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# DIOPHANTINE APPROXIMATION IN POSITIVE CHARACTERISTIC

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## § 1. Introduction

Let $k$ be a field of characteristic $p > 0$. We will be interested in the approximation of elements $y \in k[[x]]$, algebraic over $k(x)$, by elements of $k(x)$ (where $x$ is a variable) with respect to the valuation $\mathrm{ord} = \mathrm{ord}_{x=0}$.

Let $y \in k[[x]]$ and define

$$\alpha(y) = \limsup_{\substack{H(r)\to\infty \\ r\in k(x)}} \frac{\mathrm{ord}\,(y-r)}{H(r)}$$

where

$$H(P/Q) = \max\,\{\deg P,\ \deg Q\}$$

$P, Q \in k[x]$, $(P, Q) = 1$.

Define $d(y) = [k(x, y)\colon k(x)]$. Then Mahler [3], transposing a classical result of Liouville, proved that $\alpha(y) \le d(y)$, and he gave an example $\left(y = \sum_{i=0}^{\infty} x^{p^i}\right)$ which had $\alpha(y) = d(y) = p$, and thus showed that his bound was, in general, best possible.

Later, Osgood [4] showed that, if $y$ does not satisfy a Riccati equation, $y' = ay^2 + by + c$, $a, b, c \in k(x)$, then:

$$\alpha(y) \le \left[\frac{d(y)+3}{2}\right].$$

He actually showed that

$$\mathrm{ord}\,(y-r) \le \left[\frac{d(y)+3}{2}\right] H(r) + C$$

for any $r \in k(x)$ where $C$ is an effective constant depending on $y$.

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In this paper we shall be concerned with the approximation of algebraic functions $y \in k[[x]]$ satisfying

$$y = \frac{ay^q + b}{cy^q + d} \tag{1}$$

where $a, b, c, d \in k[x]$, $ad - bc \neq 0$ and $q$ is power of $p$.

We shall show that if $y$ satisfies (1) then there exists an effective constant $C$ for which $\operatorname{ord}(y - r) \leq \alpha(y)H(r) + C$ for any $r \in k(x)$. We shall also give several results that will enable us to bound $\alpha(y)$ effectively by some constant smaller than $d(y)$ in several cases. This will then give an effective improvement of the Liouville—Mahler Theorem for certain $y$ satisfying (1). Note that there are cases of $y$ satisfying (1) for which $\alpha(y) = d(y)$.

We shall also give several examples that illustrate our method and discuss the sharpness of our results.

Remark 1. If $y$ satisfies (1), then $y$ satisfies a Riccati equation.

Remark 2. If $d(y) = 3$, then $y$ satisfies (1) with $q = p$, in fact, $1, y, y^p, y^{p+1}$ are linearly dependent over $k(x)$, so one deduces immediately that $y$ satisfies (1).

## § 2. The main results

Let $y \in k[[x]]$ satisfy

$$y = \frac{ay^q + b}{cy^q + d}, \tag{1}$$

$a, b, c, d \in k[x]$, $ad - bc \neq 0$.

Let $d(y)$ be as above, note that $d(y) \leq q + 1$. Let

$$A = \max\{\deg a,\ \deg b,\ \deg c,\ \deg d\}$$

and $B = \operatorname{ord}(ad - bc)$. Assume that $d(y) > 1$.

Theorem 1. *For any $r \in k(x)$, we have either $H(r) \leq A(q - 1)$ or*

$$\operatorname{ord}(y - r) \leq \alpha(y)H(r) + \frac{\alpha(y)A}{q-1} + \frac{(B + 2\operatorname{ord} y)}{q-1}.$$

Before proving the theorem we verify a lemma.

Lemma 1. *If $r_1, r_2 \in k(x)$, $r_2 = \dfrac{ar_1 + b}{cr_2 + d}$, then*

(i) $H(r_1) - A \leq H(r_2) \leq H(r_1) + A$;
(ii) *if* $\operatorname{ord}(y^q - r_1) > 0$, *then*

$$\operatorname{ord}(y - r_2) \leq \operatorname{ord}(y^q - r_1) + B$$

*and* $\operatorname{ord}(y - r_2) > 0$;
*if* $\operatorname{ord}(y^q - r_1) > \operatorname{ord} y + B$, *then*

$$\operatorname{ord}(y - r_2) \geq \operatorname{ord}(y^q - r_1) - 2 \operatorname{ord} y - B.$$

PROOF.

(i) Let $r_1 = P/Q$, $P, Q \in k[x]$, $(P, Q) = 1$, then $r_2 = \dfrac{aP + bQ}{cP + dQ}$ and, obviously, $H(r_2) \leq H(r_1) + A$. Let

$$m = \max\{\deg(aP + bQ), \deg(cP + dQ)\}.$$

We claim that $H(r_2) \geq m - \deg(ad - bc)$. In fact, if $e \in k[X]$ divides $aP + bQ$ and $cP + dQ$, then $e$ divides $(ad - bc)P$, $(ad - bc)Q$, so $e \mid ad - bc$. This proves the claim.

We now prove that

$$m \geq H(r_1) - A + \deg(ad - bc).$$

This will complete the proof of (i).

We have that

$$\deg(adP + bdQ) \leq m + A,$$
$$\deg(bcP + bdQ) \leq m + A,$$

so $\deg(ad - bc)P \leq m + A$. Similarly, $\deg(ad - bc)Q \leq m + A$. Hence $H(r_1) + \deg(ad - bc) \leq m + A$, as desired.

(ii) We have that

$$\operatorname{ord}(y - r_2) = \operatorname{ord} \frac{(ad - bc)(y^q - r_1)}{(cy^q + d)(cr_1 + d)}. \tag{2}$$

If $\operatorname{ord}(y^q - r_1) > 0$ then, since $y \in k[[x]]$, we have $r_1 \in k[[x]]$. In particular,

$$(cy^q + d)(cr_1 + d) \in k[[x]],$$

so

$$\operatorname{ord}(y - r_2) \leq \operatorname{ord}(y^q - r_1) + B$$

by (2), and $\operatorname{ord}(y - r_2) > 0$.

Assume that

$$\operatorname{ord}(y^q - r_1) > \operatorname{ord} y + B \geq 0. \tag{3}$$

Let

$$m = \operatorname{ord}(cy^q + d),$$

then

$$\operatorname{ord}(ay^q + b) = m - \operatorname{ord} y.$$

Also

$$\operatorname{ord}(acy + ad) \geq m$$

and

$$\operatorname{ord}(acy + bc) \geq m - \operatorname{ord} y.$$

It follows that

$$\operatorname{ord}(ad - bc) \geq m - \operatorname{ord} y$$

or

(4) $$m \leq \operatorname{ord} y + B.$$

By (3) we have that

$$\operatorname{ord}(cr_1 + d) = \operatorname{ord}(cy^q + d) = m.$$

Hence, by (2),

$$\operatorname{ord}(y - r_2) \geq B - \operatorname{ord}(y^q - r_1) - 2m,$$

which by (4) completes the proof.

PROOF of Theorem 1. Let $R(X) = \dfrac{aX^q + b}{cX^q + d}$ and $R^N = R \circ R \circ \ldots \circ R$, $N$ times. Given $r \in k(x)$, define $r_N = R^N(r)$, $r_0 = r$.

Lemma 1 (i) then implies that

$$|H(r_{N+1}) - qH(r_N)| \leq A,$$

which implies

$$q^N H(r) - \left(\frac{q^N - 1}{q - 1}\right) A \leq H(r_N) \leq q^N H(r) + \left(\frac{q^N - 1}{q - 1}\right) A.$$

If $H(r) > \dfrac{A}{q-1}$, then $H(r_N) \to \infty$ as $N \to \infty$.

Further, we have either

$$\operatorname{ord}(y - r_N) \leq \frac{1}{q}(\operatorname{ord} y + B)$$

or

$$\operatorname{ord}(y - r_{N+1}) \geq q \operatorname{ord}(y - r_N) - 2 \operatorname{ord} y - B$$

by Lemma 1 (ii).

If

$$\operatorname{ord}(y - r) > \frac{2 \operatorname{ord} y - B}{q - 1}$$

then it follows by induction that ord $(y - r_N)$ is increasing with $N$. Since otherwise Theorem 1 is trivial, we may assume that this is the case. Then we have, by the above, that

$$\operatorname{ord}(y - r_{N+1}) \geq q^N \operatorname{ord}(y - r) - (2 \operatorname{ord} y + B)\frac{(q^N - 1)}{q - 1}$$

and

$$H(r_N) \leq q^N H(r) + \left(\frac{q^N - 1}{q - 1}\right) A.$$

Assuming that $H(r) > A/(q - 1)$, we have

$$\alpha(y) \geq \limsup_{N\to\infty} \frac{\operatorname{ord}(y - r_N)}{H(r_N)} \geq \liminf_{N\to\infty} \frac{\operatorname{ord}(y - r_N)}{H(r_N)} \geq$$

$$(*) \qquad \geq \lim_{N\to\infty} \frac{q^N \operatorname{ord}(y - r) - (2 \operatorname{ord} y + B)(q^N - 1)/(q - 1)}{q^N H(r) + A(q^N - 1)/(q - 1)} =$$

$$= \frac{\operatorname{ord}(y - r) - (2 \operatorname{ord} y + B)/(q - 1)}{H(r) + A/q - 1},$$

which proves Theorem 1.

THEOREM 2. *If* $r_1 \in k(x)$ *is such that*

$$(5) \qquad \operatorname{ord}(y - r_1) \geq \alpha H(r_1) - B/(q - 1) - \alpha A/(q - 1)$$

*for some* $\alpha > 2$, *then there exists some other* $r_2 \in k(x)$ *satisfying* (5) *with*

$$(6) \qquad H(r_2) \leq \frac{2A + B + q + (B + \alpha A)/(q - 1)}{\alpha - 2}$$

*and* $r_1 = R^n(r_2)$ *for some* $n \geq 0$.

PROOF. We prove that if $r \in k(x)$ satisfies (5) but not (6), then there exists $r_1 \in k(x)$ satisfying (5) and $H(r_1) < H(r)$ and $R(r_1) = r$. Since the height takes positive integer values, the theorem will follow by infinite descent.

Let $s = \dfrac{-dr + b}{cr - a}$.

By Lemma 1 (ii) we have $\left(\text{since } r = \dfrac{as + b}{cs + d}\right)$

$$\operatorname{ord}(y^q - s) \geq \operatorname{ord}(y - r) - B.$$

Let $q = p^n$ and let $m$ be an integer $(0 \leq m \leq n)$ with $s = r_1^{p^m}$ for some $r_1 \in k(x)$ and $m$ maximal. We claim that $m = n$. If not, then

$$\operatorname{ord}(y^{p^{m-n}} - r_1) \geq \frac{1}{p^m}(\operatorname{ord}(y - r) - B)$$

and

$$\begin{aligned}\operatorname{ord}(r_1') &= \operatorname{ord}\left((y^{p^{n-m}} - r_1)'\right) \geq \operatorname{ord}(y^{p^{n-m}} - r_1) - 1 \geq \\ &\geq \frac{1}{p^m}\left(\operatorname{ord}(y - r) - B\right) - 1.\end{aligned} \tag{7}$$

If $r_1' \neq 0$, we have $\operatorname{ord}(r_1') \leq H(r_1')$ but

$$H(r_1') \leq 2H(r_1) \leq \frac{2}{p^m} H(s).$$

So, by Lemma 1 (i),

$$\operatorname{ord}(r_1') \leq \frac{2}{p^m}\left(H(r) + A\right).$$

Hence, by (7) and (5),

$$H(r) \leq \frac{2A + B + q + (B + \alpha A)/q - 1}{\alpha - 2}.$$

This contradicts the hypothesis made at the beginning, so $r_1' = 0$ and therefore $m$ is not maximal. This implies that $m = n$, so $s = r_1^q$. As above we conclude that

$$\operatorname{ord}(y - r_1) \geq \frac{1}{q}\left(\operatorname{ord}(y - r) - B\right) \geq \frac{1}{q}\left[\alpha H(r) - \frac{B}{q-1} - \frac{\alpha A}{q-1} - B\right].$$

But, by Lemma 1 (i),

$$H(r) \geq H(s) - A = \frac{1}{q} H(r_1) - A,$$

so

$$\begin{aligned}\operatorname{ord}(y - r_1) &\geq \alpha H(r_1) - \frac{1}{q}\left[1 + \frac{1}{(q-1)}(B + \alpha A)\right] = \\ &= \alpha H(r_1) - (B + \alpha A)/(q - 1),\end{aligned}$$

hence $r_1$ satisfies (5).

To prove the theorem we now only need to show that $H(r_1) < H(r)$. Supposing the contrary,

$$H(r) \leq H(r_1) = \frac{1}{q} H(s) \leq \frac{1}{q}\left(H(r) + A\right),$$

so $H(r) \leq A/q - 1$, which implies (6). Since we assumed that $r$ did not satisfy (6), we arrive at a contradiction. So $H(r_1) < H(r)$ and Theorem 2 is proved.

## § 3. Examples

Some examples will be constructed based on the following proposition.

PROPOSITION 5. *Let* $y \in k[[x]]$, $y \notin k(x)$ and $r_n \in k(x)$, $r_n \to y$ *as* $n$ *tends to* $\infty$. *Assume also that for some positive constants* $\alpha$, $\beta$ *we have*

$$\lim_{n\to\infty} \frac{\operatorname{ord}(y - r_n)}{H(r_n)} = \alpha, \quad \lim_{n\to\infty} \frac{H(r_{n+1})}{H(r_n)} = \beta.$$

*If* $\alpha > \beta^{1/2} + 1$ *then* $\alpha(y) = \alpha$.

The proof is based on the following lemma.

LEMMA 2. *Let* $y \in k[[x]]$, $y \notin k(x)$. *If* $r_1 \neq r_2 \in k(x)$, $H(r_2) \geq H(r_1)$ *and* $\operatorname{ord}(y - r_i) \geq \alpha H(r_i)$ *for some* $\alpha > 0$, *then* $H(r_2) > (\alpha - 1)H(r_1)$.

PROOF.

$$\operatorname{ord}(r_2 - r_1) = \operatorname{ord}(r_2 - y + y - r_1) \geq \min\{\operatorname{ord}(y - r_1), \operatorname{ord}(y - r_2)\} \geq \alpha H(r_1).$$

On the other hand,

$$\operatorname{ord}(r_2 - r_1) \leq H(r_2 - r_1) \leq H(r_1) + H(r_2),$$

hence $H(r_2) \geq (\alpha - 1)H(r_1)$, as desired.

PROOF of Proposition 5. Obviously, $\alpha(y) \geq \alpha$. Assume that $\alpha(y) > \alpha$, then for $\varepsilon > 0$ sufficiently small there exists $s_n \to y$ with

$$\operatorname{ord}(y - s_n) \geq (\alpha + \varepsilon)H(s_n),$$

also for $n$ large

$$\operatorname{ord}(y - r_n) \geq (\alpha - \varepsilon)H(r_n), \ \operatorname{ord}(y - r_n) < (\alpha + \varepsilon)H(r_n).$$

Given $n$, choose $m$ with $H(r_m) \leq H(s_n) \leq H(r_{m+1})$, so $r_m \neq s_n \neq r_{m+1} \neq r_m$. By the lemma we have, if $n$ is large, that

$$(\alpha - 1 - \varepsilon)H(r_m) \leq H(s_n) \text{ and } H(s_n)(\alpha - 1 - \varepsilon) \leq H(r_{m+1}),$$

hence

$$(\alpha - 1 - \varepsilon)^2 \leq \frac{H(r_{m+1})}{H(r_m)}.$$

As $n \to \infty$, we have $m \to \infty$, so $(\alpha - 1 - \varepsilon)^2 \leq \beta$. Making $\varepsilon \to 0$ we get $(\alpha - 1)^2 \leq \beta$ or $\alpha \leq \beta^{1/2} + 1$; this contradicts the hypothesis, proving the result.

PROPOSITION 6.

(i) *Let* $f(x) \in k[x]$, $\deg f = m$, $\operatorname{ord} f = n > 0$, *and let* $y \in xk[[x]]$ *satisfy* $y^q - y = f(x)$. *Then* $n > m(q^{1/2} + 1)/q$ *implies* $\alpha(y) = nq/m$.

(ii) *Let* $f(x) \in k[x]$, $\deg f = m$ *and* $\operatorname{ord}(f-1) = n > 0$, *let* $d$ *be a divisor of* $q-1$ *with* $d > q^{1/2}+1$, *and let* $y \in 1 + xk[[x]]$ *satisfy* $y^d = f(x)$. *If* $n/m > (q^{1/2}+1)/d$, *then* $\alpha(y) = \dfrac{nd}{m}$.

PROOF.
(i) We have

$$y = -\sum_{i=0}^{\infty} f(x)^{q^i}.$$

Let

$$r_N = \sum_{i=0}^{N} f(x)^{q^i}.$$

Then Proposition 4 applies with $\alpha = nq/m$, $\beta = q$.
(ii) We have that

$$y = \prod_{i=0}^{\infty} f(x)^{-\frac{(q-1)}{d}\cdot q^i}.$$

If

$$r_N = \prod_{i=0}^{N} f(x)^{-\frac{q-1}{d}q^i} = f(x)^{-\frac{q-1}{d}\frac{q^{N+1}-1}{q-1}} = f(x)^{-\frac{(q^{N+1}-1)}{d}},$$

then Proposition 4 applies with $\alpha = nd/m$, $\beta = q$.

REMARK. The examples of (i) are a variation on Mahler's example [3] and the examples of (ii) with $m = n$ are a variation on Osgood's examples [4]. For $p = 2$ and $d = q-1$, examples similar to (ii) appear in [1].

Proposition 6 thus gives, when $n < m$, several examples where Theorem 1 applies, giving an effective improvement on the Liouville—Mahler Theorem. The examples of (ii) can be seen as analogues of $d$-th roots of rational numbers close to 1 in absolute value. For this class of numbers, Bombieri—Mueller [2] have recently given "good" effective improvements on Liouville's Theorem, better than those of Baker—Feldman.

For the examples in (ii) we also have

$$y^q = \frac{f(x)^{\frac{q-1}{d}} y + 0}{0 \cdot y + 1},$$

so, in the notation of Theorem 1, $A = m(q-1)/d$, $B = 0$, $\alpha(y) = \dfrac{nd}{m}$. So Theorem 1 reads

$$\operatorname{ord}(y-r) \le \frac{nd}{m} H(r) + n \quad \forall\, r \in k(x),\ H(r) > m/d. \tag{11}$$

But for $r_N$ we have, as in the proof of Proposition 5 (ii), ord $(y - r_N) = nq^{N+1}$ and

$$H(r_N) = \frac{(q^{N+1} - 1)m}{d},$$

so we have equality in (11). Therefore Theorem 1 is best possible in this case.

Another example is due to Baum and Sweet [1]. Take $P(x) \in k[x]$, $k$ a field of characteristic 2. Let $m = \deg P > 0$ and consider $y$ satisfying

$$P(x)y^3 + x^m y + P(x) = 0.$$

Then, by [1] Corollary 3, $y$ has bounded partial quotients[1] (note the change in notation, our $x$ is their $x^{-1}$), so a bound as in Theorem 1 follows. This illustrates the following result.

THEOREM 7. *If $y \in k[[x]]$, $d(y) > 1$, satisfies (1) (in particular, if $d(y) = 3$), then $y$ has bounded partial quotients if and only if $\alpha(y) = 2$.*

PROOF. The "only if" part is well known and the "if" part follows from Theorem 1.

The content of Theorem 7 is that if a "Roth" type theorem holds for $y$, i.e.

$$(\forall \varepsilon > 0)[\text{ord}\,(y - r) \le (2 + \varepsilon)H(r) + O_\varepsilon(1)],$$

then it follows that this last equation holds for $\varepsilon = 0$ and also with an effective $O(1)$.

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[1] Bounded as polynomials in $x^{-1}$.