COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 16 (2) (1996) 223–231

PACKING CIRCUITS IN EULERIAN DIGRAPHS

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Received March 29, 1994
Revised December, 1994

Let G be an eulerian digraph; let $\nu(G)$ be the maximum number of pairwise edge-disjoint directed circuits of G, and $\tau(G)$ the smallest size of a set of edges that meets all directed circuits of G. Borobia, Nutov and Penn showed that $\nu(G)$ need not be equal to $\tau(G)$. We show that $\nu(G)=\tau(G)$ provided that G has a "linkless" embedding in 3-space, or equivalently, if no minor of G can be converted to K_6 by $\Delta-Y$ and $Y-\Delta$ operations.

1. Introduction

Let G be a directed graph (briefly, a *digraph*; all graphs in this paper are finite, and may have loops or multiple edges). Let $\nu(G)$ be the maximum cardinality of a set of mutually edge-disjoint directed circuits of G, and let $\tau(G)$ be the minimum cardinality of a set of edges that meets every directed circuit of G. Clearly $\nu(G)\leq\tau(G)$, and for planar digraphs the Lucchesi–Younger theorem [2, 3] implies that $\nu(G)=\tau(G)$, but equality need not hold in general. For instance, let G_0 be the digraph with vertex set $\{a_1,a_2,a_3,b_1,b_2,b_3\}$ and edge set

$$\{(b_i,a_i) : 1\leq i\leq 3\}\cup\{(a_i,b_j) : i,\ j\in\{1,2,3\},\ i\neq j\};$$

then $\nu(G)=1$ and $\tau(G)=2$. Since the undirected graph underlying G_0 is just $K_{3,3}$, this suggests that abandoning planarity was a mistake if we want $\nu(G)=\tau(G)$.

For eulerian digraphs, the situation seems not so bad. (A digraph is *eulerian* if for every vertex v, its invalency equals its outvalency.) In an eulerian digraph, if $\{C_1,\dots,C_k\}$ is a maximal set of edge-disjoint directed circuits, then every edge belongs to one of C_1, ..., C_k; and this nice fact leads one to hope that perhaps there are deeper nice things to be discovered.

It is not always true that $\nu(G)=\tau(G)$ for eulerian digraphs, however. Borobia, Nutov and Penn [1] gave the following example: let G_0 be as before, and add to it a new vertex c, and six new edges

$$\{(c,a_i) : 1\leq i\leq 3\}\cup\{(b_i,c) : 1\leq i\leq 3\},$$

Mathematics Subject Classification (1991): 05 C 20, 05 C 70, 05 C 38

forming an eulerian digraph G_1 say. Then $\nu(G_1)=4$, but there are nine directed circuits using every edge twice, and so $\tau(G_1)>4$ (in fact $\tau(G_1)=5$).

The undirected graph underlying G_1 belongs to the "Petersen family", the seven graphs that can be obtained from K_6 by $\Delta - Y$ and $Y - \Delta$ exchanges; see Figure 1.

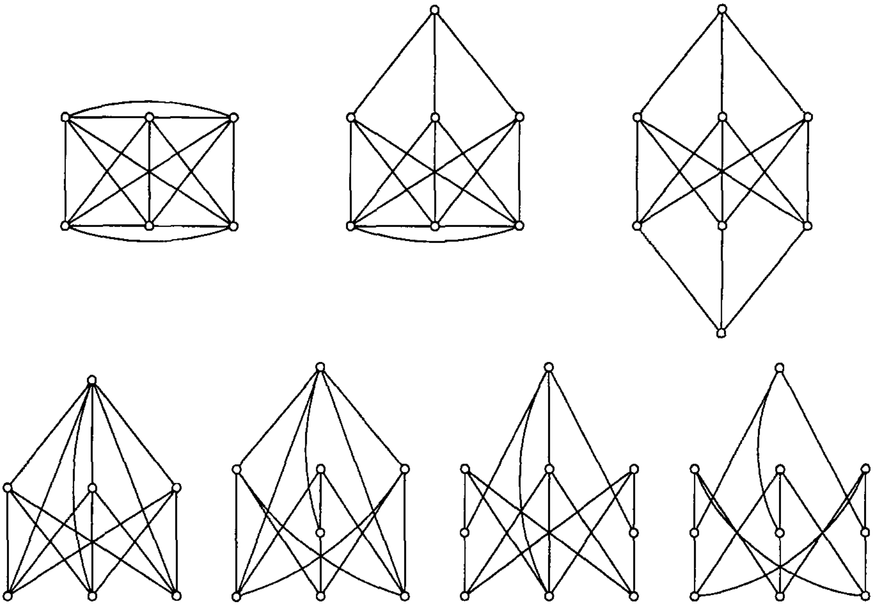

Fig. 1. The Petersen family

We say H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. In [4] we showed that every graph with no minor in the Petersen family can be embedded in 3-space so that every circuit bounds a disc disjoint from the remainder of the graph; and the example G_1 suggests that perhaps $\nu(G)=\tau(G)$ for every eulerian digraph that admits such an embedding. This turns out to be true, as we shall see. Consequently, we have

(1.1) *Let G be an eulerian digraph, such that its underlying undirected graph has no minor in the Petersen family. Then $\nu(G)=\tau(G)$.*

The proof of (1.1) is perhaps of more interest than the theorem. It seems remarkable that, for digraphs embedded in 3-space as above, one can define a notion of "uncrossing" directed circuits, analogous to Lovász's method [2] of uncrossing directed cuts to prove the Lucchesi–Younger theorem.

2. Panelled frames

We denote the unit 3-sphere by S^3. (Throughout, we assume that all embeddings and all subspaces considered are tame.) For $X \subseteq S^3$, its topological closure is denoted by $\overline{X}$. A *line* in S^3 is a subset homeomorphic to the closed unit interval, and its ends are defined in the natural way; a *circle* in S^3 is a subset homeomorphic to the unit circle; and a *disc* in S^3 is homeomorphic to $\{(z,y): x^2+y^2 \leq 1\}$. A *frame* is a pair (U,V), where

(i) $U \subseteq S^3$ is closed, and $V \subseteq U$ is finite
(ii) $U-V$ has only finitely many arc-wise connected components, called *edges*, and
(iii) for each edge e, either $\overline{e}$ is a circle with $|\overline{e} \cap V| = 1$, or $\overline{e}$ is a line and $\overline{e} \cap V$ consists of its ends.

We call V the set of *vertices* of the frame. Thus, a frame is a graph embedded in S^3 in the natural way. If Γ is a frame, we write $U(\Gamma) = U$ and $V(\Gamma) = V$, and denote its set of edges by $E(\Gamma)$.

Let Γ be a frame, and let C be a circuit of Γ. (Circuits in this paper have no "repeated" vertices or edges.) A *panel* for C is a disc $\Delta \subseteq S^3$ such that $\Delta \cap U(\Gamma) = \mathrm{bd}(\Delta) = U(C)$. A frame Γ is *panelled* if there is a panel for every circuit of Γ. Now we can state the theorem of [4] more precisely:

(2.1) *For any graph G, there is a panelled frame isomorphic to G if and only if G has no minor in the Petersen family.*

Therefore, to prove (1.1) it suffices to show that $\nu(\Gamma) = \tau(\Gamma)$ for every eulerian directed panelled frame.

Let G be a digraph. A *roll* in G is a family $(C_i : i \in I)$ of directed circuits of G, and its *length* is $\sum(|E(C_i)| : i \in I)$. A roll $(C_i : i \in I)$ *dominates* a roll $(D_j : j \in J)$ if for every edge e of G,

$$|\{i \in I : e \in E(C_i)\}| \geq \left|\{j \in J : e \in E(D_j)\}\right|.$$

(This implies that the first roll has length at least that of the second.)

Let us say two directed circuits C_1, C_2 of a digraph G are *parallel* if the common vertices of C_1 and C_2 appear in the same cyclic order in C_1 and in C_2. We need the following lemma.

(2.2) *Let C_1, C_2 be directed circuits of a digraph G. Suppose that the roll $\{C_1, C_2\}$ dominates no roll of cardinality 2 of strictly smaller length. Then C_1, C_2 are parallel.*

Proof. Suppose that C_1, C_2 are not parallel. Then there are distinct vertices x, y, $z \in V(C_1 \cap C_2)$ which occur in the order x, y, z in C_1 and in the order x, z, y in C_2. Let C_z be a directed circuit in the union of the path of C_1 from x to y and the path of C_2 from y to x; and define C_y, C_x similarly. Then $\{C_1, C_2\}$ dominates

$\{C_x,C_y,C_z\}$, and hence $\{C_x,C_y\}$, and since $E(C_z)\neq\emptyset$ it follows that $\{C_x,C_y\}$ has length strictly less than that of $\{C_1,C_2\}$, a contradiction. The result follows. ∎

Let Γ be a directed panelled frame. If $\Delta_1, \ldots, \Delta_k$ are panels for circuits of Γ, we say they are mutually *laminar* if $\Delta_i\cap\Delta_j=\mathrm{bd}(\Delta_i)\cap\mathrm{bd}(\Delta_j)$ for $1\leq i<j\leq k$.

The main result of this section is the following.

(2.3) *Let Γ be a directed panelled frame, and let $(C_i : i\in I)$ be a roll in Γ. Then there is a roll $(C'_i : i\in I)$ dominated by $(C_i : i\in I)$, and for each $i\in I$ a panel Δ_i for C'_i so that $(\Delta_i : i\in I)$ are mutually laminar.*

Proof. We proceed by induction on the length of $(C_i : i\in I)$. We may assume that $I=\{1,\ldots,k\}$ where $k\geq 1$. From the inductive hypothesis applied to $(C_i : 2\leq i\leq k)$, the latter dominates a roll $(C'_i : 2\leq i\leq k)$ such that for $2\leq i\leq k$ there is a panel Δ_i for C'_i with $\Delta_2, \ldots, \Delta_k$ mutually laminar. Thus $(C_1,C_2,\ldots,C_k)$ dominates $(C_1,C'_2,\ldots,C'_k)$, and therefore if the conclusion of the theorem holds for $(C_1,C'_2,\ldots,C'_k)$ then it holds for $(C_1,C_2,\ldots,C_k)$. Consequently, by replacing $(C_1,C_2,\ldots,C_k)$ by $(C_1,C'_2,\ldots,C'_k)$, we may assume that for $2\leq i\leq k$ there is a panel Δ_i for C_i such that $\Delta_2, \ldots, \Delta_k$ are mutually laminar. Moreover, from (2.2) and the inductive hypothesis, we may assume that C_1, C_i are parallel for $2\leq i\leq k$.

Choose a panel Δ_1 for C_1, so that for $2\leq i\leq k$ C_1 and C_i are in "general position". This means that for every point x of $(\Delta_1-\mathrm{bd}(\Delta_1))\cap(\Delta_i-\mathrm{bd}(\Delta_i))$ there is a ball $B\subseteq S^3$ with x in its interior, so that $(B,B\cap\Delta_1,B\cap\Delta_i,x)$ is homeomorphic to

$$(\{(x,y,z) : x^2+y^2+z^2\leq 1\}, \{(x,0,z) : x^2+z^2\leq 1\}, \{(0,y,z) : y^2+z^2\leq 1\}, (0,0,0)).$$

For such a set $\{\Delta_1,\ldots,\Delta_k\}$, we define $\mathscr{L}(\Delta_1,\ldots,\Delta_k)$ to be the set of all arc-wise connected components of

$$(\Delta_1-\mathrm{bd}(\Delta_1))\cap\bigcup_{2\leq i\leq k}(\Delta_i-\mathrm{bd}(\Delta_i)).$$

Then (compare [4, Theorem (2.2)]), since the sets $\Delta_i-\mathrm{bd}(\Delta_i)$ $(2\leq i\leq k)$ are mutually disjoint, we have

(1) *$\mathscr{L}(\Delta_1,\ldots,\Delta_k)$ is finite, and for each $L\in\mathscr{L}(\Delta_1,\ldots,\Delta_k)$, either*

(i) *$\overline{L}$ is a line with ends u, $v\in\mathrm{bd}(\Delta_1)$, and $L=\overline{L}-\{u,v\}$, or*

(ii) *L is a circle with one point v in $\mathrm{bd}(\Delta_1)$, and $L=\overline{L}-\{v\}$, or*

(iii) *$L=\overline{L}$ is a circle disjoint from $\mathrm{bd}(\Delta_1)$.*

We denote the cardinality of $\mathscr{L}(\Delta_1,\ldots,\Delta_k)$ by $\lambda(\Delta_1,\ldots,\Delta_k)$. If this is zero, then $\Delta_1, \ldots, \Delta_k$ are mutually laminar and the proof is complete. We thus may assume that $\lambda(\Delta_1,\ldots,\Delta_k)>0$, and we proceed by induction on $\lambda(\Delta_1,\ldots,\Delta_k)$.

If there exists $L \in \mathcal{L}(\Delta_1,\ldots,\Delta_k)$ satisfying (1)(ii) or (1)(iii), then as in the proof of [4, Theorem (2.3)], we may replace Δ_i for some $i>1$, say Δ_2, with a new panel Δ_2', so that Δ_2', Δ_3, ..., Δ_k are mutually laminar, Δ_1, Δ_2 are in general position, and

$$\lambda(\Delta_1,\Delta_2',\Delta_3,\ldots,\Delta_k) < \lambda(\Delta_1,\Delta_2,\ldots,\Delta_k),$$

and the result follows from the second inductive hypothesis. We may therefore assume that every $L\in\mathcal{L}(\Delta_1,\ldots,\Delta_k)$ satisfies (1)(i).

For each $L \in \mathcal{L}(\Delta_1,\ldots,\Delta_k)$ there are therefore two discs $D \subseteq \Delta_1$ such that $\mathrm{bd}(D) - \mathrm{bd}(\Delta_1) = L$. Since $\mathcal{L}(\Delta_1,\ldots,\Delta_k) \neq \emptyset$, we may choose $L_1 \in \mathcal{L}(\Delta_1,\ldots,\Delta_k)$ and one of its two discs D_1, so that D_1 is minimal. Let L_1 have ends u, v. Since L_1 is arc-wise connected, there exists i with $2\le i\le k$ such that $L_1 \subseteq \Delta_i - \mathrm{bd}(\Delta_i)$, say $i=2$. Thus $\overline{L}_1 \subseteq \Delta_2$. For $i=1$, 2, let the directed path of C_i from u to v be P_i, and let the directed path of C_i from v to u be Q_i. Since C_1, C_2 are parallel, it follows that

$$V(P_1\cap Q_2) = V(P_2\cap Q_1) = \{u,v\}$$

and hence $P_2\cup Q_1 = C_1'$ and $P_1\cup Q_2 = C_2'$ are both directed circuits of Γ. The roll $(C_1',C_2',C_3,\ldots,C_k)$ is dominated by $(C_1,\ldots,C_k)$, and therefore the result will follow from the second inductive hypothesis if we can find appropriate panels for this roll with λ reduced. That is therefore the objective of the remainder of the proof.

From the minimality of D_1, and since every $L\in\mathcal{L}(\Delta_1,\ldots,\Delta_k)$ satisfies (1)(i), it follows that $D_1\cap(\Delta_2\cup\ldots\cup\Delta_k)\subseteq \mathrm{bd}(D_1)$. Let Δ_2' be the union of D_1 and the disc in Δ_2 bounded by $L\cup U(Q_2)$; then Δ_2' is a panel for C_2'. Moreover, Δ_2', Δ_3, ..., Δ_k are mutually laminar. (Note that L_1 is disjoint from Δ_3, ..., Δ_k since $L_1\subseteq \Delta_2 - \mathrm{bd}(\Delta_2)$.)

Among all $L\in\mathcal{L}(\Delta_1,\Delta_2,\ldots,\Delta_k)$ with ends u, v such that $L\subseteq \Delta_2 - \mathrm{bd}(\Delta_2)$, choose one, L_2 say, so that the disc D_2 in Δ_2 bounded by $L_2\cup U(P_2)$ is minimal. (Possibly $L_2 = L_1$.)

We claim that $D_2\cap\Delta_1\subseteq \mathrm{bd}(D_2)$. For suppose not; then there is a component L of $(\Delta_1 - \mathrm{bd}(\Delta_1))\cap(\Delta_2 - \mathrm{bd}(\Delta_2))$ with $L\cap D_2 \not\subseteq \mathrm{bd}(D_2)$. Now $L\neq L_2$, and so $L\cap L_2=\emptyset$, and hence $L\subseteq D_2$. From the choice of L_2, L has an end $w\neq u$, v; and hence $w\in V(P_2)-\{u,v\}$. Since C_1, C_2 are parallel it follows that $w\in V(P_1)-\{u,v\}$, and hence $L\cap D_1\neq\emptyset$, contradicting that $D_1\cap\Delta_2\subseteq \mathrm{bd}(D_1)$. This proves that $D_2\cap\Delta_1\subseteq \mathrm{bd}(D_2)$. Let Δ_1' be the union of D_2 and the disc in Δ_1 bounded by $L_2\cup U(Q_1)$. Then Δ_1' is a panel for C_1'. Moreover, every component of

$$(\Delta_1 - \mathrm{bd}(D_1'))\cap((\Delta_2' - \mathrm{bd}(\Delta_2'))\cup\bigcup(\Delta_i - \mathrm{bd}(\Delta_i) : 3\le i\le k))$$

is a member of $\mathcal{L}(\Delta_1,\ldots,\Delta_k)$, and for $3\le i\le k$ Δ_1' and Δ_i are in general position. If $L_2\neq L_1$, then Δ_1' and Δ_2' are in general position, and since $L\notin\mathcal{L}(\Delta_1',\Delta_2',\Delta_3,\ldots,\Delta_k)$ and hence

$$\lambda(\Delta_1',\Delta_2',\Delta_3,\ldots,\Delta_k < \lambda(\Delta_1,\ldots,\Delta_k),$$

the result follows from the second inductive hypothesis. We may assume therefore that $L_2 = L_1$.

Thus Δ_1, Δ_2 touch "tangentially" along L_1; and by moving them slightly apart in a neighbourhood of L_1, we obtain a set of panels in general position, with λ reduced, and the result follows. ∎

3. The main proof

We need one more lemma about frames, the following; its proof is an elementary homotopy argument that we omit.

(3.1) *Let Γ be a frame in S^3, and let $\Delta_1, \ldots, \Delta_k$ be mutually laminar panels, so that every edge of Γ belongs to the boundary of exactly two of $\Delta_1, \ldots, \Delta_k$. Then the set of components of $S^3 - \Delta_1 \cup \ldots \cup \Delta_k$ can be partitioned into two sets X_1, X_2 so that for $1 \le i \le k$, Δ_i belongs to the closure of a member of X_1 and to the closure of a member of X_2.*

We use (3.1) and (2.3) to prove the following.

(3.2) *Let Γ be a directed, eulerian panelled frame. Let $k \ge 0$ be an integer, and let there be k directed circuits of Γ so that every edge of Γ is in at most two of them. Then $\nu(\Gamma) \ge k/2$.*

Proof. We may assume that k is maximum with the given property. From (2.3), there are k directed circuits $C_1, \ldots, C_k$ of Γ, so that every edge is in ≤ 2 of them, and for $1 \le i \le k$ there is a panel Δ_i for C_i so that $\Delta_1, \ldots, \Delta_k$ are mutually laminar. Since Γ is eulerian, it follows from the maximality of k that every edge of Γ belongs to exactly two of $C_1, \ldots, C_k$. Choose a partition X_1, X_2 as in (3.1). Fix an orientation σ of S^3. For $j = 1, 2$, let $\mathscr{C}_j$ be the set of all C_i $(1 \le i \le k)$ such that, if we orient Δ_i in the sense of the direction of C_i then the component of $S^3 - (\Delta_1 \cup \ldots \cup \Delta_k)$ on the positive side of Δ_i (defined by the product of σ and the orientation of Δ_i) belongs to X_j. Thus, $\mathscr{C}_1 \cup \mathscr{C}_2 = \{C_1, \ldots, C_k\}$, and $\mathscr{C}_1 \cap \mathscr{C}_2 = \emptyset$. Moreover, let $e \in E(\Gamma)$, and let $e \in E(C_1)$, $E(C_2)$ say. Since e belongs to the closure of only two components of $S^3 - (\Delta_1 \cup \ldots \cup \Delta_k)$, and exactly one of them is in X_1, it follows that one of C_1, C_2 belongs to $\mathscr{C}_1$ and the other to $\mathscr{C}_2$. Consequently, for $j = 1, 2$ the members of $\mathscr{C}_j$ are mutually edge-disjoint. Hence $\nu(\Gamma) \ge |C_j|$, and so $2\nu(\Gamma) \ge |\mathscr{C}_1| + |\mathscr{C}_2| = k$, as required. ∎

The sets of integers, real numbers, non-negative integers, and non-negative real numbers are denoted by $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}_+$, $\mathbb{R}_+$ respectively. Let G be a digraph. For any function $w : E(G) \to \mathbb{Z}_+$, we define $\nu(G, w)$ to be the maximum k such that there is a family $(C_1, \ldots, C_k)$ of directed circuits of G (not necessarily all distinct) so that

$$|\{i : 1 \le i \le k,\ e \in E(C_i)\}| \le w(e)$$

for every edge e of G. We say that $w: E(G) \to \mathbb{Z}$ is a *circulation* if for every vertex v of G,

$$\sum(w(e) : e \in \delta^+(v)) = \sum(w(e) : e \in \delta^-(v)),$$

where $\delta^+(v)$, $\delta^+(v)$ are the sets of edges of G with tail v and head v respectively. A digraph is *flat* if it is isomorphic to a directed panelled frame. From (3.2) we have:

(3.3) *Let G be a flat digraph, and let $w : E(G) \to \mathbb{Z}_+$ be a circulation. Then $\nu(G, 2w) = 2\nu(G, w)$.*

Proof. For each edge e, replace e by $w(e)$ parallel edges, forming a digraph H. Now H is flat, by [4, Theorem (5.1)). From (3.2) applied to H, we deduce that $\nu(G, w) \geq \nu(G, 2w)/2$. Since the reverse inequality is trivial, the result follows. ∎

It is known that for any w, the following all exist and are equal

$$\sup_{k>0} k^{-1}\nu(G, kw), \ \lim_{k\to\infty} k^{-1}\nu(G, kw), \ \lim_{k\to\infty} 2^{-k}\nu(G, 2^k w).$$

We denote their common value by $\nu^*(G, w)$. From (3.3) we have

(3.4) *Let G be a flat digraph, and let $w : E(G) \to \mathbb{Z}_+$ be a circulation. Then $\nu(G, w) = \nu^*(G, w)$.*

Proof. From (3.3), it follows by induction on k that for all $k \geq 0$,

$$2^{-k}\nu(G, 2^k w) = \nu(G, w).$$

But $\nu^*(G, w)$ is the limit of the left side, and the result follows. ∎

Henceforth it is sometimes convenient to use vector notation for functions $w : E(G) \to \mathbb{R}$. All our vectors will belong to $\mathbb{R}^{E(G)}$. A *potential* is a vector p satisfying $w^T p = 0$ for every circulation w. We need the following lemma.

(3.5) *Let G be a digraph, and let $x \in \mathbb{R}_+^{E(G)}$ such that $w^T x$ is integral for every $(0, \pm 1)$-valued circulation w. Then there is a potential p such that $x + p \in \mathbb{Z}_+^{E(G)}$.*

Proof. Define $N(x) = \{e \in E(G) : x(e) \notin \mathbb{Z}\}$; we proceed by induction on $|N(x)|$. If $N(x) = \emptyset$ then the result holds with $p = 0$; and so we may assume that there exists $f \in N(x)$. Let f have head u and tail v.

Suppose first that there is a circuit C of the undirected graph underlying G, with $E(C) \cap N(x) = \{f\}$. Let w be a circulation in G such that $w(e) = \pm 1$ for all edges e in C, and otherwise $w(e) = 0$. By hypothesis, $w^T x$ is integral; but $w(e)x(e)$ is integral for every $e \in E(G) - \{f\}$, and so $w(f)x(f)$ is integral, a contradiction. Thus there is no such C.

Consequently there is a partition X, Y of $V(G)$ such that $u \in X$, $v \in Y$ and every edge with one end in X and the other end in Y belongs to $N(x)$. For $e \in E(G)$

with head a and tail b, define $q(e)=1$ if $a\in Y$ and $b\in X$, $q(e)=-1$ if $a\in X$ and $b\in Y$, and $q(e)=0$ otherwise. Then q is a potential. Choose $\varepsilon\geq 0$ maximal such that $x(e)+\varepsilon q(e)\geq 0$ for every edge e. (This is possible since $x(e)\geq 0$ for every e, and $q(f)=-1$.) Define $x'=x+\varepsilon q$. Now $N(x')\subseteq N(x)$, for if $e\in E(G)-N(x)$ then $q(e)=0$. Moreover, $N(x')\neq N(x)$ by the maximality of ε. Now for every $(0,\pm 1)$-valued circulation w,

$$w^T x' = w^T x + \varepsilon w^T q = w^T x$$

since q is a potential, and so $w^T x'$ is integral. From the inductive hypothesis there is a potential p' so that $x'+p'\in\mathbb{Z}_+^{E(G)}$. Let $p=\varepsilon q+p'$; then p is a potential, and $x+p\in\mathbb{Z}_+^{E(G)}$, as required. ∎

Let G be a digraph and $w:E(G)\to\mathbb{Z}_+$ a function. We define $\tau(G,w)$ to be the minimum of $\sum_{e\in X} w(e)$, taken over all $X\subseteq E(G)$ such that $X\cap E(C)\neq\emptyset$ for every directed circuit C. Finally, we deduce our main result.

(3.6) *Let G be a flat digraph. Then $\nu(G,w)=\tau(G,w)$ for every circulation $w: E(G)\to\mathbb{Z}_+$.*

Proof. Let P be the set of all $x\in\mathbb{R}_+^{E(G)}$ such that

$$\sum(x(e): e\in E(C))\geq 1$$

for every directed circuit C of G. For any $w:E(G)\to\mathbb{Z}_+$ let $P(w)$ denote the set of all $x\in P$ with $w^T x$ minimum. If w is a circulation, we call $P(w)$ a *circular face* of P.

(1) *For any circulation $w: E(G)\to\mathbb{Z}_+$, $\nu(G,w)=w^T x$ for all $x\in P(x)$. In particular, $w^T x$ is integral.*

For $w^T x = \nu^*(G,w)$ from the linear programming duality theorem, via a standard argument; and $\nu^*(G,w)=\nu(G,w)$ from (3.4)

The main step in the proof is the following.

(2) *Every circular face contains an integer point.*

We prove (2) by induction on $|E(G)|$. It suffices to show that every minimal circular face contains an integer point; so let $w:E(G)\to\mathbb{Z}_+$ be a circulation so that $P(w)$ is a minimal circular face.

Suppose first that $w(f)=0$ for some edge f of G. Let G' be obtained from G by deleting f, let P' be the polyhedron in $\mathbb{R}^{E(G')}$ corresponding to P, and let w' be the restriction of w to $E(G')$. Then G' is flat and w' is a circulation in G', so from the inductive hypothesis, there is an integer point $x'\in P'(w')$. Define $x\in\mathbb{Z}_+^{E(G)}$ by $x(e)=x'(e)$ $(e\in E(G'))$ and $x(f)=1$. Then $x\in P$, and we claim that $x\in P(w)$.

For let $y \in P(w)$, and let y' be the restriction of y to $E(G')$. Then $y' \in P'$, and so $w'^T y' \geq w'^T x'$ from the choice of x'. But $w^T x = w'^T y'$ and $w^T x = w'^T x'$ since $w(f)=0$, and so $w^T y \geq w^T x$. Hence $x \in P(w)$, since $x \in P$ and $y \in P(w)$. Since x is integral, it follows that $P(w)$ contains an integer point, as required.

We may therefore assume that $w(f)>0$ for every $f \in E(G)$. Consequently, and since P is a polyhedron, there exists an integer $n>0$ such that for every $(0,\pm 1)$-valued circulation w' in G, $w+w'/n \geq 0$ and $P(w+w') \subseteq P(w)$. Since $P(nw)=P(w)$ and $P(nw+w')=P(w+w'/n)$, we may assume (by replacing w by nw) that $n=1$.

Now let $x \in P(w)$, and let w' be a $(0,\pm 1)$-valued circulation in G. By (1), $w^T x$ is integral, and since $x \in P(w)=P(w+w')$ (because $P(w+w') \subseteq P(w)$ and $P(w)$ is a minimal circular face) it follows from (1) that $(w+w')^T x$ is integral. Subtracting, we deduce that $w'^T x$ is integral, for every $(0,\pm 1)$-valued circulation w'. By (3.5) there is a potential p so that $x+p \in \mathbb{Z}_+^{E(G)}$. Since p is a potential it follows that for every directed circuit C of G, $\sum(p(e): e \in E(C))=0$ and so $x+p \in P$; and also since p is a potential it follows that $w^T p=0$, and so $x+p \in P(w)$. Hence $P(w)$ contains an integer point. This proves (2).

Now to prove the theorem, let $w: E(G) \to \mathbb{Z}_+$ be a circulation. By (2), we may choose an integer point $x \in P(w)$. Then $\tau(G,w) \leq w^T x$ since x is integral, and $w^T z = \nu(G,w)$ by (1). Since trivially $\nu(G,w) \leq \tau(G,w)$, the theorem follows. ∎

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