$ be the locus at infinity, namely

$$\mathscr{A}_\infty=\bar{\mathscr{A}}\cap\{z_0=0\}=\bigcap_1^N\{H_i(z)=0\}.$$

In appendix 1 we shall elaborate on weighted projective spaces $\mathbb{P}^n_v$ and degree considerations.

Let a system of ordinary differential equations $\dot{z}=f(z)$, $z\in\mathbb{C}^n$, have Laurent solutions in t

$$z_j(t,\alpha,D^{(r)})=t^{-k_j}(z_j^{(0)}(\alpha)+t z_j^{(1)}(\alpha)+\ldots),\quad k_j\in Z,\ \geqq 0,\ 1\leqq j\leqq n,\ r<n, \tag{1.5}$$

whose coefficients $z_j^{(k)}(\alpha)$ are rational functions on an r-dimensional algebraic variety $D^{(r)}$; assume its coordinate ring is generated by the $z_j^{(k)}$. Whenever the system possesses N polynomial invariants, the variety $D^{(N+l)}$ can be viewed as fibered over the space $\mathbb{C}^N$ of values $A_1,\ldots,A_N$ of the N invariants, with l-dimensional fibers, denoted $D^{(l)}(A)$. To see this, confine the solutions (1.5) to the invariant manifold $\mathscr{A}$ for fixed $A_1,\ldots,A_N$ and define

$$\begin{aligned}D^{(l)}(A)&\equiv\bigcap_1^N\{\text{the Laurent}^1\text{ solutions } z(t,\alpha,D^{(N+l)}),\\&\qquad\text{such that } H_i(z(t))=A_i\}\\&\equiv\bigcap_1^N\{\alpha\in D^{(N+l)}\quad\text{such that } F_i(\alpha)=A_i\}.\end{aligned} \tag{1.6}$$

The l-dimensional variety $D^{(l)}(A)$ is called a *Painlevé divisor* and has for coordinate functions the coefficients $z_j^{(\beta)}$. To see the second equality in (1.6), notice that since H is an invariant and since $z(t)$ is a solution of $\dot{z}=f(z)$, the expression $H(z(t))$ must be t-independent; thus

$$H(z(t))=\text{polynomial of }(z_k^{(r)}(\alpha))\equiv F(\alpha),$$

and so F is a rational function on $D^{(N+l)}$. Therefore, for fixed but generic A, the affine variety $D^{(l)}(A)$, with running variable p, parametrizes the Laurent solutions $z(t,p,D^{(l)}(A))$, where $0\leqq l\leqq m-1$, $m\equiv n-N$.

The following lemma shows how to glue the affine varieties $D^{(l)}(A)$ to the invariant variety $\mathscr{A}$.

Lemma 1.1. *The following map*

$$(t,p)\in(0<|t|<\varepsilon(p))\times D^l(A)\curvearrowright z(t,p,D^l(A))\in\mathscr{A}$$

is injective; it defines a biholomorphic map to its image, wherever $D^l(A)$ is smooth, and dimension $D^l(A))=l$.

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[1] whenever they make sense

Proof. To show the injectivity of the map, notice at first that different points $p \in D^{(N+l)}$ lead to different trajectories $z(t, p, D^{(N+l)})$; indeed different p's lead to different coefficients $z_k(p)$ of the series, as the latter generate the coordinate ring of $D^{(N+l)}$. Secondly, orbits on $\mathscr{A}$ cannot intersect by Picard's theorem. The holomorphy of the map and therefore the biholomorphy follow from the fact that $z(t, p, D^{(N+l)})$ is a Laurent series in t with smooth coefficients on the smooth part of $D^{(N+l)}$, again because the coefficients are the coordinates of $D^{(N+l)}$. Letting

$$\Gamma^l(A) \equiv \{z(t, p, D^{(l)}(A)), 0<|t|<\varepsilon(p), p \in D^{(l)}(A)\}$$
$$\Gamma^l \equiv \{z(t, p, D^{(N+l)}), 0<|t|<\varepsilon(p), p \in D^{(N+l)}\},$$

we have the following (in)equalities

$$\dim \Gamma^l(A) = 1 + \dim D^{(l)}(A), \qquad \dim D^{(l)}(A) \geqq l$$
$$\dim \Gamma^l = 1 + \dim D^{(N+l)} = 1 + N + l.$$

Since $\Gamma^l = \bigcup_A \Gamma^l(A)$ and since $\Gamma^l(A) \cap \Gamma^l(A') = \phi$, we have that $\dim D^l(A) = l$ for generic choices of A, ending the proof of Lemma 1.1.

Following an idea of Kowalewski [15] we have shown in [2] that for a system, $\dot{z} = f(z)$, $z \in \mathbb{C}^n$ to be algebraic complete integrable, each phase variable z_i must blow up after a finite (complex) time and the system must have one or several $n-1$-dimensional families $z(t, \alpha, D^{(n-1)})$, as described above. But, as it turns out, much more is true; this will be stated as follows.

Definition. A system $z = f(z)$ in $\mathbb{C}^n$, having N polynomial invariants H_i (set $m \equiv n - N$) has a *coherent tree of Painleve solutions*, when it possesses families of Laurent solutions $z(t, p, D_\alpha^{(m-1)}(A))$, $z(t, p, D_\beta^{(m-2)}(A))$, $\ldots$, $z(t, p, D_\gamma^{(0)}(A))$ in t, depending on $N+m-1$, $N+m-2$, $\ldots$, N free parameters respectively, such that each z_i blows up along some $D_\alpha^{(m-1)}(A)$. To be precise, the coefficients of these Laurent solutions generate the coordinate ring of the algebraic varieties $D_\alpha^{(m-1)}(A), \ldots, D_\gamma^{(0)}(A)$ of dimension $N+m-1, \ldots, N$ respectively; $A = (A_1, \ldots, A_N)$ stands for the values of the N invariants H_i. The families $z(t, p, D_\alpha^{(m-1)}(A))$ are called the *principal families*, and $z(t, p, D_\gamma^{(0)}(A))$ the *lowest families*. Henceforth we shall omit the argument A; thus $D_\alpha^{(k)} \equiv D_\alpha^{(k)}(A)$. These families organize themselves in a coherent tree, involving

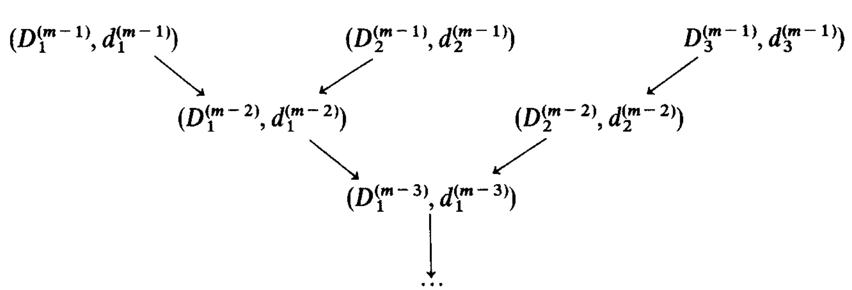

Fig. 1.1

inclusions $D_\alpha^{(k)} < D_\beta^{(l)} (k<l)$ such that each $D_\alpha^{(k)}$ has at least one $D_\beta^{(k+1)} > D_\alpha^{(k)}$. Such a tree is depicted in Fig. 1.1; the arrows "$\leftarrow$" in the tree stand for inequalities $<$ meaning "is glued onto", whereas the $d_\alpha^{(k)}$ are integers. Their precise meaning will be explained below. Each level of the tree corresponds to solutions depending on a number of parameters, given by the superscript on D; the subscripts $\alpha, \beta, \ldots, \gamma$ refer to the different families. Each $z(t, p, D_\alpha^{(k)})$ can be viewed as a *fibre bundle* over the corresponding (affine) Painleve variety $D_\alpha^{(k)}$. The graph

$$\begin{array}{ccc} z(t,p_1,D_1^{(m-1)}) & & z(t,p_2,D_2^{(m-1)}) \\ & \searrow \quad \swarrow & \\ & z(t,q,D^{(m-2)}) & \end{array}$$

means that the Painleve solution $z(t, q, D^{(m-2)})$ is *glued onto* the principal solutions $z(t, p_1, D_1^{(m-1)})$ and $z(t, p_2, D_2^{(m-1)})$, forming a new fibre bundle

$$z(t, p, D_1^{(m-1)} \amalg D_2^{(m-1)} \amalg D^{(m-2)}),$$

over the manifold $D_1^{(m-1)} \amalg D_2^{(m-1)} \amalg D_2^{(m-2)}$. One then performs the glueing procedure inductively down the levels of the tree; that this forms a fibre bundle at every step of the way, enabling one to continue down the tree, is part of the content of Theorem 1.

What the glueing means is now explained: there exists birational[2] maps

$$S_i\colon (t, p_i) \curvearrowright (t, q) \qquad i=1,2$$

and

$$T^{(m-2)}\colon z=(z_1, \ldots, z_n) \curvearrowright y=(y_1, \ldots, y_N)$$

such that

(i) both series $(i=1,2)$ (holomorphic continuation)

$$y(t, q, D_i^{(m-1)}) = T^{(m-2)} z(S_i(t, p_i), D_i^{(m-1)})$$

become Taylor series[3] in $0 \leqq t \leqq \varepsilon$, $q \in N(D^{(m-2)}) \subset D_i^{m-1}$. Also we require that

(ii) the series $y(t, q, D_i^{(m-1)})$ parametrizes an m-dimensional complex ball for at least one $D_i^{(m-1)} > D^{(m-2)}$

(iii) the families $y(t, q, D_1^{(m-1)})$ and $y(t, q, D_2^{(m-1)})$ restricted to $q \in D^{(m-2)}$ are the same.

(iv) from the above, the vector of functions

$$\begin{aligned} \tilde{z}(t, q, D_\alpha^{(m-1)}) &\equiv (T^{(m-2)})^{-1} y(t, q, D_\alpha^{(m-1)}) \\ &= \left(\frac{h_i(t,q)}{g_i(t,q)}\right)_{1 \leqq i \leqq n} \qquad \text{for } q \in N(D^{(m-2)}) \subset D_\alpha^{(m-1)} \\ &= z(t, q, D^{(m-2)}) \qquad \text{for } q \in D^{(m-2)} \end{aligned}$$

can be represented as ratios of holomorphic (relatively prime) functions h_i and g_i (in good coordinates of the ball in (ii)); also from the above the lower family

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[2] The maps are given locally in the charts of $D_\alpha^{(k)}$

[3] $N(D^{(m-2)}) \subset D_i^{(m-1)}$ means a small enough neighborhood of $D^{(m-2)}$ in $D_i^{(m-1)}$

$z(t, q, D^{(m-2)})$ for $q \in D^{(m-2)}$ is obtained from $\tilde{z}(t, q, D_\alpha^{(m-1)})$ by taking the Laurent series in t for the ratios h_i/g_i and letting q tend to the points of $D^{(m-2)}$. We now require that

$$g_i(t, q) \not\equiv 0 \text{ as a Taylor series in } t, \qquad \text{for } q \in D^{(m-2)}$$

and

$$\sum_{\text{all } D_\alpha^{(m-1)} > \text{fixed } D^{(m-2)}} (\text{order of pole of } z_i \text{ on } D_\alpha^{(m-1)}) = \text{multiplicity of } g_i(t, p);$$

the multiplicity of g is the lowest degree r appearing in the decomposition $g = (g)_r + (g)_{r+1} + \dots$ in homogeneous polynomial $(g)_j$ (see Mumford [25]).

As mentioned, working one's way inductively down the tree, using the same glueing recipe as above, we get the fibre bundle

$$z\Big(t, p, \coprod_{\substack{0 \leq k \leq m-1 \\ \text{all } \alpha}} D_\alpha^{(k)}\Big)$$

(v) Finally the global condition is required to hold

$$\sum_{\alpha, k} d_\alpha^{(k)} = \frac{\prod_1^N (\text{degrees of } H_i)(\text{l.c.m.}(v_1, \dots, v_n))^{m-1}}{v_1 \dots v_n}$$

for

$$d_\alpha^{(k)} = \text{degree} \begin{pmatrix} \text{the } m-1\text{-dimensional components of the image} \\ (\tilde{z}(0, p, D_\alpha^{(k)}), p \in D_\alpha^{(k)}) \text{ in } \mathbb{P}_v^n \end{pmatrix}$$

A system $\dot{z} = f(z)$, $z \in \mathbb{C}^n$, with N polynomial invariants and a coherent tree of Laurent solutions, will be called *regular* when for all principal Laurent solution $D_\alpha^{(m-1)}$, the image of $D_\alpha^{(m-1)}$ in $\mathbb{P}_v^n$ via the map

$$p \in D_\alpha^{(m-1)} \curvearrowright z(0, p, D_\alpha^{(m-1)}) \in \mathbb{P}_v^n$$

has dimension $m-1$, along with all the components of $\mathscr{A}_\infty \equiv \bar{\mathscr{A}} \cap \{z_0 = 0\}$. This property will be elaborated on in Appendix 2.

We now prove the three theorems stated above:

Proof of Theorem 2. Since every divisor D on an irreducible Abelian variety is known to be ample, we saw in (1.3) that some multiple kD is very ample; it suffices to pick any $k \geqq 3$. A theorem of Koizumi [14] and Mumford [26] asserts that D is projectively normal[4] whenever D is linearly equivalent on T^m to at least $3D_0$, where D_0 is an ample divisor. Therefore for any ample divisor D, the divisor $3D$ will be both very ample and projectively normal.

[4] D is projectively normal, when

$$L(kD) = L(D)^{\otimes k},$$

i.e., every function in $L(kD)$ is a homogeneous polynomial of degree k of functions in $L(D)$

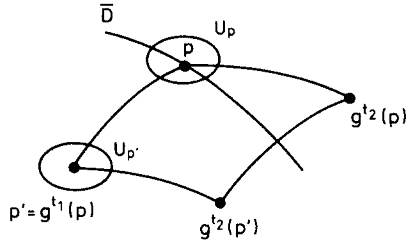

Fig. 1.2

Applying these ideas to an irreducible a.c.i. system and to the divisor $D \equiv \bigcup_1^n \{z_i^{-1}=0\} \subset T^m$, we have that $D'=D$ or $2D$ or $3D$ is very ample and projectively normal. Thus $L(D')$ consists of polynomials of $\{z_0=1, z_1, \ldots, z_n\}$; picking a basis $y_0, y_1, \ldots, y_N$ of them, we have that the Wronskian

$$\begin{aligned}\{y_i, y_j\} &= (Xy_i)y_j - y_i(Xy_j) \in L(2D') \\ &= \sum_{0 \leqq k \leqq l \leqq N} a_{kl}\, y_k y_l, \quad \text{because } L(2D') = L(D')^{\otimes 2},\end{aligned} \tag{1.7}$$

is a homogeneous quadratic function of the $y_i \in L(D')$, and also that

$$p \in T^m \curvearrowright (y_0(p), y_1(p), \ldots, y_N(p)) \in \mathbb{P}^N$$

embeds T^m smoothly into $\mathbb{P}^N$. Dividing (1.7) by an arbitrary but fixed y_α^2 leads to the statement of theorem 2.

Proof of Theorem 3. We first extend $X_1, \ldots, X_m$ to m everywhere independent holomorphic vector fields on M. Indeed, first observe X_1 is everywhere nonvanishing; by the assumptions, X_1 has no fixed points on $\bar{D}$ and does not vanish on $M \backslash \bar{D}$. We now extend X_2 to $\bar{D}$. For $p \in \bar{D}$ and small $\varepsilon > 0$, we have by hypothesis

$$p' = g^{t_1}(p) \in M \backslash \bar{D}, \qquad \text{for all } 0 < |t_1| < \varepsilon.$$

Let $U_{p'} \subset M \backslash \bar{D}$ be a neighborhood of p' as in Fig. 1.2 and let $U_p = g^{-t_1}(U_{p'})$. Define for all $m \in U_p$ and t_2 small enough the following

$$g^{t_2}(m) \equiv g^{-t_1} g^{t_2} g^{t_1}(m).$$

To show this is a definition, we must show the right hand side is independent of t_1. Indeed

$$\begin{aligned} g^{-(t_1+\varepsilon_1)} g^{t_2} g^{t_1+\varepsilon_1}(m) &= g^{-(t_1+\varepsilon_1)} g^{t_2} g^{t_1} g^{\varepsilon_1}(m) \\ &= g^{-(t_1+\varepsilon_1)} g^{\varepsilon_1} g^{t_2} g^{t_1}(m) \quad \text{(by commutativity)} \\ &= g^{-t_1} g^{t_2} g^{t_1}(m). \end{aligned}$$

Again by commutativity, this definition agrees with g^{t_2} away from $\bar{D}$. Finally $g^{t_2}(m)$ is a holomorphic function of m and t_2, because in U_p the function g^{t_1}

is holomorphic and its image is away from $\bar{D}$ (i.e., in $M \backslash \bar{D}$), where g^{t_2} is holomorphic. Thus X_2 and similarly X_i have been holomorphically extended to M. Since $X_1, \ldots, X_m$ are everywhere independent and commuting in $M \backslash \bar{D}$ and g^{t_1} is holomorphic, they will enjoy this same property near $\bar{D}$ and hence on M, and so this accomplishes our first goal.

Next we show M is conformal to a complex torus $\mathbb{C}^m/L = T^m$, and so in particular M is a Kähler manifold. Indeed, by the same argument as in the Arnold-Liouville theorem (see V. Arnold [6]), one defines a holomorphic local diffeomorphism for a fixed origin $p \in M$:

$$\mathbb{C}^m \to M: (t_1, \ldots, t_m) \curvearrowright g^{t_1} g^{t_2} \ldots g^{t_m}(p).$$

The additive subgroup Λ

$$\Lambda \equiv \{(t_1, \ldots, t_m) \in \mathbb{C}^m \quad \text{such that } g^{t_1} g^{t_2} \ldots g^{t_m}(p) = p\}$$

of $\mathbb{C}^m$ is discrete and hence is spanned by $2m$ vectors in $\mathbb{C}^m$, independent over $\mathbb{R}$, as a consequence of the compactness of M. Therefore M is conformal to a complex torus $\mathbb{C}^m/\Lambda$ as claimed, with the Kähler metric given by $\sum_{i=1}^{m} dt_i \otimes d\bar{t}_i$. But by a famous result of Mŏishezon [17], a compact, complex Kähler manifold having as many independent meromorphic functions as its dimension is a projective variety. Thus M is both a projective variety and a complex torus, and hence an Abelian variety, finishing the proof of Theorem 3.

Before proceeding to the proof of Theorem 1, we prove

Lemma 1.2. *If $D \subset \mathbb{C}^k$ is a disc about the origin with*

$$\phi_1: D \to \mathbb{C}^n, \quad \textit{holomorphic, rank} \left.\frac{\partial \phi_1}{\partial z}\right|_0 = k$$

$$\phi_2: D \to \mathbb{C}^n, \quad \textit{holomorphic, } \phi_2(D) \textit{ a k-dimensional set,}$$

and if $\phi_2(U) \subset \phi_1(D)$, with U an open set in D, then

$$\phi_1(D) = \phi_2(D)$$

and it is a smooth variety.

Proof. By the implicit function theorem, $\phi_1(D)$ is an analytic variety, locally given by holomorphic equations, $f_1 = f_2 = \ldots = f_{n-k} = 0$, and so $\phi_2(U)$ satisfies these equations as does $\phi_2(D)$ (by analytic continuation), and since by continuity $\phi_2(D)$ is connected, $\phi_1(D) = \phi_2(D)$, as claimed.

Lemma 1.3 (*Weierstrass Preparation Theorem* [18, 25]). *If $H(z)$, $z \in \mathbb{C}^{n+1}$, is a convergent power series about the origin with lowest degree monomial of degree $k \geq 1$, then after a nonsingular linear change of coordinates σ: $z \curvearrowright (s, t)$, $s \in \mathbb{C}$, $t \in \mathbb{C}^n$, we may represent H as follows:*

$$\begin{aligned} H &= \textit{Weierstrass polynomial} \cdot \textit{unit} \\ &= (s^k + a_1(t) s^{k-1} + \ldots a_k(t)) \cdot (1 + 0(s, t)), \end{aligned} \tag{1.8}$$

with $O(s,t)$ *and* $a_i(t)$ *being convergent power series with* $a_i(0)=0$; *moreover, the monomials of degree k of the Weierstrass polynomial are obtained from the monomials of degree k of H, upon performing the linear transformation* σ.

Sketch of proof. First observe that after a nonsingular linear transformation of the form σ, the hypothesis guarantees the presence of a term of the form $c\cdot s^k$ in H; indeed, almost any transformation σ will produce that effect, and upon rescaling s, set $c=1$. Next observe that if the identity (1.8) holds, we have the following Laurent representation for its constituents[5]:

$$\begin{aligned}&\text{Weierstrass polynomial} = s^k \exp\{\text{polar part}_s(\log s^{-k} H(s,t))\}\\ &\text{Unit} = s^k \exp\{\text{holomorphic part}_s(\log s^{-k} H(s,t))\},\end{aligned} \tag{1.9}$$

where we view the above as Laurent series in s over the coefficient ring $\mathscr{R}$ of Taylor series in t with nonconstant terms. Indeed (1.8) implies

$$\begin{aligned}\log(s^{-k} H(s,t)) &= \log\left(1+\frac{a_1(t)}{s}+\dots\frac{a_k(t)}{s^k}\right)+\log(1+O(s,t))\\ &= (\text{series in } s^{-1} \text{ over } \mathscr{R}) + (\text{series in } s \text{ over } \mathscr{R}).\end{aligned} \tag{1.10}$$

Thus we have shown (1.8) yields (1.9); to deduce formulas (1.8) from the hypothesis of the lemma, use formulas (1.9) to define the Weierstrass decomposition (1.8) and Hartog's Lemma to prove convergence. Formula (1.9) then enables one to deduce the last assertion in the lemma.

Proof of Theorem 1, part I. It is assumed that the flow X_1: $\dot{z}=f(z)$, $z\in\mathbb{C}^n$, is a.c.i.; therefore the coordinates z_i are meromorphic functions on the Abelian varieties T_A^m, where $m\equiv n-N=n-$ (number of invariants). Also

$$\mathscr{A}=\bigcap_1^N\{H_i=A_i, z\in\mathbb{C}^n\}=T_A^m\backslash\bar{D}^{(m-1)}(A)$$

where the ($m-1$-dimensional) effective divisor

$$\bar{D}^{(m-1)}(A)\equiv\bigcup_{i=1}^n\{p\in T^m, z_i^{-1}(p)=0\}=\sum_\alpha \text{components } \bar{D}_\alpha^{(m-1)}(A)$$

may consist of several components. Each function z_k will be given by the ratio of two theta-functions θ'_k and θ_k of the variables $(t_1,\dots,t_m)\in\mathbb{C}^m$, where t_i is the time variable of each of the commuting flows X_i on T^m. Assume now the vector field X_1 is transversal[6] to every component $\bar{D}_\alpha^{(m-1)}(A)$. For fixed k, consider one of the components $\bar{D}_\alpha^{(m-1)}(A)$ along which z_k blows up; thus z_k has at

[5] $\exp x=1+x+\dots+\frac{x^n}{n!}+\dots$, $\log(1+x)=x-\frac{x^2}{2}+\dots$

[6] at least on a Zariski open set

least a simple pole along that component $\bar{D}_\alpha^{(m-1)}(A)$. If $\phi^t(p, A)$ is the trajectory of X_1 starting at $p\in T_A^m$, we have

$$z_k(\phi^t(p, A)) = \frac{\theta'_k(t_1+t, t_2, \ldots, t_m)}{\theta_k(t_1+t, t_2, \ldots, t_m)}, \qquad (t_1, \ldots, t_m) = p \in \bar{D}_\alpha^{(m-1)}(A),$$

$$= \frac{\sum_r^\infty t^j U_k^{(j)}(p)}{\sum_s^\infty t^j V_k^{(j)}(p)},$$ upon expanding in a Taylor series in t, and using the fact that z_k has a pole along $\bar{D}_\alpha^{(m-1)}(A)$.

$$(U_k^{(r)}, V_k^{(s)} \not\equiv 0, s > r \geqq 0) \tag{1.11}$$

$$= \frac{1}{t^{s-r}} \frac{U_k^{(r)}(p)}{V_k^{(s)}(p)} + 0(t),$$ upon dividing the two Taylor series. (1.12)

Observe that the function $\theta_k(t_1+t, t_2, \ldots, t_m)$ is not identically zero in t, since we have assumed (for most A) the flow X_1 is transversal to every component of D_A, at least in a Zariski open set.

We now define the affine variety

$$D_\alpha^{(m-1)}(A) \equiv \bar{D}_\alpha^{(m-1)}(A) \cap \{p, V_k^{(s)}(p) \neq 0 \text{ for each } k=1, \ldots, n \text{ with } s \text{ as in (1.11)}\},$$

which supports a fibre bundle, whose fibres are given by the vector of Laurent series (1.12); they will be denoted by $z_k(t, p, D_\alpha^{(m-1)}(A))$, $1 \leqq k \leqq n$, for $p\in D_\alpha^{(m-1)}(A)$. Next we show the coefficients of the Laurent series generate the coordinate ring of the variety $D_\alpha^{(m-1)} \equiv \bigcup_A D_\alpha^{(m-1)}(A)$, obtained by varying A.

Since the z_i generate the field of meromorphic functions of an affine chart of T_A^m, they generate the field of T_A^m and so suitable rational functions of the z_i, restricted to the component $D_\alpha^{(m-1)}(A)$, generate its field of rational functions and provide a coordinatization in any chart, to wit

$$B_j = R_j(z)|_{D_\alpha^{(m-1)}(A)}, \qquad j=1, \ldots, r,^7 \tag{1.13}$$

while on T_A^m we have

$$A_j = H_j(z), \qquad j=1, \ldots, N. \tag{1.14}$$

Given the component $D^{(m-1)}(A)$ and the corresponding Laurent series

$$z_l(t, p, D^{(m-1)}(A)) = t^{-k_l} \sum_{j=0}^{\infty} z_l^{(j)}(p, A) t^j, \tag{1.15}$$

we claim that the map

$$\{z_l^{(j)}\}_{\substack{0\leqq j\leqq\infty \\ 1\leqq l\leqq n}} \curvearrowright (B_1, \ldots, B_r, A_1, \ldots, A_N) \in D^{(n-1)} \tag{1.16}$$

[7] r may be much larger than $m-1$

is birational. Indeed putting the series (1.15) into (1.13) and (1.14) and setting $t=0$ shows this map is rational. To show it is one-to-one, we compute its inverse: by definition, the functions A and B specify a unique point (p, A) on $D^{(n-1)}$; consider the unique trajectory running through that point and the functions z_l evaluated along the trajectory, yielding the Laurent series $z_l(t, p, D^{(m-1)}) = t^{-k_l} \sum_{j=0}^{\infty} z_l^{(j)} t^j$ and thus the set of coefficients $z_l^{(j)}$ constructing the inverse map (1.16). But a rational one-to-one map is automatically birational, concluding the proof that the coefficients of the Laurent solutions generate the coordinate ring of $D^{(n-1)}$.

The arguments given above do not hold if X_1 leaves invariant a component D' of $\bar{D}^{(m-1)}(A)$, at least for generic A_i. Then for any $p \in D'$, the Zariski closure $\overline{\{\phi^t(p), t \in \mathbb{P}^1\}}$ of the group $\{\phi^t(p), t \in \mathbb{P}^1\}$ is an Abelian subvariety of T_A^m, and so we have $\overline{\{\phi^t(p), t \in \mathbb{P}^1\}} \subset D'$. If $\overline{\{\phi^t(p), t \in \mathbb{P}^1\}} \neq D'$, the divisor D' would contain a continuously varying (parametrized by p) family of Abelian subvarieties, which is impossible. Therefore we have that $\overline{\{\phi^t(p), t \in \mathbb{P}^1\}} = D'$ for $p \in D'$, and so by Poincaré's reducibility theory of Abelian varieties, we have

$$T_A^m = D' \oplus \text{elliptic curve (up to an isogeny).}$$

This shows that if a component of $D^{(m-1)}(A)$ is invariant under the flow X_1, the Abelian variety T_A^m must contain an elliptic curve, which contradicts the hypothesis.

So far we have shown the existence of $n-1$-dimensional families of Laurent solutions $z(t, p, D_\alpha^{(m-1)}(A))$ such that each coordinate function z_l blows up on one of them, i.e., we have verified the criterion as initially used by Kowalewski.

We now show that the next level of Laurent solutions $z(t, p, D^{(m-2)})$ form a fibre bundle over $m-2$-dimensional affine[8] subvarieties $D_\beta^{(m-2)}(A)$ of $\bar{D}_\alpha^{(m-1)}(A)$, with

$$\bar{D}_\beta^{(m-2)}(A) \equiv \bar{D}_\alpha^{(m-1)}(A) \cap \{V_k^{(s)} = 0\} \qquad \text{(a component).}$$

Indeed[9], setting $V_k^{(s)}(p)=0$ in (1.12) forces us to recompute the series (1.11), as then (1.12) becomes meaningless, yielding a new series for z. Let $s_1 > s$ be the first integer for which $V_k^{(s_1)}(p) \neq 0$ for generic p in $\bar{D}_\beta^{(m-2)}(A)$; then we have from (1.11)

$$z_k(\phi^t(p, A)) = \frac{\sum_{r_1}^{\infty} t^j U_k^{(j)}(p)}{\sum_{s_1}^{\infty} t^j V_k^{(j)}(p)} \qquad \text{along } \bar{D}_\beta^{(m-2)} \cap \{V_k^{(s_1)}(p) \neq 0\}$$

$$= \frac{1}{t^{s_1 - r_1}} \frac{U_k^{(r_1)}(p)}{V_k^{(s_1)}(p)} + 0(t);$$

8 algebraic by analytic extension and Chow's lemma

9 the locus where $V_k^{(A)}(p)=0$ corresponds to places where, for instance, $\bar{D}_\alpha^{(m-1)}(A)$ becomes singular or where the flow becomes tangent to $\bar{D}_\alpha^{(m-1)}(A)$

this Laurent series – which actually may be a Taylor series – forms a fibre bundle above the $m-2$-dimensional affine subvariety

$$D_{\beta}^{m-2}(A) \equiv \bar{D}_{\beta}^{(m-2)}(A) \cap \{V_k^{(s_1)}(p) \neq 0, k=1, \ldots, n\}$$

of $\bar{D}_{\alpha}^{(m-1)}(A)$. This defines the "inequality":

$$\{D_{\beta}^{(m-2)}(A) < D_{\alpha}^{(m-1)}(A))\} \leftrightarrow \{D_{\beta}^{(m-2)} \text{ is glued onto } D_{\alpha}^{(m-1)}\}.$$

And so it goes, yielding a tree of $m-1$, $m-2$, $m-3, \ldots, 0$-dimensional affine varieties $D_{\alpha}^{(m-1)}$, $D_{\beta}^{(m-2)}$, ... $D_{\gamma}^{(0)}$, each of them supporting a fibre bundle of Laurent series, constructed inductively by the above method.

Moreover assembling all members of the tree, in the way specified by the inequality $<$ yields the original divisor on T^m:

$$\coprod_{\substack{k=0 \\ \text{all } \beta}}^{m} D_{\beta}^{(k)} = \bigcup_{i=1}^{n} \{p \in T^m, z_i^{-1}(p) = 0\} = \bar{D}^{(m-1)}(A).$$

Observe that the variety $D^{(m-k)}$ may be glued onto one or several divisors $D^{(m-1)}$. The requirements (i) to (v) of the coherence condition is the reflection of how the Abelian variety T^m looks in good coordinates near these varieties $D^{(m-k)}$; this we shall now verify. In a neighborhood of a point of $p \in D_{\beta}^{(k)}$ of T^m, some rational functions $y_1, \ldots, y_N$, of z form a good system of holomorphic coordinates on the Abelian variety, and upon enlarging the system if necessary, they define a birational map $T^{(k)}$ from $(z_1, \ldots, z_n)$ to $(y_1, \ldots, y_N)$:

$$(y_1, \ldots, y_N) = T^{(k)}(z_1, \ldots, z_n).$$

In these new coordinates, the flow X_1 is holomorphic, and the function

$$T^{(k)}(z(t, p, D_{\alpha}^{(m-1)})), \qquad p \in N(D_{\beta}^{(k)}) \subset D_{\alpha}^{(m-1)}, \; 0 \leqq |t| < \varepsilon$$

is holomorphic in t and p for all $\bar{D}_{\alpha}^{(m-1)} \supset D_{\beta}^{(k)}$, after perhaps a birational change of coordinate p. The orbits in T^m emanating from $D_{\beta}^{(k)}$ must of course be independent of $\bar{D}_{\alpha}^{(m-1)} \supset D_{\beta}^{(k)}$, verifying (i) and (iii); in addition, since the neighborhood $N(D_{\beta}^{(k)}) \subset \bar{D}_{\alpha}^{(m-1)}$ is a section for the X_1 flow (as X_1 is only tangent to $\bar{D}_{\alpha}^{(m-1)}$ on a subvariety), we have that

$$\{T^{(k)}(z(t, p, D_{\alpha}^{(m-1)})), p \in N(D_{\beta}^{(k)}) \subset D_{\alpha}^{(m-1)}, |t| < \varepsilon\}$$

parametrizes an m-dimensional complex ball in T^m, verifying (ii), and what follows will be expressed in good coordinates on the ball.

Now observe that

$$(T^{(k)})^{-1}[y(z(t, p, D_{\alpha}^{(m-1)}))] = \left[\frac{h_i(t, p)}{g_i(t, p)}\right]_{1 \leqq i \leqq n}$$

provides a meromorphic representation of the variables z on T^m in the neighborhood $N(D_{\beta}^{(k)}) \subset \bar{D}_{\alpha}^{(m-1)} \subset T^m$. If $D_{\beta}^{(k)}$ belongs to several $\bar{D}_{\alpha}^{(m-1)}$, then the local behavior of z_i along each $\bar{D}_{\alpha}^{(m-1)}$ (away from $D_{\beta}^{(k)}$) will be reflected as a factor in the Weierstrass polynomial of the denominator $g_i(t, p)$, leading to the summation formula stated in (iv) (see also Lemma 1.3). Also note that $g_i(p, t) \not\equiv 0$ in t for any $p \in D_\beta^{(k)}$; otherwise the orbit $\{z(t, p, D_\alpha^{(m-1)}), t \in \mathbb{P}^1\}$ through $p \in D_\beta^{(k)}$ would be contained in $D_\alpha^{(m-1)}$ and thus by a previous argument T_A^m would contain an elliptic curve, which has been ruled out. All together, this verifies (iv).

We now verify (v); indeed, first remember that the invariant manifolds $\mathscr{A}$ can always be embedded into some weighted projective space $\mathbb{P}_\nu^n$ (see (1.4)), yielding $\bar{\mathscr{A}}$. Thus the correspondence $T^m \simeq \mathscr{A}$ leads to the map:

$$\Phi: T^m \simeq \bar{\mathscr{A}} \subset \mathbb{P}_\nu^n.$$

Since under this correspondence

$$\Phi: \bar{D}^{(m-1)}(A) = \bigcup_{i=1}^{n} \{z_i^{-1} = 0\} = \coprod D_\alpha^{(k)}(A) \simeq \mathscr{A}_\infty = \bar{\mathscr{A}} \cap \{z_0 = 0\},$$

we conclude that

$$\Phi: \coprod_{k,\alpha} D_\alpha^{(k)} \simeq \sum_{k,\alpha} \{\tilde{z}(0, p, D_\alpha^{(k)}), p \in D_\alpha^{(k)}\} = \mathscr{A}_\infty.$$

But by Bézout's theorem (see Appendix 1), we have

$$\begin{aligned} \text{degree}(\text{hyperplane} \cap \bar{\mathscr{A}}\} &= \text{degree}(\mathscr{A}_\infty) \\ &= \sum \text{degree}(\text{components of } \mathscr{A}_\infty) \\ &= \sum \text{degrees}\{\tilde{z}(0, p, D_\alpha^{(k)}), p \in D_\alpha^{(k)}\} \\ &\equiv \sum_{k,\alpha} d_\alpha^{(k)}, \end{aligned}$$

concluding the verification of (v) and part I of Theorem 1.

Proof of Theorem 1, part II. The strategy is to glue the Painlevé varieties $D_\alpha^{(k)}$, appearing in the coherent tree, onto the smooth affine variety $\mathscr{A} = \bigcap_1^N \{H_i = A_i\}$, thus creating a m-dimensional complex manifold M, such that

$$\mathscr{A} = M \backslash \bar{D}^{(m-1)}, \quad \text{with } \bar{D}^{(m-1)} = \coprod_{\substack{0 \leq k \leq m-1 \\ \text{all } \alpha}} D_\alpha^{(k)}.$$

The coherence condition enables one to assemble the Painlevé varieties $D_\alpha^{(k)}$ and the expansions $z(t, p, D_\alpha^{(k)})$ into a fibre bundle on M. Then the independent commuting vector fields $X_1, \ldots, X_m$ defined on it and generated by the Hamiltonians H_{k+j}, $1 \leq j \leq m$, all extend to commuting holomorphic vector fields on M. Thus we will satisfy the conditions of Theorem 3, by showing that M is compact and that it has m algebraically independent meromorphic functions. Indeed m of the functions $z_1, \ldots, z_n$ on $\mathscr{A}$ will be extended to m independent meromorphic functions on M, again using the coherence condition. Finally, upon employing the valuative criterion of properness (see Hartshorne [24]), condition (v) will yield the compactness of M and then Theorem 3 will do the rest. For the sake of simplicity we will refer to the case $m = 2$.

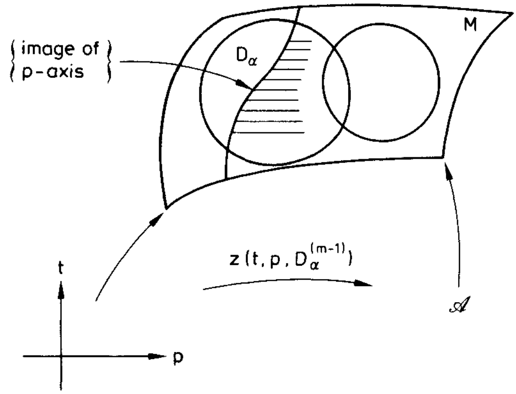

Fig. 1.3

The complex manifold M will be assembled by taking the affine chart $\mathscr{A}$, and by glueing on various charts "at infinity", using the Laurent solutions $z(t, p, D_\alpha^{(k)})$; the glueing recipe is explained in Lemma 1.1 and illustrated in Fig. 1.3.

The lines $p=\text{constant}$[10] in the (t, p) plane map to X_1-trajectories of M and the p-axis maps to the smooth variety $D_\alpha^{(m-1)}$. Moreover the intersection of the affine chart and the one at infinity, for t small and p in an open set $\subset D_\alpha^{(m-1)}$ corresponds to

$$\{z(t, p, D_\alpha^{(m-1)}), 0<|t|<\varepsilon, p\in \text{affine part} \subset D_\alpha^{(m-1)}\}.$$

By Lemma 1.1, the glueing map $(t, p) \curvearrowright z(t, p, D_\alpha^{(m-1)})$ is biholomorphic in the intersection of the charts. Observe $z(t, p, D_\alpha^{(m-1)})$ defined in the chart at infinity is a meromorphic continuation of z as defined in the affine chart. A change of parametrization on the smooth manifold $D_\alpha^{(m-1)}$ will trivially yield a change of parametrization of the chart at infinity.

We now come to the main point, building charts about the $D_\beta^{(k)}$ for $k<m-1$. As announced, consider the case $m=2$, where the upper level divisors $D_\alpha^{(m-1)}$ are curves and the next level varieties $D_\alpha^{(m-2)}$ are points, as illustrated in Fig. 1.4.

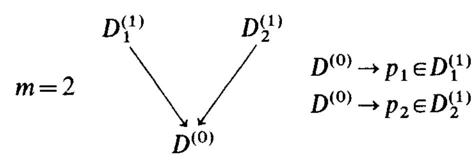

Fig. 1.4

Then the coherence condition yields the existence of a birational map $T: z \curvearrowright y$ depending on $D^{(0)}$ having the properties

[10] We shall purposely confuse the point p with its local parametrization so as to reduce notation

(i) $y(t, p, D_\alpha^{(1)}) \equiv Tz(t, p, D_\alpha^{(1)})$ is a Taylor series in the disc $|t|<\varepsilon$, $p \in N(p_\alpha) \subset D_\alpha^{(1)}$, for $\alpha = 1, 2$,

(ii) $\{y(t, p, D_\alpha^{(1)}), |t|<\varepsilon, p \in D_\alpha^{(1)}\}$ is a $\mathbb{C}^2$-disc, say, for $\alpha = 1$, meaning that

$$\operatorname{rank}\left[\frac{\partial y}{\partial t}(t, p, D_1^{(1)}), \frac{\partial y}{\partial p}(t, p, D_1^{(1)})\right]\Bigg|_{\substack{t=0\\ p=p_1}} = 2,$$

possibly after a reparametrization[11],

(iii) $y(t, p_1, D_1^{(1)}) = y(t, p_2, D_2^{(1)}) \equiv y(t, D^{(0)}), \quad 0 \leqq |t| < \varepsilon$

(iv) $\tilde{z}(t, p, D_1^{(1)}) \equiv T^{-1}(y(t, p, D_1^{(1)}))$

$$\begin{aligned} &= \text{rational functions in } y, \text{ evaluated at } y(t, p, D_1^{(1)}) \\ &= \left(\frac{h_i(t, p)}{g_i(t, p)}\right)_{1 \leqq i \leqq n}, \quad \text{for } p \in N(p_1) \subset D_1^{(1)} \\ &= z(t, D^{(0)}), \quad \text{for } p = p_1 \end{aligned} \tag{1.17}$$

with the h_i and g_i being relatively prime holomorphic functions satisfying

$$g_i(t, p_1) \neq 0 \quad \text{for } 0 < |t| < \varepsilon;$$

upon defining

$$v_i(D_\alpha^{(1)}) \equiv \text{order of pole of } z_i \text{ on } D_\alpha^{(1)}, \tag{1.18}$$

we have for all i that

$$v_i(D_1^{(1)}) + v_i(D_2^{(1)}) = \text{multiplicity of the series } g_i(p, t),$$

(v) The same statement as in the general coherence condition, but with $m = 2$.

We first show $\tilde{z}(t, p, D_1^{(1)})$ is a meromorphic extension of $z(t, p, D_1^{(1)})$ in (t, p)-space. Observe that $z(t, p, D_1^{(1)})$ is defined in a sector of the form

$$S = \{(t, p) \mid 0 < |t| < \varepsilon(p), p \sim p_1\},$$

where possibly $\varepsilon(p) \to 0$ as $p \to p_1$. By Lemma 1.1, the dimension of the image $z(S, D_1^{(1)})$ of S, by means of the map $(t, p) \curvearrowright z(t, p, D_1^{(1)})$, is two-dimensional, and so $T(z(s, p, D_1^{(1)}))$ is two dimensional, because the birational map T is biholomorphic off a divisor in $z(S, D_1^{(1)})$, the latter being two-dimensional. Thus the image $T^{-1}T(z(S, D_1^{(1)}))$ will be biholomorphic to $z(S, D_1^{(1)})$ off a divisor; therefore $T^{-1}T$ will be the identity on $z(S, D_1^{(1)})$ off a divisor and thus we have

$$\tilde{z}(t, p, D_1^{(1)}) \equiv T^{-1}Tz(t, p, D_1^{(1)}) = z(t, p, D_1^{(1)}) \quad \text{on } S\backslash\text{subvariety and thus on } S,$$

proving the assertion. This argument works equally well for $z(t, p, D_2^{(1)})$, yielding its meromorphic extension $\tilde{z}(t, p, D_2^{(1)})$. Also by meromorphic extension, we have

$$\tilde{z}(t, p, D_1^{(1)}) \in \mathscr{A},$$

[11] of the form $(t, p) \curvearrowright (t', p') = (t + \phi_1(p), \phi_2(p))$, ϕ_1 and ϕ_2 being holomorphic; this is necessary when the flow is tangent to $\bar{D}_1^{(1)}$ at p_1

wherever it is defined in (t, p)-space. Defining

$$\Gamma_1 \equiv \left\{ \begin{matrix} y(t, p, D_1^{(1)}) \\ 0 \leqq |t| < \varepsilon \\ p \sim p_1 \end{matrix} \right\} \quad \text{and} \quad \Gamma_2 \equiv \left\{ \begin{matrix} y(t', p', D_2^{(1)}) \\ 0 \leqq |t'| < \varepsilon \\ p' \sim p_2 \end{matrix} \right\},$$

we show the images Γ_1 and Γ_2 coincide. Indeed, set

$$W_1 \equiv \bigcap_{i=1}^{n} \{(t, p) | g_i(t, p) \neq 0\} \supset \{(t, p_1) | 0 < |t| < \varepsilon\},$$

with the last inclusion a consequence of (iv). By the Weierstrass Preparation Theorem (Lemma 1.3) W_1 is an open set with a boundary composed of analytic subvarieties. Thus

$$y(W_1, D_1^{(1)}) = T(z(W_1, D_1^{(1)}))$$

is a 2-dimensional set by the same dimension argument given earlier.

Now using the other expansion $\tilde{z}(t', p', D_2^{(1)})$ associated with $D_2^{(1)}$, we also construct an open set W_2 analogous to W_1; indeed, $\tilde{z}(t', p', D_2^{(1)})$ is a ratio of holomorphic functions similar to (1.17), which by condition (iii) coincides with $\tilde{z}(t, p_1, D_1^{(1)})$ at $p' = p_2$, and thus its denominator $g_i'(t, p')$ does not vanish identically. As in the above, this is all one needs to show that W_2 is open; thus $y(W_2, D_2^{(1)})$ is two-dimensional as is $y(W_1, D_1^{(1)})$. Moreover, since $y(W_1, D_1^{(1)})$ and $y(W_2, D_2^{(1)})$ both contain the orbit

$$\{y(t, D^{(0)}), |t| < \varepsilon\} = T\{\tilde{z}(t, D^{(0)}), |t| < \varepsilon\}, \tag{1.19}$$

we have the following inclusions

$$\mathscr{A} \supset T^{-1}(y(W_1, D_1^{(1)})) \cap T^{-1}(y(W_2, D_2^{(1)})) \supset \{\tilde{z}(t, D^{(0)}), |t| < \varepsilon\}; \tag{1.20}$$

the first follows from the definition of W_i, whereas the second follows from (1.19). Also the sets $T^{-1}(y(W_i, D_i^{(1)})) = \tilde{z}(W_i, D_i^{(1)})$ are two-dimensional by the usual arguments, since the $y(W_i, D_i^{(1)})$ are two-dimensional. Thus (1.20) turns the smooth affine variety $\mathscr{A}$ into the union of two 2-dimensional components meeting along $\{\tilde{z}(t, D^{(0)}), |t| < \varepsilon\}$; therefore the sets $\tilde{z}(W_1, D_1^{(1)})$ and $\tilde{z}(W_2, D_2^{(1)})$ must agree, at least on the connected component containing $\tilde{z}(t, D^{(0)})$, and so must their image under T, namely $y(W_1, D_1^{(1)}) \subset \Gamma_1$ and $y(W_2, D_2^{(1)}) \subset \Gamma_2$, at least on a connected component containing $y(t, D^{(0)})$. But by condition (ii), the map $(t, p) \curvearrowright y(t, p, D^{(1)})$ has rank 2 at $(t, p) = (0, p_1)$, and since $y(W_2, D_2^{(1)}) \subset \Gamma_2$ is 2-dimensional with a component of it contained in $\{y(t, p, D_1^{(1)}) \,|\, |t| < \varepsilon, p \sim p_1\} \subset \Gamma_1$, we conclude by Lemma 1.2 that the sets Γ_1 and Γ_2 coincide, as promised.

The variables $\{(t, p), |t| < \varepsilon, p \sim p_1\}$ form a one-to-one coordinatization of Γ_1 by the rank condition (ii); since $\Gamma_1 = \Gamma_2$, the set Γ_2 will carry the same coordinization. The connection between the coordinates (t, p) and (t', p') (the latter incidentally may not be good coordinates) is given by solving the equation in t, p:

$$y(t, p, D_1^{(1)}) = y(t', p', D_2^{(1)}), \quad |t'| < \varepsilon, p' \sim p_2; \tag{1.21}$$

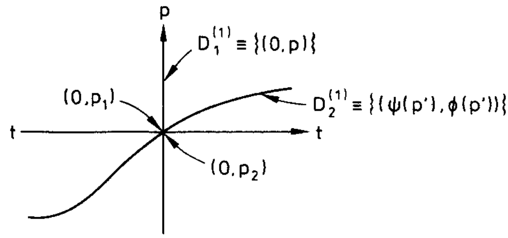

Fig. 1.5

indeed, using the implicit function theorem, which is possible, again by the rank condition (ii), the equation (1.21) can be solved in (t, p):

$$\begin{aligned}(t,p)&=(g(t',p'),h(t',p')) \quad \text{(holomorphic functions)}\\ &=(t'+\psi(p'),\phi(p')).\end{aligned} \tag{1.22}$$

To see the latter identity, let $\hat{\phi}_1^t$ be the extension of the flow ϕ_1^t generated by X_1 on Γ_1 and $\hat{\phi}_2^t$ the same X_1 generated flow extended to Γ_2; then we have the following identities

$$\begin{aligned}\phi_1^{\bar{t}}(y(t,p,D_1^{(1)}))&=y(t+\bar{t},p,D_1^{(1)}) \qquad \text{on } \Gamma_1\\ \phi_2^{\bar{t}}(y(t',p',D_2^{(1)})&=y(t'+\bar{t},p',D_2^{(1)}) \qquad \text{on } \Gamma_2,\end{aligned} \tag{1.23}$$

which moreover are equivalent. Clearly we have $\phi_i^t(z(t,p,D_i^{(1)})=z(t+\bar{t},p,D_i^{(1)})$, for small $\bar{t}$, and by meromorphic extension the same holds for $\tilde{z}$ as well; applying the birational map T the latter relation lifts, yielding (1.23). Since the two flows ϕ_1^t and ϕ_2^t agree in the affine $\mathscr{A}$, and in particular on the two-dimensional set $\mathscr{A}\cap T^{-1}(\Gamma_1)\cap T^{-1}(\Gamma_2)$, $\hat{\phi}_1^t$ and $\hat{\phi}_2^t$ must agree on their two-dimensional image $T(\mathscr{A})\cap\Gamma_1\cap\Gamma_2$ under T and hence, by analytic continuation, on $\Gamma_1\cap\Gamma_2$. This yields the equivalence of the two flows given in (1.23), and thus orbits in $\Gamma_2\ni(t',p')$ are orbits in $\Gamma_1\ni(t,p)$, yielding the second relation in (1.22). The map (1.22) enables us to put the divisors $D_1^{(1)}$ and $D_2^{(1)}$ in the same chart. (See Fig. 1.5.)

The construction of the charts along $D_1^{(1)}$ starting from the chart about (t, p_1) proceeds as before. In order to continue this construction along $D_2^{(1)}$, notice that the map $(t',p')\curvearrowright(t,p)$ given by (1.22) – although not necessarily one-to-one about $(t',p')=(0,p_2)$ – will be one-to-one along any branch near $(0, p_2)$, because of the special form of the map (1.22).

Finally we show that all points in the Γ_1-chart which are neither on $D_1^{(1)}$ nor on $D_2^{(1)}$ correspond to points in the affine, i.e.,

$$\tilde{z}(t,p,D_1^{(1)})\in\mathscr{A} \qquad \text{for } (t,p)\notin D_1^{(1)}\cup D_2^{(1)}. \text{ (See Fig. 1.5.)}$$

Indeed, in the Γ_1-chart, the functions $z(t,p,D_i^{(1)})$ and the meromorphic extension $\tilde{z}(t,p,D_i^{(1)})$ coincide near $D_i^{(1)}$, but away from (t,p_i). Therefore, also in view of (1.22), $\tilde{z}_i$ has a pole of order $\nu_i(D_1^{(1)})$ along $D_1^{(1)}$ and $\nu_i(D_2^{(1)})$ along $D_2^{(1)}$; for notation see (1.18). By assumption (iv) in the coherence condition, we have that each coordinate $\tilde{z}_i$ has the following representation:

$$\tilde{z}_i(t, p, D_1^{(1)}) = \frac{h_i(t, p)}{g_i(t, p)} = \frac{P_i(t, p)\cdot \text{unit}}{Q_i(t, p)\cdot \text{unit}},$$

with h_i and g_i holomorphic functions which are relative prime, with P_i and Q_i their respective Weierstrass polynomials (see Lemma 1.3) in t, with

$$Q_i(t, p) = t^r + a_1(p)t^{r-1} + \ldots a_r(p), \quad \text{with } r = v_i(D_1^{(1)}) + v_i(D_2^{(1)}); \tag{1.24}$$

the $a_j(p)$ are holomorphic functions satisfying $a_j(p_1)=0$. Moreover, since we know that the $\tilde{z}_i$ blow up along $D_1^{(1)}$ and $D_2^{(1)}$ to order $v_i(D_1^{(1)})$ and $v_i(D_2^{(1)})$ respectively, the Weierstrass polynomial $Q_i(t, p)$ of the denominator g_i must have the form[12]

$$Q(t, p) = (t + b_1(p))^{v_i(D_1^{(1)})}(t + b_2(p))^{v_i(D_2^{(1)})}(t^s + \ldots). \tag{1.25}$$

Then combining (1.24) and (1.25), we must have $s=0$; this shows that the only poles of $\tilde{z}_i$ in the chart Γ_1 are along $D_1^{(1)}$ and $D_2^{(1)}$ of order $v_i(D_1^{(1)})$ and $v_i(D_2^{(1)})$ respectively and thus the functions $\tilde{z}_i$ are finite away from these divisors, as claimed.

To sum up, we have built a complex manifold M, with meromorphic functions $\tilde{z}_i$, $1 \leq i \leq n$, by glueing onto the affine chart $\mathscr{A}$ the varieties $\bar{D}^{(m-1)} \equiv \coprod_{k,\alpha} D_\alpha^{(k)}$ appearing in the tree, such that the Laurent solutions $z(t, p, D_\alpha^{(k)})$ form a fibre bundle on M; therefore $\mathscr{A} = M \backslash \bar{D}^{(m-1)}$. We now show M is compact. By the valuative criterion of properness (see Hartshorne [24]), it suffices to show that every punctured analytic arc

$$\gamma\{\tau \mid 0 < |\tau| < \varepsilon\} \subset M$$

has a completion $\bar{\gamma} \subset M$, upon taking the limit $\tau \to 0$. Since $\mathscr{A}$ is an affine variety and $\bar{D}^{(m-1)}$ is compact, we only have to show this for arcs $\gamma \subset \mathscr{A}$ terminating at ∞, i.e., in terms of the embedding $\mathscr{A} \subset \mathbb{P}_\gamma^n$, we have

$$\operatorname*{limit}_{|\tau| \to 0} \gamma(\tau) \in \mathscr{A}_\infty \subset \mathbb{P}_\nu^n. \tag{1.26}$$

But condition (v), together with Bézout's theorem (see appendix 1), shows that the meromorphic extension

$$\Phi\colon \; \mathscr{A} \coprod (\coprod_{k,\alpha} D_\alpha^{(k)}) \curvearrowright \bar{\mathscr{A}} = \mathscr{A} \cup \mathscr{A}_\infty \subset \mathbb{P}_\gamma^n$$

of $\tilde{z}$ restricted to $\coprod D_\alpha^{(k)}$, namely

$$\Phi\colon \; \coprod_{k,\alpha} D_\alpha^{(k)} \to \mathscr{A}_\infty,$$

12 if the variable t is too special, we may have to replace $t \curvearrowright \bar{t} = t + h(p)$ in (1.25)

is *onto*. By hypothesis, the arc $\gamma \subset \mathscr{A}$ has a limit point on $\mathscr{A}_\infty$, as $\tau \to 0$; then taking the inverse map Φ^{-1} this limit point of the arc belongs to $\bar{D}^{(m-1)}$. This concludes the proof that M is compact and thus the proof of Theorem 1, part II.

§ 2. The convergence of the formal Laurent solutions

All systems of differential equations $\dot{z} = f(z)$, $z \in \mathbb{C}^n$, discussed here will be polynomial and weight-homogeneous with weights $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_i \in \mathbb{Z}$, > 0, i.e., they satisfy

$$f_i(\alpha^{\nu_1} z_1, \ldots, \alpha^{\nu_n} z_n) = \alpha^{\nu_i + 1} f_i(z_1, \ldots, z_n), \qquad \text{for all } \alpha \in \mathbb{C}.$$

See §4 and appendix 2 for specific examples. In general, a function f is called (weight) homogeneous of degree N whenever

$$f(\alpha^{\nu_1} z_1, \ldots, \alpha^{\nu_n} z_n) = \alpha^N f(z_1, \ldots, z_n), \qquad \text{for all } \alpha \in \mathbb{C}^*,$$

from which it follows that $\partial f / \partial z_j$ has weight $N - \nu_j$. Thus f_i has weight $\nu_i + 1$, and $\partial f_i / \partial z_j$ weight $\nu_i + 1 - \nu_j$. Observe that a basis for the constants of motion of such a system can always be chosen weight-homogeneous. All invariants in this paper will be assumed polynomial, and so their weights will be positive integers.

The following lemma states that *formal* asymptotic solutions of weight-homogeneous systems consistent with the weights $\nu = (\nu_1, \ldots, \nu_n)$ are actually convergent Laurent series:

Lemma 2.1. *The formal Laurent solutions*

$$z_i(t) = \frac{1}{t^{\nu_i}} (z_i^{(0)} + z_i^{(1)} t + \ldots), \qquad z^{(0)} \neq 0, \tag{2.1}$$

of a weight-homogeneous system $\dot{z} = f(z)$ are convergent series; the coefficients $z^{(0)}$ belongs to the indicial locus

$$\mathscr{C} \equiv \bigcap_{i=1}^{n} \{\nu_1 z_i^{(0)} + f_i(z^{(0)}) = 0\} \tag{2.2}$$

and the subsequent coefficients $z^{(k)}$ satisfy

$$(\mathscr{L} - kI) z^{(k)} = \textit{some polynomial in the } z^{(j)}, \qquad 0 \leq j < k, \tag{2.3}$$

where the Kowalewski matrix $\mathscr{L}$ is the Jacobian matrix of the locus (2.2):

$$\mathscr{L} = \mathscr{L}_{z^{(0)}} = \left(\frac{\partial f_i}{\partial z_j} (z^{(0)}) + \delta_{ij} \nu_i \right)_{1 \leq i, j \leq n}. \tag{2.4}$$

Proof. The proof inspired by J.P. Françoise [23], is an application of the majorant method. Putting the formal series (2.1), written in short

$$z_i(t)=t^{-\nu_i}(z_i^{(0)}+U_i(t))$$

into $\dot{z}=f(z)$ yields on the left hand side, upon multiplying the i^{th} component by t^{ν_i+1},

$$\sum_{k\geqq 0} t^k(k-\nu_i)\,z_i^{(k)}$$

and on the right hand side[13]

$$\begin{aligned}
t^{\nu_i+1}f_i(z(t)) &= f_i(z^{(0)}+U(t)) \quad \text{(using the homogeneity)}\\
&= f_i(z^{(0)})+\sum_{j=1}^{n}\frac{\partial f_i}{\partial z_j}(z^{(0)})\,U_j(t)+\sum_{|\beta|>1}\frac{1}{\beta!}\frac{\partial^\beta f_i}{\partial z^\beta}(z^{(0)})\,U^\beta(t)\\
&= f_i+\sum_{j=1}^{n}\frac{\partial f_i}{\partial z_j}(z_j^{(1)}t+z_j^{(2)}t^2+\ldots)\\
&\quad +\sum_{k=2}^{\infty}t^k\sum_{\substack{\sigma_i>0\\ \langle\beta,\sigma\rangle=k\\ |\beta|>1}}\frac{1}{\beta!}\frac{\partial^\beta f_i}{\partial z^\beta}(z_1^{(\sigma_1)})^{\beta_1}\ldots(z_n^{(\sigma_n)})^{\beta_n}.
\end{aligned}$$

In this formula, the f_i and its derivatives are evaluated at $z^{(0)}$. Comparing the coefficients of the various powers of t in the above two expression, we get (2.2) for $k=0$; for $k=1$,

$$(\mathscr{L}-I)\,z^{(1)}=0$$

and $k\geqq 2$,

$$((\mathscr{L}-k)\,z^{(k)})_i=-\sum_{\substack{\sigma_i>0\\ \langle\beta,\sigma\rangle=k\\ |\beta|>1}}\frac{1}{\beta!}\frac{\partial^\beta f_i}{\partial z^\beta}(z_1^{(\sigma_1)})^{\beta_1}\ldots(z_n^{(\sigma_n)})^{\beta_n},\qquad k\geqq 2, \tag{2.5}$$

which establishes (2.3). The coefficients $z^{(0)}$ may or may not contain free parameters, depending on whether the indicial locus is a continuum or not. The matrix $\mathscr{L}$ may have some integer eigenvalues $\lambda_i=k\geqq 1\,(1\leqq i\leqq n)$ and this may lead to free parameters in the determination of $z^{(k)}$ by the linear equation (2.5). Then the coefficients can be viewed as rational functions on an affine variety W, fibered over the indicial locus. Fix a compact set $W_0\subset W$ and let

$$M_1=1+\max_{\substack{1\leqq i\leqq n\\ 1\leqq j\leqq\lambda_n\\ \text{free parameters}\in W_0}}|z_i^{(j)}|. \tag{2.6}$$

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[13] $\beta=(\beta_1,\ldots,\beta_n),\ |\beta|=\sum_1^n\beta_i$

By analyticity, there are constants M_2 and $M_3 > M_1$ such that in W_0

$$\left|\frac{\partial^\beta f_i}{\partial z^\beta}(z^{(0)})\right| \leqq \beta!\, M_2^{|\beta|} \quad \text{and} \quad |(\mathscr{L}-kI)^{-1}| \leqq M_3,\; k \geqq \lambda_n + 1.$$

Applying these estimates to (2.5) leads to the recursive estimate

$$|z_i^{(k)}| \leqq M_3 \sum_{\substack{\langle\beta,\sigma\rangle = k \\ |\beta| \geqq 2 \\ \sigma_i > 0}} M_2^{|\beta|} |z_1^{(\sigma_1)}|^{\beta_1} \dots |z_n^{(\sigma_n)}|^{\beta_n} \quad \text{for } k \geqq \lambda_n + 1. \tag{2.7}$$

Define now the series (see Françoise [23])

$$V(t) = M_1 t + \sum_{k=2} \alpha_k t^k, \qquad \alpha_k \in \mathbb{C},$$

where the coefficients α_k are defined inductively by

$$\alpha_k = M_3 \cdot \sum_{\substack{\langle\beta,\sigma\rangle = k \\ |\beta| \geqq 2 \\ \sigma_i > 0}} M_2^{|\beta|} \alpha_{\sigma_1}^{\beta_1} \dots \alpha_{\sigma_n}^{\beta_n}, \qquad k \geqq 2. \tag{2.8}$$

The series $V(t)$ majorizes $U_i(t)$ for all $1 \leqq i \leqq n$. Indeed, from the definition of α_k, M_2 and M_3 and the estimate (2.6), $|z_i^{(j)}| \leqq M_1 \leqq \alpha_j$, $1 \leqq i \leqq n$, $1 \leqq j \leqq \lambda_n$; beyond that we proceed by induction: assume $|z_i^{(j)}| \leqq \alpha_j$, for $j < k$, all i with $k \geqq \lambda_n + 1$, and comparing (2.7) and (2.8) and using the inductive assumption deduce $|z_i^{(k)}| \leqq \alpha_k$, and the inequality then holds for all $k \geqq 1$. Finally from the definition of the α_k one observes that the series $V(t)$ satisfies

$$V = M_1 t + M_3 M_2^2 \frac{(nV)^2}{1 - M_2 n V},$$

which amounts to a quadratic equation for V; solving for V yields the desired majorant for the functions U_i, which therefore converge for t sufficiently small.

§ 3. The spectrum of the Kowalewski matrix, and the nature of the free parameters

Let a weight-homogeneous system $\dot{z} = f(z)$ have weight-compatible Laurent solutions

$$z_i(t) = t^{-\nu_i}(z_i^{(0)} + z_i^{(1)} t + \dots), \qquad z^{(0)} \neq 0.$$

Then, according to Lemma 2.1, $z^{(0)} \in \mathscr{C}$ and $z^{(k)}$, $k > 0$ satisfies

$$(\mathscr{L} - kI)\, z^{(k)} = \text{some polynomial in } z^{(j)}, \qquad 0 \leqq j \leqq k,$$

where $\mathscr{L}$ is defined in (2.4). Note that by the weight-homogeneity of $\dot{z}=f(z)$, we have the following equivalence

$$\{c\in\mathscr{C}\}\Leftrightarrow \text{the } z_i(t)=c_i t^{-\nu_i} \quad (i=1, \ldots, n) \quad \text{solve } \dot{z}=f(z). \tag{3.1}$$

As already observed, arbitrary polynomial invariants of $\dot{z}=f(z)$ are sums of weight-homogeneous invariants; therefore, given a constant of the motion H, the affine invariant manifolds $\{H(z)-A=0\}$ are naturally embedded into weighted projective space $\mathbb{P}^n_\nu$ with $\nu_0=1$, namely

$$\{H(z)-Az_0^d=0\}\subset\mathbb{P}^n_\nu, \qquad d=\text{weighted degree of } H.$$

Also the indicial locus $\mathscr{C}$ is a subvariety of its hyperplane section,

$$\mathscr{C}\subset\{H(z)-Az_0^d=0\}\cap\{z_0=0\}, \tag{3.2}$$

whatever be the weight-homogeneous invariant H. Indeed, picking $z^{(0)}\in\mathscr{C}$ and substituting the solution $z(t)$ mentioned in (3.1) into H yield

$$H(z(t))=H(c_1 t^{-\nu_1}, \ldots, c_n t^{-\nu_n})=t^{-d}H(c)=\text{constant},$$

implying $H(c)=0$ and thus assertion (3.2). For a completely integrable weight-homogeneous system, define the invariant manifold

$$\bar{\mathscr{A}}=\bigcap_1^{k+m}\{H_i(z)=A_i z_0^{d_i},\ z\in\mathbb{P}^n_\nu\}\subset\mathbb{P}^n_\nu$$

and

$$\mathscr{A}_\infty=\bar{\mathscr{A}}\cap\{z_0=0\}.$$

From the above it follows that

$$\mathscr{C}\subset\mathscr{A}_\infty.$$

Given $c\in\mathscr{C}$, we adopt the following notation:

$$\mathscr{H}^M=\{\text{invariants } H \text{ of } \dot{z}=f(z), \text{ having degree } M\}, \qquad \alpha^M=\dim\mathscr{H}^M$$

Let

$$\nabla_c\mathscr{H}^M=\left\{\frac{\partial H}{\partial z}(c),\ H\in\mathscr{H}^M\right\}, \qquad \beta_c^M=\dim\nabla_c\mathscr{H}^M,$$

be the gradients of the constants of motion along $\mathscr{C}\subset\mathscr{A}_\infty$ and let $F_1, \ldots, F_{\beta_c^M}\in\mathscr{H}^M$ be independent functions such that

$$\nabla_c\mathscr{H}^M=\operatorname{span}\left\{\frac{\partial F_i}{\partial z}(c),\ i=1, \ldots, \beta_c^M\right\}.$$

Let[14]

$$\begin{aligned} &E_\rho(c)=\{w\in\mathbb{C}^n \text{ such that } (\mathscr{L}_c-\rho)^k w=0, \text{ for some } k\geqq 1\},\\ &E_\rho^\perp(c)=\{v\in\mathbb{C}^n \text{ such that } \langle v, E_\rho(c)\rangle=0\}, \qquad (3.3)\\ &\quad\gamma_c^M=\dim E_M(c)=\text{multiplicity of } M \text{ in spectrum } (\mathscr{L}_c).\\ &\quad\delta_c=\#\{\text{non-negative integers}\in\text{spectrum } \mathscr{L}_c\}\\ &\quad\varepsilon_c=\#\{\text{free parameters in the Laurent solution } z(t)=(t^{-\nu_1}c_1,\dots,t^{-\nu_n}c_n)+\dots\} \end{aligned}$$

It is then possible to find $G_{\beta+1},\dots,G_\alpha\in\mathscr{H}^M$ ($\beta=\beta_c^M$, $\alpha=\alpha^M$) such that the invariants $F_1,\dots,F_\beta, G_{\beta+1},\dots,G_\alpha$ span $\mathscr{H}^M$ and such that

$$\left.\frac{\partial G_i}{\partial z}\right|_c=0, \qquad i=\beta+1,\dots,\alpha, \tag{3.4}$$

where the invariants G_i will (in general) depend on $c\in\mathscr{C}$.

Throughout this paper, given a component of $\mathscr{C}$, the G's will stand for invariants with vanishing gradient along $\mathscr{C}$, whereas the F's will be part of a set of invariants whose gradients span the spaces $\mathscr{H}^M$. We wish to emphasize that the partition of the set of invariants H in F's and G's is always relative to a component of $\mathscr{C}$. Clearly if $\alpha^M-\beta_c^M>0$ for some degree M, the variety of $\bar{\mathscr{A}}$ will be singular at $\mathscr{C}$ and indeed

$$\sum_M(\alpha_M-\beta_c^M)\geqq\text{deficiency in rank of the Jacobian matrix of } \bar{\mathscr{A}} \text{ at } c,$$

and so it is a measure of the severity of the singularity of $\bar{\mathscr{A}}$ at c.

The next theorem discusses the precise relationship between

- the spectrum of the Kowalewski matrix $\mathscr{L}$
- the degrees of the invariants
- the singularity at ∞ of the invariant manifolds $\bar{\mathscr{A}}$ in $\mathbb{P}_\nu^n$
- the free parameters.

It also shows that each family of Laurent solutions leads to a natural affine variety, the *Painlevé variety* D; the coefficients $z_i^{(k)}$ of the series (2.1) can be viewed as holomorphic functions $z_i^{(k)}(p)$ on D. One of the important points of this paper is to assemble the Painlevé varieties, thus forming a compact variety with the Laurent solutions

$$z_i(t)=t^{-\nu_i}(z_i^{(0)}(p)+z_i^{(1)}(p)\,t+\dots) \tag{3.5}$$

being the fibres of a fibre bundle over it.

Theorem 4. *Consider the weight-homogeneous system $\dot z=f(z)$, $z\in\mathbb{C}^n$ and their weight-compatible Laurent solutions* (3.5). *Decompose the indicial locus $\mathscr{C}$ of lead-*

[14] $v, w\in\beta^n$, $\langle v,w\rangle=\sum_1^n v_i w_i$

ing terms of the Laurent solutions (3.5) *into its components (not necessarily of the same dimension):*

$$\mathscr{C}=\sum_1^l C_i,$$

then

(i) $\mathscr{C}\subset \bigcap_{\text{all } H\text{'s}} \{H(z)-A z_0^d=0\}\cap\{z_0=0\}\subset P_v^n$, *and if the system is algebraic integrable, then all components C_i of $\mathscr{C}$ have the following property:*

$$\begin{cases}\mathscr{L}_{C_i} \text{ is diagonalisable}\\ (\text{spectrum } \mathscr{L}_{C_i})\subset \mathbb{Z}\end{cases}$$

(ii) *for $c\in\mathscr{C}$, $-1\in$ spectrum $\mathscr{L}_c$, and if $\dot z=f(z)$ is divergence free*[15], *then trace $\mathscr{L}=\sum v_i$. Moreover if the component C_i leads to Laurent solutions depending on $n-1$ free parameters, then*

$$\mathscr{L}_{C_i} \text{ is diagonalisable}$$
$$\text{spectrum } \mathscr{L}_{C_i}=\{-1\}\cup\{n-1 \text{ integers}, \geqq 0\}$$

(iii) *the tangent space $T_c\mathscr{C}$ to $\mathscr{C}$ at a point $c\in\mathscr{C}$ is a subspace of $E_0(c)$, i.e., $T_c\mathscr{C}\subset E_0(c)$; therefore* $\dim T_c\mathscr{C}\leqq\gamma_c^0$ *for all $c\in\mathscr{C}$.*

(iv) *the coefficients $z_i^{(k)}$ of the Laurent solutions $z(t)$ are rational functions of the parameters and $\mathscr{C}$.*

(v) $\beta_c^M=\dim \nabla_c H^M\leqq\gamma_c^M\equiv$ *multiplicity of M in spectrum $\mathscr{L}_c$ for $c\in\mathscr{C}$.*

(vi) $\nabla_c\mathscr{H}^M\subset \bigcap_{\rho\neq M} E_\rho^\perp(0)$ *for $c\in\mathscr{C}$.*

(vii) *for each degree M, for each component C_j of C, such that $\delta(C_j)=\varepsilon(C_j)$ and for each $F_1,\ldots,F_\beta\in\mathscr{H}$ such that $\left.\frac{\partial F_i}{\partial z}\right|_{C_j}$ form a basis of $\nabla_{c_j}\mathscr{H}_M$, there exist appropriately chosen M-eigenvectors of $\mathscr{L}$ and free parameters $A_1,\ldots,A_\beta$ which appear for the first time and linearly in the coefficient $z^{(M)}$ of the Laurent series so that*

$$F_i^M(z_1(t),\ldots,z_n(t))=A_i,\qquad i=1,\ldots,\beta^M$$

holds. The $\sum_M \beta^M$ parameters A_i thus obtained are called trivial parameters, whereas the $\delta(C_j)-\sum_M\beta^M$ remaining parameters in the expansion (3.5) *are called the effective parameters.*

(viii) *Assume now the system satisfies condition* (i) *in the definition of algebraic complete integrability; then for each of the irreducible components C_α of C, such*

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[15] $\dot z=f(z)$ is divergence free or volume preserving if $\sum\frac{\partial f_i(z)}{\partial z_i}\equiv 0$

that spectrum $\mathscr{L}_{C_\alpha} \in \mathbb{Z}$ *and* $\delta(C_\alpha)=\varepsilon(C_\alpha)$, *we have that using the asymptotic solutions* $z(t)$ *obtained in* (vii), *the set of solutions,*

$$\bigcap_{i=1}^{k+m} \{\text{Laurent solutions } z(t) \text{ with } z^{(0)} \in C_a \text{ such that } H_i(z(t))=A_i\}$$

$$= \bigcap_{\substack{\sum_M (\alpha^M - \beta^M_{C_\alpha}) \text{ invariants } G \\ \text{such that } \left.\frac{\partial G}{\partial z}\right|_{C_\alpha}=0}} \{\text{Laurent solutions } z(t) \text{ such that } G(z(t))=A\}$$

$$= \sum (\alpha^M - \beta^M_{C_\alpha}) \text{ polynomial equations between the } \delta(C_\alpha) - \sum \beta^M_{C_\alpha} \text{ effective parameters,}$$

parametrizes an affine variety of dimension

$$\delta(C_\alpha)\text{-(\# independent invariants),}$$

called the Painlevé variety D_α *(associated with* C_α*), along which all the coefficients* $z_j^{(k)}$ *in the Laurent solution* $z_j(t)$ *are holomorphic. These coefficients generate the coordinate ring of* D_α.

Remark 1. The deficiency index $\sum_M (\alpha^M - \beta^M_{C_\alpha})$ is a measure of the severity of the singularity of $\bar{\mathscr{A}}$ along C_α; it also determines the number of nonlinear relations between the $\delta(C_\alpha) - \sum \beta^M_{C_\alpha}$ effective parameters; these relations define the Painlevé variety D_α associated with a component C_α of C.

Remark 2. The arguments used to prove (v) and (vi) are inspired by Yoshida [21].

Proof. The first part of (i) has been shown in the considerations preceding this theorem, whereas the second part will be shown later.

(ii) and (iii): We first check the eigenvector equation

$$(\mathscr{L}_c + I)(v_1 c_1, \ldots, v_n c_n) = 0, \qquad c \in \mathscr{C}.$$

Indeed, differentiating $f_i(\alpha^{v_1} z_1, \ldots, \alpha^{v_n} z_n) = \alpha^{v_i+1} f_i(z_1, \ldots, z_n)$ by α and setting $\alpha = 1$, one deduces, the identity

$$\sum_j v_j z_j \frac{\partial f_i}{\partial z_j} - (v_i + 1) f_i(z) = 0, \qquad \text{for all } z \quad \text{and} \quad i = 1, \ldots, n;$$

then substitute in this equation $z = c \in \mathscr{C}$, using the relation $f_i(c) = -v_i c_i$, to yield the eigenvector equation; thus we have $-1 \in \text{Spectrum } \mathscr{L}_c$, for $c \in \mathscr{C}$. Remembering that the coefficients in the Laurent solution satisfy (2.3) and (2.4) (Lemma 2.1), the only source of free parameters is

(I) the indicial locus $\mathscr{C}$, giving rise to a number of degrees of freedom equal to dimension $\mathscr{C}$. Then 0 is an eigenvalue of $\mathscr{L}_c$ with at least multiplicity $\gamma_c^0 \geqq \dim T_c \mathscr{C}$, as $\mathscr{L}_c$ is the Jacobian matrix of the defining relations of $\mathscr{C}$, establishing (iii).

(II) the integers $k \in$(spectrum $\mathscr{L}_c$), $k>0$, giving rise to at most γ_c^k degrees of freedom in the determination of $z^{(k)}$. Since $\mathscr{L}_c$ is an n by n matrix and already $-1 \in$spectrum $\mathscr{L}_c$, statement (ii) must hold in order to have the full $n-1$ degrees of freedom in the Laurent solutions; thus $\mathscr{L}$ must be diagonalizable along C.

(iv) is obvious from the fact that $z^{(k)}$ is found inductively as a solution of the linear problem (2.3) and from the way in which the parameters arise, as discussed above (whether there be $n-1$ parameters or less).

(v) and (vi) will be shown in a number of steps:

Step (*a*). If $c \in \mathscr{C}$, then a special solution of $\dot{z}=f(z)$ is given by (see (3.1))

$$z_i(t)=c_i t^{-\nu_i}, \qquad i=1, \ldots, n. \tag{3.6}$$

Step (*b*). The variational equation about $z_i(t)=c_i t^{-\nu_i}$, namely

$$\dot{\xi}_i=\sum_{j=1}^{n} \frac{\partial f_i}{\partial z_i}(c_1 t^{-\nu_1}, \ldots, c_n t^{-\nu_n})\, \xi_j, \qquad i=1, \ldots, n, \tag{3.7}$$

has for solution

$$\xi_i(t)=\eta_i t^{\rho-\nu_i}, \tag{3.8}$$

where η and the constant ρ satisfy

$$\frac{d\eta}{ds}=(\mathscr{L}_c-\rho I)\eta, \qquad s=\ln t. \tag{3.9}$$

Indeed, substituting (3.8) for ξ_i in (3.9) leads to

$$\begin{aligned}\dot{\eta}_i t^{\rho-\nu_i}+\eta_i(\rho-\nu_i) t^{\rho-\nu_i-1} &=\sum_j \frac{\partial f_i}{\partial z_j}(c_1 t^{-\nu_1}, \ldots, c_n t^{-\nu_n}) \eta_j t^{\rho-\nu_j} \\ &=\sum_j \frac{\partial f_i}{\partial z_j}(c_1, \ldots, c_n)(t^{-1})^{(\nu_i+1-\nu_j)} \eta_j t^{\rho-\nu_j},\end{aligned}$$

using weight $\left[\dfrac{\partial f_i}{\partial z_j}\right]=\nu_i+1-\nu_j$; then dividing both sides by $t^{\rho-\nu_i-1}$ yields

$$t\dot{\eta}_j=\sum_j \frac{\partial f_i}{\partial z_j}(c)\eta_j+(\nu_i-\rho)\eta_i=[(\mathscr{L}_c-\rho I)\eta]_i.$$

This establishes Step (b).

Step (*c*). If $z(t)$ is a solution of $\dot{z}=f(z)$ and $H(z)$ is an invariant of $\dot{z}=f(z)$, then $\left\langle \dfrac{\partial H}{\partial z}(z(t)), \xi(t)\right\rangle$ is a constant of motion of the variational equation

$$\dot{\xi}=\frac{\partial f}{\partial z}(z(t))\,\xi.$$

Indeed, using the fact $\dot{H}(z)=\left\langle\frac{\partial H}{\partial z}(z), f(z)\right\rangle\equiv 0$, one computes[16]

$$\frac{d}{dt}\left\langle\frac{\partial H}{\partial z}(z(t)), \xi(t)\right\rangle=\left\langle\left[\frac{\partial^2 H}{\partial z\,\partial z}\right]f\Big|_{z=z(t)}, \xi(t)\right\rangle+\left\langle\frac{\partial H}{\partial z}(z(t)), \frac{\partial f}{\partial z}(z(t))\,\xi(t)\right\rangle$$
$$=\left\langle\frac{\partial}{\partial z}\left\langle\frac{\partial H}{\partial z}, f\right\rangle\Big|_{z(t)}, \xi(t)\right\rangle=0.$$

Step (*d*). If $H^M(z)$ is an invariant of degree M and η is defined by (3.8) and (3.9), then

$$\left\langle\frac{\partial H^M}{\partial z}(c), \eta(\ln t)\right\rangle t^{\rho-M} \tag{3.10}$$

is constant in t by (*c*). Indeed, since H^M is an invariant of $\dot{z}=f(z)$, we have that the following expression is constant in t:

$$\left\langle\frac{\partial H}{\partial z}(z(t)), \xi(t)\right\rangle=\sum_i \frac{\partial H}{\partial z_i}(c_1 t^{-\nu_1}, \ldots, c_n t^{-\nu_n})\,\eta_i\, t^{\rho-\nu_i}$$

(using the solution (3.7) of the variational equation around the solution (3.6))

$$=\left\langle\frac{\partial H}{\partial z}(c), \eta(\ln t)\right\rangle t^{\rho-M}$$

using the fact that $\partial H^M/\partial z_i$ has weight $M-\nu_i$; this establishes Step (d).

Step (*e*). Whenever $\rho\in\text{spectrum}(\mathscr{L}_c)$ and $\rho\neq M$, we have that

$$\left\langle\frac{\partial H^M}{\partial z}(c), \eta\right\rangle=0, \qquad \eta\in E_\rho. \tag{3.11}$$

Indeed, by the spectral theorem, (3.7) and (3.8) have solution of the form

$$\eta(\ln t)=p(\ln t)\in E_\rho, \qquad t\in\mathbb{C},$$

where $p(s)$ is a vector polynomial in s of degree γ_c^M-1 and where the vector $p(o)$ can be picked arbitrarily in E_ρ, regardless of whether $\mathscr{L}_c$ is diagonalizable or not. Substituting these special solutions into (3.10), Step (d) and the inequality $\rho-M\neq 0$ force upon us the relation

$$\left\langle\frac{\partial H^M}{\partial z}(c), \eta(\ln t)\right\rangle=0 \quad \text{for all } t;$$

[16] $\frac{\partial^2 H}{\partial z\,\partial z}=\left(\frac{\partial^2 H}{\partial z_i\,\partial z_j}\right)_{1\le i,j\le n}$ is the Hessian of H

then setting $t=1$ yields (3.11). Hence $\frac{\partial H^M}{\partial z}(c) \in \bigcap_{\rho \neq M} E_\rho^\perp$ for any invariant H^M of weight M, proving (vi). Since the spaces $\bigcap_{\rho \neq M} E_\rho^\perp$ and E_M have the same dimension, (v) follows from (vi)..

We now prove the second part of (i). By the definition of a.c.i., the solutions of $\dot{z}=f(z)$ are holomorphic and single-valued on each of the affine varieties $\mathscr{A}$; the latter fill up a Zariski open set in $\mathbb{C}^n$. The solution to the variational equation about *every* solution $z(t)$ must be single-valued, or else in a full neighborhood of $z(t)$, the solutions to $\dot{z}=f(z)$ would not be single-valued (Haine [12]). Therefore any solution $\xi_i(t)=\eta_i(\ln t)\, t^{\rho-\nu_i}$, $1 \leqq i \leqq n$, $\rho \in$ spectrum $\mathscr{L}_c$, of the variational equation about $z(t)=(c_1 t^{-\nu_1}, \ldots, c_n t^{-\nu_n})$ is single-valued. Thus ρ must be an integer and the $\eta_i(\ln t)$ must be time-independent, leading to the conclusion that $\mathscr{L}_c$ is diagonalisable and (spectrum $\mathscr{L}_c) \subset \mathbb{Z}$.

(vii) At first substitute the Laurent solutions (3.5) into a constant of motion $H=H^M$ of weight M,

$$
\begin{aligned}
& H^M(z_1(t), \ldots, z_n(t)) \\
&\quad = t^{-M} H^M(z_1^{(0)}+z_1^{(1)} t+\ldots, z_2^{(0)}+z_2^{(1)} t+\ldots, \ldots, z_n^{(0)}+z_n^{(1)} t+\ldots) \\
&\quad \equiv t^{-M} H^M(z^{(0)}+U(t)) \\
&\quad = t^{-M}\left[H(z^{(0)})+\left\langle\frac{\partial H}{\partial z}(z^{(0)}), U(t)\right\rangle+\left\langle U, \frac{\partial^2 H}{\partial z\, \partial z}(z^{(0)}) U\right\rangle+\ldots\right] \\
&\quad = t^{-M} \cdot t^M\left[\left\langle\frac{\partial H}{\partial z}(z^{(0)}), z^{(M)}\right\rangle+\sum_{\substack{i+j=M \\ i, j \geqq 1}}\left\langle z^{(i)}, \frac{\partial^2 H}{\partial z\, \partial z}(z^{(0)}) z^{(j)}\right\rangle+\ldots\right] \\
&\quad = \left\langle\frac{\partial H}{\partial z}(z^{(0)}), z^{(M)}\right\rangle+(\text{terms involving } z^{(i)}, 0 \leqq i<M). \qquad (3.12)
\end{aligned}
$$

We claim that on a Zariski open set of any component C_α of C, such that $\delta(C_\alpha)=\varepsilon(C_\alpha)$, there exists a basis of $E_M(C_\alpha)$

$$v_1, \ldots, v_\beta, v_{\beta+1}, \ldots, v_\gamma \qquad (\beta=\beta_{C_\alpha}^M, \gamma=\gamma_{C_\alpha}^M)$$

such that

$$\left\langle v_i, \frac{\partial F_j^M}{\partial z}(z^{(0)})\right\rangle=\delta_{ij} \qquad 1 \leqq i, j \leqq \beta. \qquad (3.13)$$

Indeed, remembering that $\nabla_{C_\alpha} \mathscr{H}^M$ is (generically) β-dimensional and that $\bigcap_{\rho \neq M} E_\rho^\perp(C_\alpha)$ is $\gamma-d$ imensional, we have, using (vi) and (v), the inclusion

$$\nabla_{C_\alpha} \mathscr{H}^M=\operatorname{span}\left\{\left.\frac{\partial F_i^M}{\partial z}\right|_{C_\alpha}, i=1, \ldots, \beta\right\} \subset \bigcap_{\rho \neq M} E_\rho^\perp(C_\alpha) \equiv \operatorname{span}\langle\bar{v}_1^M, \ldots, \bar{v}_\beta^M, \ldots, \bar{v}_\gamma^M\rangle.$$

Since $\mathbb{C}^n = \oplus E_M(C_\alpha)$, we may pick (for generic $c \in C_\alpha$) a basis $v_1^M, \ldots, v_\gamma^M$ of $E_M(C_\alpha)$, such that $\langle v_i^M, \bar{v}_j^M \rangle = \delta_{ij}$, $1 \leqq i, j \leqq \beta$, from which (3.13) follows. By (2.3), the general coefficient $z^{(M)}$ in the Laurent expansion (3.5) is a solution of

$$(\mathscr{L} - MI)\, z^{(M)} = \text{prior information};$$

it is a linear combination of the vectors v_i^M $(1 \leqq i \leqq \gamma)$ and a special solution u^{M-1} depending on previous data. Thus we write z^M as follows:

$$z^{(M)} = \sum_1^\beta (A_i + d_i)\, v_i^M + \sum_{\beta+1}^{\gamma} e_i v_i^M + u^{M-1} \tag{3.14}$$

with γ^M free parameters $A_i + d_i$ and e_i; the quantities d_i can now be chosen such that

$$\begin{aligned}
&F_j^M(z_1(t), \ldots, z_n(t)) \\
&= \left\langle \frac{\partial F_j^M}{\partial z}(z^{(0)}), z^M \right\rangle + (\text{terms involving } z^{(i)},\ 0 \leqq i < M), \qquad \text{using (3.12)} \\
&= \sum_1^\beta (A_i + d_i) \left\langle \frac{\partial F_j^M}{\partial z}(z^{(0)}), v_i^M \right\rangle + \sum_{\beta+1}^{\gamma} e_i \left\langle \frac{\partial F_j^M}{\partial z}(z^{(0)}), v_i^M \right\rangle + \left\langle \frac{\partial F_j^M}{\partial z}(z^{(0)}), u^{M-1} \right\rangle \\
&\quad + (\text{terms involving } z^{(i)},\ 0 \leqq i < M), \qquad \text{using (3.14)} \\
&= A_j + d_j + \sum_{\beta+1}^{\gamma} e_i \left\langle \frac{\partial F_j^M}{\partial z}(z^{(0)}), v_i^M \right\rangle + \left\langle \frac{\partial F_j^M}{\partial z}(z^{(0)}), u^{M-1} \right\rangle \\
&\quad + \text{terms involving } z^{(i)},\ 0 \leqq i < M, \qquad \text{using (3.13)} \\
&= A_j, \qquad 1 \leqq j \leqq \beta,
\end{aligned}$$

for an appropriate choice of d_j. Note $z^{(M)}$ is linear in u^{M-1} and the $\gamma - \beta$ *effective parameters* $e_{\beta+1}, \ldots, e_\gamma$, which remain free.

(viii) To restrict the Laurent solutions (3.5) specified by (3.14) to the m-dimensional invariant manifolds $\bigcap \{H_i = A_i\}$, we clearly must constrain the $\sum_M (\gamma^M - \beta_{C_j}^M) = \delta(C_j) - \sum \beta_{C_j}^M$ effective parameters e_i (see (3.14)) by substituting the expansions into the remaining relations G_i (having vanishing gradient along C_α), giving rise to the (affine) Painlevé variety D_α. Before doing this, first observe that for $k > \sigma \equiv \max(\text{spectrum } \mathscr{L}_{z^{(0)}})$, one has

$$z^{(k)}(p) = \text{polynomial in } z^{(j)}(p), \qquad 0 \leqq j \leqq \sigma;$$

this is an immediate consequence of the recursion relations (2.3) for $z^{(k)}$ and since (spectrum α)$\in \mathbb{Z}$, we have

$$\det(\mathscr{L} - kI) \in \mathbb{Z} \backslash \{0\}, \qquad \text{for } k > \delta.$$

Define the Painlevé divisor

$$D_j = \{\text{Laurent solutions } z(t) \text{ satisfying } H_r(z(t)) = A_r,\ r = 1, \ldots, k+m \text{ and } z^{(0)} \in C_j\}$$

$$= \bigcap_{\substack{\text{all } G \text{ such that} \\ \frac{\partial G}{\partial z}\big|^{\cdot}_{C_j} = 0}} \{\text{effective parameters } e_i \text{ such that } G(z(t)) = A\}^{17}.$$

In view of (3.4), the Painlevé divisor D_j going with C_j is defined by $\sum_M (\alpha^M - \beta^M_{C_j})$ nonlinear relations, between the $\delta(C_j) - \sum_M \beta^M_{C_j}$ effective parameters e_i, hence

$$\dim D_j \geqq \delta(C_j) - \sum \alpha^M = \delta(C_j) - (\#\text{independent invariants}), \tag{3.15}$$

and equality follows from Lemma 1.1. This establishes Theorem 4.

§ 4. Example: a geodesic flow on $SO(4)$

The free motion of a solid body in an ideal fluid can be interpreted as geodesic motion on the dual Lie algebra $T^* E_3$ of the group $E_3 = SO(3) \times \mathbb{R}^3$. By means of a reduction to the coadjoint orbits of $e_3^* \simeq e_3 = so(3) \times \mathbb{R}^3 \ni (l, p)$, this motion can be written

$$\dot{p} = p \wedge \frac{\partial H}{\partial l} \qquad \dot{l} = l \wedge \frac{\partial H}{\partial l} + p \wedge \frac{\partial H}{\partial p}, \tag{4.1}$$

$H(l, p)$ being the sum of the kinetic energies of both the rigid body and the surrounding ideal fluid. Lyapunov and Steklov have considered the following Hamiltonian

$$H(l, p) = \tfrac{1}{2} \sum_1^3 (l_i - (a_1 + a_2 + a_3 - a_i) p_i)^2.$$

This flow on E_3 turns out to be a limit of a geodesic flow

$$\dot{x}' = x' \wedge \frac{\partial H}{\partial x'}, \qquad \dot{x}'' = x'' \wedge \frac{\partial H}{\partial x''},$$

$$x' = (x_1, x_2, x_3), \qquad x'' = (x_4, x_5, x_6) \tag{4.2}$$

on the group $SO(3) \times SO(3) \simeq SO(4)$, for a left-invariant metric

$$H = \tfrac{1}{2} \sum_1^6 \lambda_i x_i^2 + \sum_1^3 \lambda_{i,i+3} x_i x_{i+3}$$

---

[17] the coefficients $z^{(k)}$ of the expansions $z(t)$ used here must have the form (3.14)

satisfying ($A_{ij}\equiv\lambda_i-\lambda_j$)

$$(\lambda_{14}^2,\lambda_{25}^2,\lambda_{36}^2)=\frac{A_{13}A_{46}A_{21}A_{54}A_{32}A_{65}}{(A_{46}A_{32}-A_{65}A_{13})^2}\cdot\left(\frac{(A_{65}-A_{32})^2}{A_{65}A_{32}},\frac{(A_{46}-A_{13})^2}{A_{46}A_{13}},\frac{(A_{54}-A_{21})^2}{A_{54}A_{21}}\right)$$

with the following sign specification

$$\lambda_{14}\lambda_{25}\lambda_{36}=\frac{A_{13}A_{46}A_{21}A_{54}A_{32}A_{65}}{(A_{46}A_{32}-A_{65}A_{13})^3}(A_{65}-A_{32})(A_{46}-A_{13})(A_{54}-A_{21}).$$

In the classification of algebraic integrable geodesic flows for left-invariant metrics there appear three strata of metrics, a first one first considered by Manakov, a second described above and a third discovered by us in 1984 [4]. Algebro-geometrical considerations lead to a natural (linear) change of variables, which transforms any of the flows above into a new (much simpler) flow X_1: $\dot z=f_1(z)$ and a flow X_2: $\dot z=f_2(z)$ commuting with X_1, with all parameters scaled out:

$$\begin{array}{ll}
X_1: \dot z_1=z_2z_6 & X_2: \dot z_1=z_5z_6\\
\dot z_2=\tfrac12 z_3(z_1+z_4) & \dot z_2=z_3z_4\\
\dot z_3=\tfrac12 z_2(z_1+z_4) & \dot z_3=z_2z_4\\
\dot z_4=z_3z_5 & \dot z_4=z_5(2z_3-z_6)\\
\dot z_5=z_3z_4 & \dot z_5=z_4(2z_3-z_6)\\
\dot z_6=z_1z_2 & \dot z_6=z_1z_5
\end{array}$$

with four quadratic invariants:

$$H_1=-z_4^2+z_5^2=A_1=A$$

$$H_2=-z_1^2+z_6^2=A_2=B$$

$$H_3=z_2^2-z_3^2=A_3=C/4$$

$$H_4=-(z_1-z_4)^2+2(z_2-z_5)^5+2(z_3-z_6)^2=A_4=D.$$

The system is purely homogeneous, with z_i having weight 1, the invariants have degree 2, and so the invariant surfaces $\mathscr{A}=\bigcap_1^4\{H_i=A_iz_0^2,\ z_0=1\}$ naturally embed into $\mathbb{P}^6$; yielding

$$\bar{\mathscr{A}}\equiv\bigcap_1^4\{H_i(z)=A_iz_0^2,\ z=(z_0,z_1,\dots,z_6)\in\mathbb{P}^6\}\subset\mathbb{P}^6.$$

Consider a fixed vector field $\alpha X_1+\beta X_2$. Because of the weights of the z_i, it is natural to search for Laurent solutions having simple poles:

$$z(t)=t^{-1}(z^{(0)}+z^{(1)}t+\dots). \tag{4.3}$$

The leading term is given by the indicial locus

$$\begin{aligned}\mathscr{C} &= \{z^{(0)} + \alpha f_1(z^{(0)}) + \beta f_2(z^{(0)}) = 0\} \\ &= C + \tilde{C} = \sum_{X^2, Y^2 = 1} C_{X,Y} + \sum_{X^2, Y^2 = 1} \tilde{C}_{X,Y}(\alpha, \beta),\end{aligned}$$

consisting of four lines

$$C_{X,Y} = \{(-2XY(1-Z), X, Y, -2XYZ, 2XZ, 2Y(1-Z)), Z \in \mathbb{C}\}, \tag{4.4}$$

independent of (α, β) and four points

$$\begin{aligned}\tilde{C}_{X,Y}(\alpha, \beta) \equiv \Bigg(&\frac{(\alpha+2\beta)}{\alpha\beta} XY, \frac{-2\beta X}{\alpha(\alpha+2\beta)}, \frac{2\beta Y}{\alpha(\alpha+2\beta)}, \frac{-\alpha XY}{\beta(\alpha+2\beta)}, \\ &\frac{-\alpha X}{\beta(\alpha+2\beta)}, \frac{(\alpha+2\beta)Y}{\alpha\beta}\Bigg),\end{aligned} \tag{4.5}$$

depending on (α, β). Along $C_{X,Y}$ the gradients

$$\begin{aligned}&\frac{\partial H_1}{\partial z} = 4XYZ(0,0,0,1,Y,0), \quad &&\frac{\partial H_2}{\partial z} = 4XY(1-Z)(1,0,0,0,0,X) \\ &\frac{\partial H_3}{\partial z} = 2(0, X, -Y, 0, 0, 0), \quad &&\frac{\partial H_4}{\partial z} = 4XY(1-2Z)(1, Y, -X, -1, -Y, X)\end{aligned} \tag{4.6}$$

span a 3-dimensional space showing that $\bar{\mathscr{A}}$ is singular along each line $C_{X,Y}$; incidentally, the gradient of the Z-dependent quadrics

$$\begin{aligned}H &\equiv \frac{H_1}{Z} - \frac{H_2}{1-Z} - 2H_3 + \frac{H_4}{1-2Z} \\ &= \frac{1}{1-2Z}\Bigg(-\left(z_1\sqrt{\frac{Z}{1-Z}} - z_4\sqrt{\frac{1-Z}{Z}}\right)^2 + 2\left(z_2\sqrt{2Z} - z_5\frac{1}{\sqrt{2Z}}\right)^2 \\ &\quad + 2\left(z_3\sqrt{2(1-Z)} + z_6\frac{1}{\sqrt{2(1-Z)}}\right)^2\Bigg)\end{aligned} \tag{4.7}$$

which is a family of rank 3 quadrics, vanishes along $C_{X,Y}$. At the 4 points $\tilde{C}_{X,Y}$ the gradients (4.6) are independent, showing smoothness of $\bar{\mathscr{A}}$ along $\tilde{C}_{X,Y}$. According to Theorem 4, (ii), (iii) and (v), the considerations above lead to *a priori* information on the $\mathscr{L}$-matrix (the Jacobi matrix of the equations defining $\mathscr{C}$):

$$\text{along } C_{X,Y}\colon -1, 0, 2, 2, 2 \in \text{spectrum } \mathscr{L}$$

$$\text{at } \tilde{C}_{X,Y}\colon -1, 2, 2, 2, 2 \in \text{spectrum } \mathscr{L}.$$

Since Tr $\mathscr{L}=\Sigma$ weights$=6$, it follows at once that

$$\text{along } C_{XY}\text{: spectrum } \mathscr{L}=(-1,0,1,2,2,2),$$
$$\text{at } \tilde{C}_{XY}\text{: spectrum } \mathscr{L}=(-1,-1,2,2,2,2).$$

In view of (vii) in Theorem 4, and since the gradients (4.6) span a 3-dimensional space along $C_{X,Y}$, the null-vectors of $\mathscr{L}-2I$ may be chosen such that the corresponding free parameters in the Laurent solution are exactly A, B, C; they are trivial parameters. Besides the parameter Z in the leading term $z^{(0)}$ of the Laurent series (4.3), there is one other effective parameter U, leading at once to the Laurent solution

$$z(t)=\frac{\zeta}{t}\cdot\left(\mathbb{1}+\frac{t}{4}U\begin{pmatrix}-1\\1\\1\\-1\\-1\\-1\end{pmatrix}+\frac{t^2}{48}\left(U^2\mathbb{1}-\frac{A}{2}\begin{pmatrix}1\\-2\\-2\\\frac{Z+3}{Z}\\\frac{Z-3}{Z}\\1\end{pmatrix}-\frac{B}{(1-Z)}\begin{pmatrix}\frac{Z-4}{Z-1}\\-2\\-1\\1\\1\\\frac{Z+2}{Z-1}\end{pmatrix}-C\begin{pmatrix}-(Z-2)\\2(Z-2)\\2(Z+1)\\-(Z+1)\\-(Z+1)\\-(Z-2)\end{pmatrix}\right)+O(t^3)\right), \tag{4.8}$$

(with $\mathbb{1}=(1,1,1,1,1,1)$, and $\zeta=\text{diagonal}\,(z^{(0)})$)

convergent by means of lemma 2.1. We have at once $H_i(z(t))=A_i$ for $i=1,2,3$ and the Painlevé divisor $D(X,Y)$ is obtained by putting, in view of Theorem 4 (viii), the Laurent solution $z(t)$ into the constant of the motion (4.7) with zero gradient; an elementary computation shows

$$H(z(t))=\frac{U^2}{1-2Z}$$

and since the value of the invariant H equals

$$\frac{A}{Z}-\frac{B}{1-Z}-\frac{C}{2}+\frac{D}{1-2Z},$$

the Painlevé variety consists of 4 isomorphic hyperelliptic curves, $D = \sum_{X^2=Y^2=1} D(X, Y)$,

$$D(X, Y)\colon U^2 + (2Z-1)\left(\frac{A}{Z} - \frac{B}{1-Z} - \frac{C}{2}\right) = D \tag{4.9}$$

covering the 4 lines $C(X, Y)$. For the general vector field $\alpha X_1 + \beta X_2$ the Laurent solutions (4.3) could be computed as well; although the expressions (4.8) would be different, the Painlevé divisor would have the same form (4.9); for $\alpha X_1 + \beta X_2$, the leading term ζ in (4.8) would become $\zeta(\alpha + 2\beta Z)^{-1}$. Observe that some of the coefficients of the Laurent solutions (4.8) blow up and thus the solutions cease to make sense at the points $Z=0$, 1 and ∞ on the curve. Therefore $D(X, Y)$ is actually the curve (4.9) with the three points $Z=0, 1, \infty$ removed, whereas $\bar{D}(X, Y)$ is the full curve; on the latter the coordinates U and Z behave as follows

$\underline{Z=0}$	$\underline{Z=1}$	$\underline{Z=\infty}$	
(branch point)	(branch point)	(branch point)	
$Z = \frac{A}{4} s^2$	$1 - Z = \frac{B}{4} s^2$	$Z^{-1} = \frac{C}{4} s^2$	
$U = -\frac{2}{s}(1 + O(s^2))$	$U = -\frac{2}{s}(1 + O(s^2))$	$U = -\frac{2}{s}(1 + O(s^2))$.	(4.10)

To check how the four 5-dimensional families of Laurent solutions hang together in a coherent way, we need to construct birational maps T for each of the points $Z=0, 1, \infty$ on the four curves $D(X, Y)$, $X, Y = \pm 1$. Notice this system is *regular*, in the sense of § 1, because the four 5-dimensional families of Laurent solutions have a leading term parametrized by a curve. To verify the coherence, it is more efficient to search for functions of increasing degree behaving like $1/t$ and then make ratios of such functions; this idea is related to Kodaira's embedding theorem, which states that the functions having a k-fold pole along an "ample" divisor of a compact variety embed the variety smoothly into some projective space. The following functions constitute a basis of polynomials having degree $\leqq 3$, and behaving like $1/t$ (modulo the constants of motion):

$$z_0 = 1, \qquad z_1, \ldots, z_6,$$

$$z_7 = -2 z_2 z_3 + z_2 z_6 + z_3 z_5$$

$$z_8 = -z_1 z_6 + 2 z_1 z_3 - z_4 z_6 \qquad z_9 = -z_4 z_5 + 2 z_2 z_4 - z_1 z_5$$

$$z_{10} = -z_6^2 + (2 z_2 - z_5)^2 \qquad z_{11} = -z_5^2 + (2 z_3 - z_6)^2$$

$$z_{12} = 2(z_1 z_3 z_5 - z_2 z_4 z_6)$$

$$z_{13} = 2(z_5 z_3 z_6 - z_2 z_1 z_4) \qquad z_{14} = 2(z_6 z_2 z_5 - z_3 z_1 z_4)$$

$$z_{15} = 2(z_1 z_2 z_5 - z_4 z_3 z_6). \tag{4.11}$$

The leading terms in

$$z_i = t^{-1}(z_i^{(0)} + z_i^{(1)}t + \dots), \qquad 1 \leqq i \leqq 15$$

are the following:

$$\begin{aligned}
(z_1^{(0)}, \dots, z_6^{(0)}) &= (-2XY(1-Z), X, Y, -2XYZ, 2XZ, 2Y(1-Z))\\
(z_7^{(0)}, \dots, z_{11}^{(0)}) &= -U(XY, 2X(1-Z), 2YZ, -4(1-Z), -4Z)\\
(z_{12}^{(0)}, \dots, z_{15}^{(0)}) &= (-A_0 + B_0 + C_0, X(A_0 + B_0 - C_0),\\
&\qquad Y(A_0 + B_0 + C_0), XY(-A_0 + B_0 - C_0))
\end{aligned} \tag{4.12}$$

with

$$A_0 \equiv \frac{A(1-Z)}{Z}, \qquad B_0 \equiv \frac{BZ}{1-Z}, \qquad C_0 \equiv CZ(1-Z), \qquad X, Y = \pm 1$$

Notice that the involution

$$\begin{aligned}
\Pi: &(z_1, z_2, z_3, z_4, z_5, z_6, t, U, X, Y, Z, A, B, C, D)\\
&\sim (z_4, z_3, z_2, z_1, z_6, z_5, t, U, Y, X, 1-Z, B, A, -C, D)
\end{aligned}$$

cuts down the computing labour by roughly half.

The birational map $T: z \curvearrowright y$, which will be used to check coherence reads as follows:

$$\begin{aligned}
T&: (z_0, \dots, z_6) \curvearrowright (y_0, \dots, y_{15}), \qquad \text{where } y_i = z_i/z_{12}\\
T^{-1}&: (y_0, \dots, y_{15}) \curvearrowright (z_0, \dots, z_6), \qquad \text{where } z_i = y_i/y_0
\end{aligned} \tag{4.13}$$

and turns out to be the same for all points $Z = 0, 1, \infty$. Since we expect $D = \Sigma D(X, Y)$ to be an ample and projectively normal divisor on an Abelian surface, we may also expect the functions $(y_0, \dots, y_{15})$ to form a closed system of quadratic differential equations, whatever be the flow $\alpha X_1 + \beta X_2$, as explained in Theorem 2. Indeed, one verifies

$$\left(\frac{z_i}{z_{12}}\right)^{\bullet} = X_1\left(\frac{z_i}{z_{12}}\right) = \text{quadratic polynomial}\left(\frac{z_0}{z_{12}}, \dots, \frac{z_{15}}{z_{12}}\right), \quad 0 \leqq i \leqq 15$$

and thus

$$\dot{y}_i = X_1(y_i) = \text{quadratic polynomial}\,(y_0, \dots, y_{15}), \qquad 0 \leqq i \leqq 15. \tag{4.14}$$

To do this, it suffices to compute the Wronskian $\{z_i, z_j\} = z_j X_1 z_i - z_i X_1 z_j$ of the functions z_i with z_{12}. Again the labour involved is considerably reduced by means of the involution Π. The actual calculation proceeds as follows: express the leading term of the Wronskian

$$\{z_i, z_j\} = \frac{(z_j^{(0)} z_i^{(1)} - z_j^{(1)} z_i^{(0)})}{t^2} + \dots, \text{ using (4.8)},$$

in terms of quadratic polynomials of $z_i^{(0)}$:

$$(z_i^{(0)} z_j^{(1)} - z_i^{(1)} z_j^{(0)}) = \text{quadratic polynomial } (z_0^{(0)}, \ldots, z_{15}^{(0)}),$$

implying that the following difference has a simple pole along D:

$$\{z_i, z_j\} - \text{same quadratic polynomial } (z_0, \ldots, z_{15}) = \frac{c}{t} + \ldots;$$

one then expresses this function as a linear combination $\sum_0^{15} c_i z_i$, with constant coefficients. This leads to the following result:

$$\{z_0, z_{12}\} = 2H_3 z_1 z_4 - (z_1 + z_4)\frac{z_{15}}{2}$$

$$\{z_1, z_{12}\} = z_7 z_{12} + \frac{z_{15}}{4}[2z_{10} + z_{11} + H_1 + 3H_2 + 2H_3 - H_4] - H_3 z_1 [z_{11} + H_1 + H_2 + 2H_3]$$

$$\{z_2, z_{12}\} = H_3 z_1 z_9, \{z_5, z_{12}\} = \frac{z_9 z_{15}}{2}, \{z_7, z_{12}\} = -H_3 z_8 z_9$$

$$\begin{aligned}\{z_8, z_{12}\} = &-4H_3 z_4\left(H_2 z_3 + \frac{z_6}{2}\left(-\frac{H_1}{2} - H_2 + H_3 + \frac{H_4}{4}\right)\right) \\ &-\frac{z_{12}}{2}(z_{13} + z_2(-H_1 + 2H_3)) \\ &+ z_{15}\left(z_{14} + z_3\left(-\frac{H_1}{2} + H_2 + H_3 + \frac{H_4}{2}\right) - z_6\left(H_1 + H_2 + H_3 + \frac{H_4}{2}\right)\right)\end{aligned}$$

$$\begin{aligned}\{z_{10}, z_{12}\} = &H_2(-2z_1 z_4 H_3 + \tfrac{1}{2}(z_1 + z_4) z_{15}) + z_{15}^2 \\ &-\frac{z_{15}}{2}(z_1(3H_1 + 2H_2 + 6H_3 - H_4) + z_4(-H_1 - 2H_2 + 2H_3 + H_4)) \\ &+4H_3 z_1\left(z_1 H_1 + z_4\left(H_3 + \frac{H_4 - H_1}{2}\right)\right)\end{aligned}$$

$$\{z_{13}, z_{12}\} = 2H_2 H_3 z_4 z_9, \quad \{z_{15}, z_{12}\} = 2H_3(H_1 z_6 z_8 - H_2 z_5 z_9)$$

with the missing equations deduced from the involution Π and the observation that $z_{12}^{\pi} = -z_{12}$; this establishes (4.14).

Putting the expansions for $z_0, \ldots, z_{15}$ into the rational map T, letting $t \searrow 0$, and using the local behavior of the curves $D(X, Y)$ near the points p_0, p_1, p_∞ (corresponding to $Z = 0, 1, \infty$), one computes that a neighborhood on $D(X, Y)$

of the points $p_0, p_1, p_\infty(X, Y)$ is mapped to the following segments of curve in $\mathbb{P}^{15}$ (remember $X, Y = \pm 1$)

$$\begin{aligned}
&y(0,p,D(-X,Y))|_{Z\sim 0}\\
&\quad=\lim_{t\to 0} T(z(t,p,D(-X,Y))|_{Z=\frac{A}{4}s^2\sim 0}\\
&\quad=(0,\dots,0,1,X,-Y,-XY)+s\left(0,\dots,0,\frac{XY}{2},X,0,2,0,0,0,0,0\right)\\
&\qquad-\frac{s^2}{4}(0,2XY,-X,Y,0,0,2Y,0,\dots,0)+O(s^3),\\
&y(0,p,D(X,-Y))|_{Z\sim 1}\\
&\quad=\lim_{t\to 0} T(z(t,p,D(X,-Y)))|_{1-Z=\frac{B}{4}s^2\sim 0}\\
&\quad=(0,\dots,0,1,X,-Y,-XY)\\
&\qquad+s\left(0,\dots,0,-\frac{XY}{2},0,-Y,0,-2,0,0,0,0\right)+O(s^2),\\
&y(0,p,D(-X,-Y))|_{Z\sim\infty}\\
&\quad=\lim_{t\to 0} T(z(t,p,D(-X,-Y)))|_{\frac{1}{Z}=\frac{C}{4}s^2\sim 0}\\
&\quad=(0,\dots,0,1,X,-Y,-XY)\\
&\qquad+s(0,\dots,0,0,-X,Y,-2,2,0,0,0,0)+O(s^2).
\end{aligned}\tag{4.15}$$

One first observes that the following three points coincide

$$\begin{aligned}
\lim_{p\to p_0} y(0,p,D(-X,Y)) &= \lim_{p\to p_1} y(0,p,D(X,-Y))\\
&= \lim_{p\to p_\infty} y(0,p,D(-X,-Y))
\end{aligned}$$

i.e., $p_0(-X,Y)=p_1(X,-Y)=p_\infty(-X,-Y)$ (see figures 4.2 and 4.3 below). Moreover, the tangents to the three branches at this point lie in a 2-dimensional plane, because the sum of the coefficients of s vanish. The function $y(t,p,D(-X,Y))$ is the solution to the differential equation (4.14) in t, with initial condition given by the segment $y(0,p,D(-X,Y))$ above; therefore its solution $y(t,p,D(-X,Y))$ is a holomorphic function in (t,p) for $0\leqq|t|<\varepsilon$ and $p\sim p_0$, confirming *condition* (*i*) of the coherence; for instance, near p_0 the first few terms can be computed from the differential equations:[18]

$$\begin{aligned}
&y(t,p,D(X,-Y))\\
&\quad=\tfrac{1}{8}(-st(t+2s)+O(4),\,-XY(t+2s)^2+O(3),\\
&\qquad 2Xs^2+O(3),\,-2Ys^2+O(3),\,XYt^2+O(3),\\
&\qquad Xt^2+O(3),\,-Y(t+2s)^2+O(3),\dots).
\end{aligned}\tag{4.16}$$

[18] $O(n)$ denotes a Taylor series in s and t of multiplicity n

Because of the uniqueness of the solution of differential equations, the solution running through the point $p_0(-X, Y)=p_1(X, -Y)=p_\infty(-X, -Y)$ is the same whichever expansion $D(-X, Y)$, $D(X, -Y)$ or $D(-X, -Y)$ is used, confirming *condition (iii)*; to be specific we have

$$\lim_{p\to p_0} y(t, p, D(-X, Y)) = \lim_{p\to p_1} y(t, p, D(X, -Y)) = \lim_{p\to p_\infty} (t, p, D(-X, -Y))$$

$$= \left(-\frac{H_3}{192}t^5, -\frac{XY}{8}t^2, \frac{XH_3}{64}t^4, \frac{YH_3}{64}t^4, \frac{XY}{8}t^2, \frac{X}{8}t^2, -\frac{Y}{8}t^2, \frac{XYH_3}{12}t^3, \frac{X}{2}t, -\frac{Y}{2}t, t, -t, 1, X, -Y, -XY\right)$$

$$+\text{higher order terms.} \tag{4.17}$$

Combining the formulas (4.15) and (4.17), one readily checks rank $\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial s}\right)\Big|_{s=t=0}=2$, confirming *condition (ii)*. As pointed out in (4.13), the inverse map T^{-1}: $y \curvearrowright z=(y_j/y_0)_{0\le j\le 15}$ yields the functions z_i $(0\le i\le 6)$ as meromorphic functions on T^2 near the point $p_0(-X, Y)$. Computing the ratios y_j/y_0 of the Taylor series appearing in (4.16), one finds

$$T^{-1}(y(t, p, D(-X, Y)))_{Z=\frac{A}{4}s^2}$$
$$=(z_1, \dots, z_6)(s, t)$$
$$=\left(\frac{XY(t+2s)^2+O(3)}{st(t+2s)+O(4)}, \frac{-2Xs^2+O(3)}{st(t+2s)+O(4)}, \frac{2Ys^2}{st(t+2s)+O(4)}, \frac{-XYt^2+O(3)}{st(t+2s)+O(4)}, \frac{-Xt^2+O(3)}{st(t+2s)+O(4)}, \frac{Y(t+2s)^2+O(3)}{st(t+2s)+O(4)}\right). \tag{4.18}$$

These new expressions for $z_i=h_i(t, s)/g_i(t, s)$ are the analytic extensions of the expressions z_i defined by the original Laurent series; notice that the Taylor series

$$g_i(t, s)=st(t+2s)+O(4),$$

of multiplicity 3, accounts for the simple poles of z_i along the three curves $D(-X, Y)$, $D(X, -Y)$ and $D(-X, -Y)$ intersecting in one point $p_0(-X, Y)$.

By making ratios in (4.17), it is easy to compute the Laurent series for z_i along the trajectory running through the common point $p_0(-X, Y)$, namely

$$(z_1, \dots, z_6)(t, 0)$$
$$=3\left(\frac{8XY}{A_3t^3}, -\frac{X}{t}, -\frac{Y}{t}, -\frac{8XY}{A_3t^3}, -\frac{8X}{A_3t^3}, \frac{8Y}{A_3t^3}\right)$$
$$+\text{higher order terms;} \tag{4.19}$$

in fact, the leading terms are readily seen to satisfy the system $\dot{z}=f_1(z)$, going with X_1, as they should (see the beginning of this section). These expressions also show that $g_i(t, 0)\not\equiv 0$; this ends the verification of *condition (iv)*.

To check *item* (*v*) in the coherence condition, we investigate the image of the origin $(t, s)=(0, 0)$, under the map

$$(t, s) \curvearrowright (z_1, \ldots, z_6)(t, s) \in \mathbb{P}^6 \tag{4.20}$$

given by (4.18). Since the ratios are 0/0 for $s=t=0$, one examines how the limits along the lines running through the origin $(t, s)=(0, 0)$ are mapped by means of (4.20); to do this consider the map of a disc Δ around $(0, 0)$ to $\Delta \times \mathbb{P}^1$, defined by

$$\Delta \to \Delta \times \mathbb{P}^1$$
$$(t, s) \curvearrowright (t, s, \beta, \alpha) \quad \text{with} \quad s\beta = t\alpha.$$

Making the substitution $s=t\alpha/\beta$, and letting $t \to 0$ in (4.18), the vector $(z_1, \ldots, z_6)$, viewed projectively, tends to the line

$$E_{XY}=\{\tilde{C}_{X,Y}(\alpha, \beta) \mid (\alpha, \beta) \in \mathbb{P}^1\} \qquad (\tilde{C}_{X,Y} \text{ as in (4.5)}),$$

or in other terms, the birational map T blows down the line $E_{X,Y}$, i.e. it is an exceptional divisor. The line $E_{X,Y}$ has degree two, because setting a linear combination of the coordinates in $E_{X,Y}$ equal to zero leads to a quadratic equation in α/β. Moreover the generic Laurent solutions are double covers of the lines $C_{X,Y}$. To summarize, we have

$$\sum_{X,Y=\pm 1} [\text{degree}(D(X, Y)) + \text{degree } T^{-1}(p_0(-X, Y))] = 16,$$

which coincides with the degree of the generic hyperplane section of $\mathscr{A} = \bigcap_1^4 \{H_i = A_i\}$, thus confirming *condition* (*v*). By Theorem 1, this system is algebraic completely integrable on tori T^2; the tori are Jacobi varieties of the hyperelliptic curves $D(X, Y)$.

The behavior of the variety $\bar{\mathscr{A}}$ in $\mathbb{P}^6$ is straightforward; we have

$$\mathscr{A}_\infty = \sum_{X,Y=\pm 1} (C(X, Y) + E(X, Y))$$

with the intersection pattern of figure 4.1. Moreover $\tilde{\mathscr{A}}$ is smooth along the E's and has a double crossing along the C's.

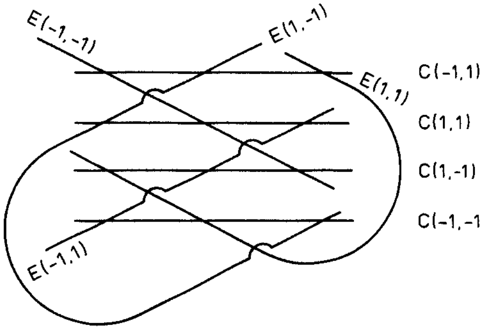


Fig. 4.1 $\mathscr{A}_\infty = \sum_{X,Y=\pm 1} (C(X, Y) + E(X, Y))$

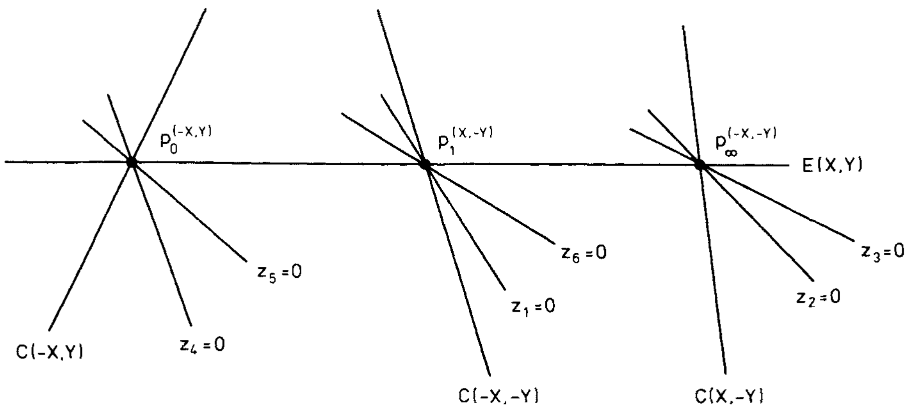


Fig. 4.2 detailed view of Fig. 4.1

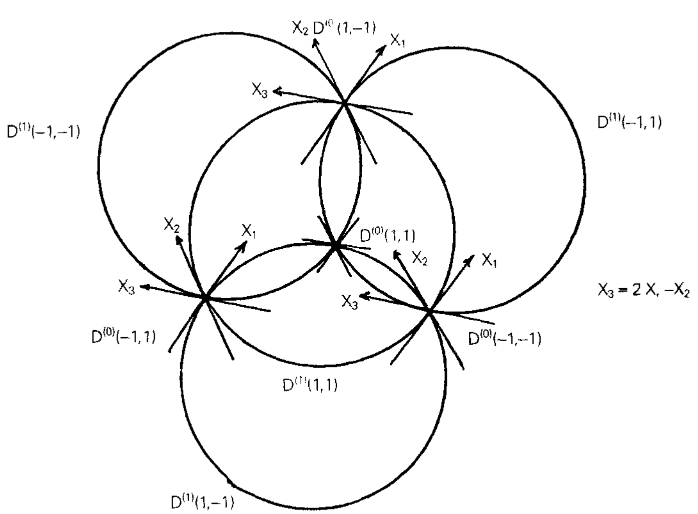
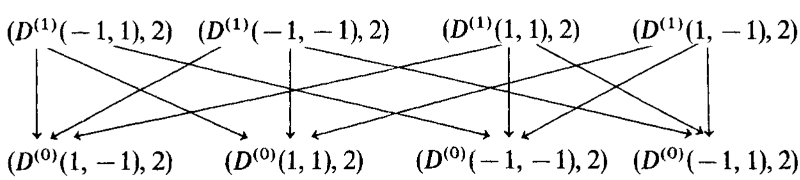


Fig. 4.3 four hyperelliptic curves $D^{(1)}(X, Y)$ and four points $D^{(0)}(X, Y)$, all parametrizing Laurent solutions

To conclude, the variety $\mathscr{A}$ in $\mathbb{P}^6$ is transformed into a Jacobi variety by blowing down the exceptional divisors $E(X, Y)$ and by blowing up $\mathscr{A}$ along the lines $C(X, Y)$; this leads to T^2, with 4 hyperelliptic curves on it, intersecting as in Fig. 4.3.

Associated with each flow $\alpha X_1 + \beta X_2$, there are four principal Laurent solutions $D^{(1)}(X, Y) \equiv D(X, Y)$ and four lowest solutions $D^{(0)}(X, Y)$, which fit together according to the tree,



as represented by means of divisors in Fig. 4.3.

The Laurent solutions in the tree are as follows:
Flow X_1:

$$z(t, Z, D^{(1)}(X, Y)) = t^{-1}(-2XY(1-Z), X, Y, -2XYZ, 2XZ, 2Y(1-Z)) + O(1)$$

$$z(t, D^{(0)}(X, Y)) = 3\left(\frac{8XY}{A_3 t^3}, -\frac{X}{t}, -\frac{Y}{t}, \frac{-8XY}{A_3 t^3}, \frac{-8X}{A_3 t^3}, \frac{8Y}{A_3 t^3}\right)$$

$$+ \text{higher order terms.}$$

Flow $\alpha X_1 + \beta X_2$: (generic α, β)

$$z(t, Z, D^{(1)}(X, Y))$$
$$= t^{-1}(-2XY(1-Z), X, Y, -2XYZ, 2XZ, 2Y(1-Z))(\alpha + 2\beta Z)^{-1} + O(1)$$

$$z(t, D^{(0)}(X, Y))$$
$$= t^{-1}\left(\frac{\alpha+2\beta}{\alpha\beta} XY, \frac{-2\beta X}{\alpha(\alpha+2\beta)}, \frac{2\beta Y}{\alpha(\alpha+2\beta)}, \frac{-\alpha XY}{\beta(\alpha+2\beta)}, \frac{-\alpha X}{\beta(\alpha+2\beta)}, \frac{(\alpha+2\beta)Y}{\alpha\beta}\right) + O(1)$$

Appendix 1

We give some basic facts about weighted projective spaces $\mathbb{P}^n_\nu$ for the integer weights $\nu = (\nu_0, \nu_1, \ldots, \nu_n)$, $\nu_i \geqq 1$. A standard projective space $\mathbb{P}^n$ is obtained by identifying all points on the lines running through the origin in $\mathbb{C}^{n+1}$. A weighted projective space $\mathbb{P}^n_\nu$ with variables z_i having weights ν_i is defined by identifying all points on the curve

$$(z_0 t^{\nu_0}, z_1 t^{\nu_1}, \ldots, z_n t^{\nu_n}), \qquad t \in \mathbb{C}^*$$

running through the origin. There is a natural surjection

$$\Phi: \mathbb{P}^n \to \mathbb{P}^n_\nu: (y_0, y_1, \ldots, y_n) \curvearrowright (y_0^{\nu_0}, y_1^{\nu_1}, \ldots, y_n^{\nu_n}) \tag{1}$$

whose kernel in $\mathbb{P}^n$ is given by a discrete group action μ and thus

$$\mathbb{P}^n_\nu = \mathbb{P}^n/\mu, \qquad \mu = \bigoplus_{i=0}^{n} \mathbb{Z}^{\nu_i}.$$

In analogy with $\mathbb{P}^n$, the space $\mathbb{P}^n_\nu$ is covered by $n+1$ charts specified by $z_i \neq 0$ ($i = 0, 1, \ldots, n$). In $\mathbb{P}^n_\nu$, it will be convenient to consider the following "hyperplane"

$$\sum_{i=0}^{n} a_i x_i^{d_i} = 0, \qquad x \in \mathbb{P}^n_\nu, \tag{2}$$

where

$$d = \text{l.c.m.}(\nu_1, \ldots, \nu_n) \quad \text{and} \quad d_i = d/\nu_i. \tag{3}$$

This is the generic equation in $\mathbb{P}_v^n$ of smallest degree. Through the map Φ^{-1}, the hyperplane equation (2) in $\mathbb{P}^n{}_v$ corresponds to a homogeneous equation of degree d,

$$\sum_{i=0}^{n} a_i x_i^{d_i} = \sum_{i=0}^{n} a_i y_i^{v_i d_i} = \sum a_i y_i^d = 0.$$

It is also via the map Φ that most degree questions in $\mathbb{P}_v^n$ can ultimately be reduced to questions in $\mathbb{P}^n$. The *degree of an m-dimensional variety* $V \subset \mathbb{P}_v^n$ is defined as the number of intersection points of V with an $n-m$-dimensional "plane", the latter being the intersection of m "hyperplanes" (2).

Example. Consider an $n-m$-dimensional variety V in $\mathbb{P}_v^n$ defined by

$$V = \bigcap_{i=1}^{m} \{F_i(x)=0, x \in \mathbb{P}_v^n\}$$

and let

$$p \in V \curvearrowright (x_0(p), x_1(p), \ldots, x_n(p)) \in \mathbb{P}_v^n$$

be a parametrization. Then the degree of V can be computed in two different ways: on the one hand by the definition above we have

$$\text{degree } V = \text{number of points in } \bigcap_{s=1}^{\dim V} \left\{ p \in V, \sum_{i=0}^{n} a_i^{(s)} x_i(p)^{d_i} = 0 \right\}; \tag{4}$$

on the other hand, the degree of V in $\mathbb{P}_v^n$ can be computed by means of Bézout's theorem: namely one first computes the degree of $\Phi^{-1}(V)$ in $\mathbb{P}^n$, using the map Φ defined in (1) and then one takes account of the quotient by the discrete group action μ, yielding

$$\text{degree } V = \frac{\prod_{i=1}^{m} \{\text{degrees of } F_i\} \cdot d^{n-m}}{v_0 v_1 \ldots v_n}. \tag{5}$$

Appendix 2. *Regular systems*

In most lower-dimensional examples, we have the equality

$$\dim C_i = \text{dimension (invariant manifolds)} - 1$$

for all components C_i of $\mathscr{C}$ in the indicial locus; this is equivalent to the relation

$$\gamma_{C_i}^0 \equiv \dim(\ker \mathscr{L}_{C_i}) = m - 1 \qquad \text{(see (3.3))};$$

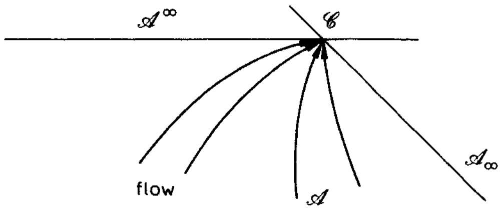


Fig. A.1

in other terms, the divisor D on T^m is a finite cover of $\mathscr{C}$. However in higher-dimensional situations, almost always we have

$$\gamma^0_{C_i} < m-1,$$

i.e., the trajectories, which in the affine cover the whole of the invariant manifold $\mathscr{A}$, flow towards a lower dimensional $(<m-1)$ manifold $\mathscr{C} \subset \mathscr{A}_\infty = \bar{\mathscr{A}} \cap \{z_0 = 0\}$, creating a singularity of codimension $\geqq 2$, as depicted in Fig. A.1. Also, the leading term $z^{(0)}$ of the Laurent solutions contain strictly less than $m-1$ free parameters. Usually this situation is easily remedied by preparing the system; this is done by adjoining $\delta = m-1-\gamma^0$ polynomials $P_1, \ldots, P_\delta$ in the phase variables $z_1, \ldots, z_n$ of weighted degrees $l_1, \ldots, l_\delta$, such that the full set of leading terms $z_1^{(0)}, \ldots, z_{n+\delta}^{(0)}$ of $z_1, \ldots, z_n$ and

$$P_i(z(t))|_{D_j} = t^{-\nu_{n+i}}(z^{(0)}_{n+i} + z^{(1)}_{n+i} t + \ldots), \qquad i = 1, \ldots, \delta,$$

depend on $m-1$ free parameters on each D_j, where ν_{n+i} is defined as the degree of the pole of P_i along the expansion D_j; we assume this can be done independently of j. The polynomials P_k must be chosen such that the dimension of the variety

$$\bigcap_1^{k+m} \{H_i = 0\} \cap \bigcap_1^{\delta} \{P_k = 0\}$$

is what it should be. In order to respect the weight homogeneous structure of the problem, we now adjoin δ variables $z_{n+1}, \ldots, z_{n+\delta}$ of weight $\nu_{n+1}, \ldots, \nu_{n+\delta}$ defined by

$$z_{n+i} z_0^{l_i - \nu_{n+i}} - P_i(z) = 0, \qquad i = 1, \ldots, \delta.$$

These δ equations of degree l_i are now added to the relations expressing the constants of motion. The reason for multiplying z_{n+i} by $z_0^{l_i - \nu_{n+1}}$ is that z_{n+i} blows up like $t^{-\nu_{n+i}}$, which respects the weight ν_{n+i} of z_{n+i}. Consider now the enlarged system: the new locus at infinity $\mathscr{A}_\infty$ contains the image of D under the map

$$D \to \tilde{C} \subset \mathbb{P}_\nu^{n+\delta}$$

$$p \curvearrowright (z_1^{(0)}(p), \ldots, z^{(0)}_{n+\delta}(p)) \in \tilde{C}.$$

Then D will become a finite covering of $\tilde{C}$ of degree $[D/\tilde{C}]$. To conclude this trick, which is crude, remedies the situation depicted in Fig. A.1; in the new picture the trajectories all flow to $m-1$-dimensional components of $\mathscr{A}_\infty$, still leaving us with a discrete number of trajectories flowing to a generic point and still leaving us with components of $\mathscr{A}_\infty$ not reached by the Laurent solutions of $\dot{z}=f(z)$.

Example. We consider the following system of differential equations in $z_1, \ldots, z_6$

$$\begin{aligned} \dot{z}_1 &= z_1(z_5-z_4) \qquad & \dot{z}_4 &= z_3-z_1 \\ \dot{z}_2 &= z_2(z_6-z_5) \qquad & \dot{z}_5 &= z_1-z_2 \\ \dot{z}_3 &= z_3(z_4-z_6) \qquad & \dot{z}_6 &= z_2-z_3, \end{aligned}$$

related to the 3-body periodic Toda lattice. Clearly the variables z_1, z_2, z_3 have weight 2 and z_4, z_5, z_6 weight 1; the system has the following constants of motion

$$\begin{aligned} H_1 &= z_1 z_2 z_3 = A_1 z_0^6 \\ H_2 &= z_4+z_5+z_6 = 2A_2 z_0 \\ H_3 &= \tfrac{1}{2}(z_4^2+z_5^2+z_6^2)-z_1-z_2-z_3 = A_3 z_0^2 \\ H_4 &= z_4 z_5 z_6 + z_1 z_6 + z_2 z_4 + z_3 z_5 = A_4 z_0^3. \end{aligned} \tag{1}$$

The indicial locus

$$v_i z_i^{(0)} + f_i(z^{(0)}) = 0, \qquad i=1, \ldots, 6,$$

for these equations consists of three points

$$\begin{aligned} \mathscr{C} &= \{1, 0, 0, 1, -1, 0\} \cup \{0, 1, 0, 0, 1, -1\} \cup \{0, 0, 1, -1, 0, 1\} \\ &= C_1 + C_2 + C_3, \\ &\text{with spectrum } \mathscr{S} = (-1, 1, 1, 2, 3, 3); \end{aligned}$$

each one leads to a Laurent solution depending on 5 free parameters

$$\left.\begin{aligned} z_i(t) &= t^{-2}(z_i^{(0)} + z_i^{(1)} t + \ldots) \\ z_{i+3}(t) &= t^{-1}(z_{i+3}^{(0)} + z_{i+3}^{(1)} t + \ldots) \end{aligned}\right\} \quad i=1,2,3,$$

with leading terms

	on C_1	on C_2	on C_3
$(z_1^{(0)}, \ldots, z_6^{(0)})$:	$(1, 0, 0, 1, -1, 0)$	$(0, 1, 0, 0, 1, -1)$	$(0, 0, 1, -1, 0, 1)$
$(z_1^{(1)}, \ldots, z_6^{(1)})$:	$(0, 0, 0, Y, Y, Z)$	$(0, 0, 0, Z, Y, Y)$	$(0, 0, 0, Y, Z, Y)$.

The parameter Z is effective, since there are two degrees of freedom at the first step in the expansions and only one invariant H_2 of weighted degree one. From this table, it is seen at once that both $z_1 + z_4 z_5$ and $z_2 + z_5 z_6$ never blow up worse than t^{-1}; therefore taking any arbitrary linear combination of them, for instance the difference, and defining z_7 by the equation

$$z_7 = z_0^{-1} P(z) \equiv z_0^{-1} [(z_1 + z_4 z_5) - (z_2 + z_5 z_6)],$$

one sees it has weight 1, behaves like t^{-1} in the affine chart $z_0 = 1$ and has the leading behavior

$$\begin{array}{cccc} & \underline{\text{on } C_1} & \underline{\text{on } C_2} & \underline{\text{on } C_3} \\ z_7^{(0)}: & Z & Z & -2Z. \end{array}$$

Therefore the regularized system is expressed in the variables $z_1, \ldots, z_7$ with equations (1) and

$$z_0 z_7 - (z_1 + z_4 z_5) + (z_2 + z_5 z_6).$$

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