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On a method of Pi-Calleja for describing additive generators of associative functions

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Summary. We review and extend a method introduced by Pi-Calleja in 1954 for describing additive generators of some associative functions on closed intervals.

1. Introduction

In 1954 Professor Pedro Pi-Calleja published ([10]) an interesting paper on functional equations and the theory of magnitudes in which several previous results (e.g. [6]) were studied, corrected and extended. In doing this, an original idea for describing additive generators of associative functions was introduced. The paper was included in the volume of the "Segundo Symposium de Matemáticas" held in Mendoza, República Argentina, but his result remained unnoticed in the literature dealing with the associativity equation on closed intervals. The main aim of this paper is to present Pi-Calleja's method.

2. Historical background

The main goal in the study of the associativity equation

$$T(x, T(y, z)) = T(T(x, y), z), \tag{1}$$

is to find a representation theorem for T of the form

$$T(x, y) = \phi(\phi^{-1}(x) + \phi^{-1}(y)). \tag{2}$$

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The story of and references to this subject may be found in ([2], [5], [12]). A crucial point is the fundamental result published by J. Aczél in 1949 where he obtains (2) when T satisfies (1) on a proper real interval I and is continuous and strictly increasing (or cancellative). In ([1]) the additive generator ϕ appearing in (2) is defined on a real interval J with values in I, starting with the values of ϕ on rational numbers m/n by means of the formula $\phi(m/n) = (\sqrt[n]{c})^m$, where c is a non-idempotent element; one then defines the function $x^m = T(x, .\,^m.\,., x)$ and the function $\sqrt[n]{x}$ as the inverse function of x^n; and lastly, ϕ is extended by continuity to the real interval J (see [4]). Recently, R. W. Craigen and Zs. Páles ([7]) have given an alternative approach by representing T in the form $T(x, y) = F^{-1}(F(x) + F(y))$ where F from I into a real interval J is given by

$$F(x) = \inf\left\{\frac{m-n}{k}\,\middle|\, m, n, k \in N \text{ and } c^m > T(x^k, c^n)\right\}.$$

This fundamental idea of J. Aczél of finding the additive generators of associative functions T by playing with iterates of the diagonal function $T(x, \dots, x)$ and its inverse (and applying suitable extensions to the reals) has become the keystone of the subject. Thus, while in the study of (1) on closed intervals with an endpoint as a unit ([8], [9]) the representation (2) has been obtained under different assumptions than those quoted above, the idea of J. Aczél for describing the additive generators has always been a basic tool (see [12]).

The idea of Pi-Calleja adapted to the case of t-norms T on $[0, 1]$ (his initial case was $[0, +\infty]$) is the following: If one has the representation $T(x, y) = f^{-1}(f(x) + f(y))$ with f: $[0, 1] \to [0, +\infty]$ strictly decreasing, continuous and $f(1) = 0, f(0) = +\infty$, then for any x, y in $[0, 1)$ there always exists a unique natural number $[x \mid y]$ such that $y^{[x|y]+1} < x \leqslant y^{[x|y]}$ and consequently:

$$f^{-1}(([x \mid y] + 1)f(y)) < x \leqslant f^{-1}([x \mid y]f(y)),$$

whence

$$[x \mid y] + 1 > \frac{f(x)}{f(y)} \geqslant [x \mid y],$$

i.e., denoting the integer part of any real z by $[z]_+$, we have

$$[x \mid y] = \left[\frac{f(x)}{f(y)}\right]_+;$$

and from this one shows that if $f(c) = 1$, then

$$\lim_{y \to 1-} \frac{[x \mid y]}{[c \mid y]} = \lim_{y \to 1-} \left[\frac{f(x)}{f(y)}\right]_+ \Big/ \left[\frac{f(c)}{f(y)}\right]_+ = \lim_{t \to \infty} \frac{[tf(x)]_+}{[t]_+} = f(x),$$

i.e., it is possible to describe $f(x)$ directly as a limit of rational numbers (related, of course, to T). As one can see, this is somewhat similar to the definition of F above (cf. [7]).

The aim of the paper is to present the representation theorem for strict t-norms following this approach, but simplifying some of the arguments and assumptions made by Pi-Calleja in [10] (e.g. we will not need full continuity or commutativity).

3. Representation theorem following Pi-Calleja's method

Let T be an associative binary operation on $[0, 1]$ such that T is strictly increasing on $(0, 1]^2$, 1 is a unit and $T(x, x)$ is continuous. For any x in $(0, 1)$ we define $x^0 = 1$, $x^1 = x$, $x^n = T(x, x^{n-1})$, i.e., $x^n = T(x, \overset{n}{\ldots}, x) > 0$, which makes sense by virtue of the associativity. It is immediate that, for any $m, n \geqslant 0$ and x in $[0, 1]$,

$$(x^n)^m = x^{n \cdot m} \quad \text{and} \quad T(x^n, x^m) = x^{n+m}, \tag{3}$$

and that, for any x in $(0, 1)$, the sequence $\{x^n\}$ is strictly decreasing with $\lim_{n \to \infty} x^n = 0$, since $T(x, x) < T(x, 1) = x$ and $T(x, x)$ is continuous. Thus, taking any y in $(0, 1)$, we induce a partition of $(0, 1] = \bigcup_{n=0}^{\infty} (y^{n+1}, y^n]$. Under these conditions we define $[\,|\,]_T$: $(0, 1] \times (0, 1) \to N$ to be the function which assigns to each pair (x, y) the unique natural number $[x \mid y]_T$ such that

$$y^{[x|y]_T + 1} < x \leqslant y^{[x|y]_T}. \tag{4}$$

When there is no danger of misunderstanding, we will write $[x \mid y]$ instead of $[x \mid y]_T$. The next lemma lists some basic properties of the mapping $[\,|\,]$.

LEMMA. *Under the above assumptions on T the mapping* $[\,|\,]$ *satisfies the following conditions for all* x, x', y, z, z' in $(0, 1)$:

(*i*) $y^m < x$ *for some* m *in* N *implies* $m \geqslant [x \mid y] + 1$;
(*ii*) $x \leqslant y^n$ *for some* n *in* N *implies* $n \leqslant [x \mid y]$;
(*iii*) $[x \mid x] = 1$ *and* $[x \mid z] = 0$ *if* $z < x$;
(*iv*) $[x \mid z] \geqslant [x' \mid z]$ *whenever* $x \leqslant x'$; $[x \mid z] \leqslant [x \mid z']$ *whenever* $z \leqslant z'$;
(*v*) $\lim_{z \to 1-} [x \mid z] = +\infty$ *for any* x *in* $(0, 1)$;
(*vi*) $[x \mid z] + [y \mid z] \leqslant [T(x, y) \mid z] \leqslant [x \mid z] + [y \mid z] + 1$;
(*vii*) $[y \mid z] \cdot [x \mid y] \leqslant [x \mid z] \leqslant ([x \mid y] + 1)([y \mid z] + 1)$.

Proof. Conditions (i), (ii), (iii) and (iv) are obvious consequences of the definition of $[\ |\ ]$. To prove (v), suppose there is an $M > 0$ in $\mathbb{N}$ such that $[x \mid z] \leqslant M$ for all $z \in (0, 1)$. Then

$$T(z^M, z^M) < T(z, z^M) \leqslant T(z, z^{[x|z]}) = z^{[x|z]+1} < x \leqslant z^{[x|z]} \leqslant 1. \tag{5}$$

Thus, letting $z \to 1^-$ and using the continuity of T on the diagonal, we get $x = 1$ which is a contradiction.

Next, since T is strictly increasing, using (3) we have

$$z^{[x|z]+[y|z]+2} = T(z^{[x|z]+1}, z^{[y|z]+1}) < T(x, y) \leqslant T(z^{[x|z]}, z^{[y|z]}) = z^{[x|z]+[y|z]}$$

whence by (i) and (ii) we obtain (vi). Finally we have

$$z^{([x|y]+1)([y|z]+1)} = (z^{[y|z]+1})^{([x|y]+1)} < y^{[x|y]+1} < x \leqslant y^{[x|y]} \leqslant (z^{[y|z]})^{[x|y]} = z^{[y|z]\cdot[x|y]}$$

whence by (ii), $[y \mid z] \cdot [x \mid y] \leqslant [x \mid z]$ and by (i), $[x \mid z] + 1 \leqslant ([x \mid y] + 1)([y \mid z] + 1)$ from which (vii) follows.

THEOREM. *Let T be a binary operation on $[0, 1]$ which is associative, strictly increasing and $(0, 1]^2$ and has 1 as a unit. Suppose further that $T(x, x)$ is continuous and that there exists an α in $(0, 1)$ such that T is continuous on $[0, 1] \times [\alpha, 1]$. Then there exists a function f_c: $(0, 1] \to R^+$ defined by*

$$f_c(x) = \lim_{z \to 1^-} \frac{[x \mid z]}{[c \mid z]}, \tag{6}$$

where c is a fixed element in $(0, 1)$, such that f_c is strictly decreasing, continuous on $(0, 1]$, $f_c(1) = 0$, $\lim_{x \to 0^-} f_c(x) = +\infty$, and

$$f_c(T(x, y)) = f_c(x) + f_c(y).$$

Moreover the full continuity of T as well as its commutativity follows.

Proof. Fix c and x in $(0, 1)$ and consider any ϵ in $(0, 1)$. Let $M = \epsilon/8([x \mid c] + 1)$. Since by (v) we have $\lim_{z \to 1^-} [x \mid z] = +\infty$, we can find a z_0 in $(0, 1)$ such that $[x \mid z_0] > 1/M$ and $[c \mid z_0] > 1/M$. Once we have captured such a z_0, it follows from (iv) and (v) that for the given $\epsilon > 0$ there exists a $\delta \in (0, 1)$ such that for all z in $(1 - \delta, 1)$ we have $[z_0 \mid z] > 1/M$. Using (vii) and the above bounds we have the following inequalities for all z in $(1-\delta, 1)$:

$$\frac{[x \mid z]}{[c \mid z]} \geqslant \frac{[x \mid z_0]}{[c \mid z_0]} \cdot \frac{[c \mid z_0]}{[c \mid z_0]+1} \cdot \frac{[z_0 \mid z]}{[z_0 \mid z]+1} > \frac{[x \mid z_0]}{[c \mid z_0]} \cdot \frac{1}{(1+M)^2}$$

and

$$\frac{[x \mid z]}{[c \mid z]} \leqslant \frac{[x \mid z_0]}{[c \mid z_0]} \cdot \frac{[x \mid z_0]+1}{[x \mid z_0]} \cdot \frac{[z_0 \mid z]+1}{[z_0 \mid z]} < \frac{[x \mid z_0]}{[c \mid z_0]}(1+M)^2.$$

In view of these inequalities and (vi), we have, for all z, z' in $(1-\delta, 1)$,

$$\begin{aligned}\left|\frac{[x \mid z]}{[c \mid z]} - \frac{[x \mid z']}{[c \mid z']}\right| &< \frac{[x \mid z_0]}{[c \mid z_0]}\left[(1+M)^2 - \frac{1}{(1+M)^2}\right] \\ &\leqslant \frac{([x \mid c]+1)([c \mid z_0]+1)}{[c \mid z_0]} \cdot \frac{(1+M)^4-1}{(1+M)^2} \\ &< ([x \mid c]+1)(1+M) \cdot \frac{(1+M)^4-1}{(1+M)^2} \\ &< ([x \mid c]+1)8M = \epsilon,\end{aligned}$$

i.e., f_c, as given by (6), is well-defined. Obviously $f_c(1) = 0$.

By reversing the roles of x and c the above argument also shows that $\lim_{z \to 1^-}([c \mid z]/[x \mid z])$ exists, and that consequently we must have $f_c(x) > 0$ for $x > 0$. From (iv) it is clear that f is non-increasing and by (vi), after dividing by $[c \mid z]$, we have

$$\frac{[x \mid z]}{[c \mid z]} + \frac{[y \mid z]}{[c \mid z]} \leqslant \frac{[T(x, y) \mid z]}{[c \mid z]} \leqslant \frac{[x \mid z]}{[c \mid z]} + \frac{[y \mid z]}{[c \mid z]} + \frac{1}{[c \mid z]},$$

whence, using (v) and (6), we obtain at once that

$$f_c(T(x, y)) = f_c(x) + f_c(y), \tag{7}$$

for all x, y in $(0, 1)$. If $x = 1$ or $y = 1$ (7) holds because $f_c(1) = 0$.

Next we will show that f must be strictly decreasing. For any $x < y$, there exists $z > 0$ such that $x < T(y, z_0) < y$ since the continuity of T as $[0, 1] \times [\alpha, 1]$ implies that $\lim_{z \to 1^-} T(y, z) = y$. Since f_c is non-increasing and we know already that (7) holds, we deduce that

$$f_c(x) \geqslant f_c(T(y, z_0)) = f_c(y) + f_c(z_0) > f(y),$$

and since $f_c(z_0) > 0$ (because $z_0 > 0$) we must have $f_c(x) > f_c(y)$.

Finally, we will show that f_c is continuous on $(0, 1]$. If $x_0 \in (0, 1]$ were a discontinuity point of f_c, then the function $f_c(x) + f_c(y)$ would be discontinuous on the set $\{(x_0, y) \mid y \in [0, 1]\}$ and by (7) and the fact that T and f_c are strictly monotonic we would conclude that f_c is discontinuous on the set $\{T(x_0, y) \mid y \in [0, 1]\}$ which has positive measure due to the continuity of T on $[0, 1] \times [\alpha, 1]$. This contradicts the well-known fact that the set of points of discontinuity for a strictly decreasing function has measure zero.

Thus f_c is continuous on $(0, 1]$. It is immediate that $\lim_{x \to 0^-} f_c(x) = +\infty$ (just note that for x in $(0. 1)$, $x^n = T(x, .^n.., x)$ tends to 0 and $f_c(x^n) = nf_c(x)$ tends to infinity). The proof is complete.

REMARK 1. The previous theorem applies to the case of any closed interval $[a, b]$ with one endpoint as a unit element of the operation. Thus instead of $[0, 1]$ we can repeat mutatis mutandis the same arguments in the case $[0, +\infty]$ or $[0, 1]$ with 0 as a unit ("zero"). But note that in the definition of the generating function the limit will be taken when the variable z appearing in it tends to the corresponding unit element.

REMARK 2. Since the additive function of an associative operation representable in the form (7) is unique up to a positive multiplicative constant ([2]), in the definition (6) if we consider another value c', $c \neq c'$ we will obtain $f_{c'}(x) = kf_c(x)$ for some $k > 0$.

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