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On the continuous solutions of the Gołąb–Schinzel equation

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Summary. In this paper theorems of P. Javor and N. Brillouët–J. Dhombres will be completed and a theorem of S. Wołodźko generalized, by describing complex-valued continuous solutions defined on a complex topological vector space of the Gołąb–Schinzel equation

$$f(x + yf(x)) = f(x)f(y). \tag{$*$}$$

The main result reads as follows.

THEOREM. *Assume that X is a linear topological Hausdorff space over the field $\mathbb{K}$ of all real or complex numbers.*

A function $f: X \to \mathbb{K}$ is a continuous solution of this equation $(*)$ if and only if

$$f = \phi \circ x^*,$$

where $\phi: \mathbb{K} \to \mathbb{K}$ is a continuous solution of $(*)$ and x^* is a continuous linear functional on the space X.

1. The aim of this paper is to complete the theorems of P. Javor [6; Theorem 2] N. Brillouët–J. Dhombres [3; Proposition 3] by describing complex-valued continuous solutions, defined on a complex topological vector space, of the Gołąb–Schinzel equation

$$f(x + yf(x)) = f(x)f(y). \tag{1}$$

Complex-valued continuous solutions of this equation have been considered previously in two papers by S. Wołodźko: in paper [9] for functions defined on the complex plane and in paper [10] for functions defined on a complex normed

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space. In paper [10] the author notes also that his result on continuous complex-valued solutions of the Gołąb–Schinzel equation [9; Théorème 7] does not establish all such solutions and he quotes a theorem [10; Twierdzenie 1] which already describes all of them. However, the paper, to which the reader is referred there, has probably never appeared. All complex continuous solutions $f: \mathbb{C} \to \mathbb{C}$ have been determined by P. Plaumann and K. Strambach [7; Satz 1, Bemerkung 2, Hilfssatz 2, Satz 2]. We prove here S. Wołodźko's theorem the way he had suggested it: from the theorem on the general solutions $f: \mathbb{C} \to \mathbb{C}$ of (1).

2. Assume that X is a linear topological Hausdorff space over the field $\mathbb{K}$ of all real or complex numbers.

THEOREM. *A function $f: X \to \mathbb{K}$ is a continuous solution of equation* (1) *if and only if*

$$f = \phi \circ x^*,$$

where $\phi: \mathbb{K} \to \mathbb{K}$ is a continuous solution of equation (1) *and x^* is a continuous linear functional on the space X.*

Because of the above-mentioned theorem of N. Brillouët and J. Dhombres, only the complex case (i.e. the case where $\mathbb{K} = \mathbb{C}$) requires a proof. We shall start with describing all continuous and complex-valued solutions defined on the complex plane. Regarding continuous and real-valued solutions defined on the real line see [1; Section 2.5.1, Theorem 2].

3. Three tools will be used in getting all continuous solutions $f: \mathbb{C} \to \mathbb{C}$ of equation (1). First of all, like in the first paper of S. Wołodźko, a theorem on the general solution $f: \mathbb{C} \to \mathbb{C}$ will be exploited. But, additionally, also the form of continuous solutions $f: \mathbb{C} \to \mathbb{R}$ of this equation will be used. Finally, the following property of closed subgroups of the group $(\mathbb{C}, +)$ will be very helpful (cf. [2; Chapter VII, §2, Proposition 3]): each such subgroup is either discrete or contains a straight line passing through the origin. At this place I would like to thank Professor Roman Ger for calling my attention to this property of closed subgroups.

As to the general solution of equation (1), they are determined in the papers [5] by P. Javor and [9] by S. Wołodźko for functions acting between a vector space and the (commutative) field over which the space is considered (cf. also [4; Satz 13.3.2, Satz 13.3.3]). In our case, if $f: \mathbb{C} \to \mathbb{C}$ is a non-constant solution of equation (1), then

$$f(x)=\begin{cases}\alpha, & \text{if } x\in N+w(\alpha) \text{ and } \alpha\in M,\\ 0, & \text{if } x\in\mathbb{C}\setminus[N+w(M)],\end{cases} \tag{2}$$

where N is a subgroup of the group $(\mathbb{C}, +)$, M is a subgroup of the group $(\mathbb{C}\setminus\{0\}, \cdot)$,

$$M\cdot N=N \tag{3}$$

and the function $w\colon M\to\mathbb{C}$ fulfills the following two conditions:

$$w(\alpha\beta)-w(\alpha)-\alpha w(\beta)\in N \tag{4}$$

for all $\alpha, \beta\in M$, and, for every $\alpha\in M$,

$$\text{if}\quad w(\alpha)\in N,\qquad \text{then } \alpha=1. \tag{5}$$

In order to get all continuous solutions $f\colon\mathbb{C}\to\mathbb{R}$ of equation (1), we can apply the theorem of Z. Daróczy on real-valued continuous solutions of the Gołąb–Schinzel equation defined on a real Hilbert space (cf. [4; Satz 13.4.3]) or the above-mentioned theorem of N. Brillouët and J. Dhombres. It follows that, if $f\colon\mathbb{C}\to\mathbb{R}$ is a continuous solution of equation (1), then either $f=0$, or

$$f(x)=1+c\operatorname{Re}x+d\operatorname{Im}x,\qquad x\in\mathbb{C}, \tag{6}$$

for some real constants c and d, or

$$f(x)=\begin{cases}1+c\operatorname{Re}x+d\operatorname{Im}x, & \text{if } 1+c\operatorname{Re}x+d\operatorname{Im}x\geqslant 0,\\ 0, & \text{if } 1+c\operatorname{Re}x+d\operatorname{Im}x<0,\end{cases} \tag{7}$$

for some real constants c and d.

Now we are able to prove the following theorem of S. Wołodźko.

PROPOSITION. *A function* $f: \mathbb{C} \to \mathbb{C}$ *is a continuous solution of equation* (1) *iff either* $f = 0$, *or* f *has the form* (6) *or* (7) *with some real constants* c *and* d, *or*

$$f(x) = 1 + cx, \qquad x \in \mathbb{C}, \tag{8}$$

with a complex constant c.

Proof. Let $f: \mathbb{C} \to \mathbb{C}$ be a continuous solution of equation (1) and represent it in form (2), where M and N are subgroups of the groups $(\mathbb{C}\backslash\{0\}, \cdot)$ and $(\mathbb{C}, +)$, respectively, so that equality (3) holds, and $w: M \to \mathbb{C}$ is a function fulfilling conditions (4) and (5) for all $\alpha, \beta \in M$. Of course

$$N = f^{-1}(\{1\}) \tag{9}$$

and

$$M = f(\mathbb{C})\backslash\{0\}. \tag{10}$$

If the function f takes real values only, then either $f = 0$, or there exist real constants c and d such that f has either form (6) or (7). In the sequel we shall assume that f does not take only real values. Hence (cf. also (10))

$$M\backslash\mathbb{R} \neq \varnothing. \tag{11}$$

We shall show that

$$N = \{0\}. \tag{12}$$

Let us consider two cases. If N is not a discrete subgroup then it contains a straight line passing through the origin. This together with properties (3) and (11) implies that $N = \mathbb{C}$ but then, because of equality (9), $f = 1$ which was excluded. Hence the subgroup N is discrete. Suppose that it is nontrivial. Then, in view of equality (3), the subgroup M is discrete, too. Because $f(\mathbb{C})$ is a connected set and equality (10) holds true, this is possible only in the case where f is a constant function. But this is impossible, as constant solutions of equation (1) are real-valued. Property (12) has been proved. Taking it into account we can rewrite the form (2) of the function f as follows

$$f(x) = \begin{cases} \alpha, & \text{if } x = w(\alpha) \text{ and } \alpha \in M, \\ 0, & \text{if } x \in \mathbb{C}\backslash w(M). \end{cases} \tag{13}$$

Moreover (cf. (4) and (5)), the function w is a solution of the equation

$$w(\alpha\beta) - w(\alpha) - \alpha w(\beta) = 0 \tag{14}$$

such that, for every $\alpha \in M$,

$$\text{if } w(\alpha) = 0, \qquad \text{then } \alpha = 1. \tag{15}$$

It is easy to see (cf. also [9; Lemma 13]) that each solution $w: M \to \mathbb{C}$ of equation (14) has the form

$$w(\alpha) = d(1 - \alpha), \qquad \alpha \in M,$$

where d is a complex constant. Of course, in our case (cf. (15) and (11)), $d \neq 0$. Hence, putting $c = -1/d$ and making use of formula (13), we get the following form of our solution:

$$f(x) = \begin{cases} 1 + cx, & \text{if } 1 + cx \in M, \\ 0, & \text{if } 1 + cx \in \mathbb{C} \setminus M. \end{cases} \tag{16}$$

In particular,

$$f^{-1}(\{0\}) = \frac{1}{c}[(\mathbb{C} \setminus M) - 1],$$

which shows that M is an open subset of the space $\mathbb{C}$, and being a subgroup of the group $(\mathbb{C} \setminus \{0\}, \cdot)$ it must be equal to this group:

$$M = \mathbb{C} \setminus \{0\}.$$

This equality together with formula (16) leads to form (8) of the function f and concludes the proof of the Proposition as each of the functions given there is a continuous solution of equation (1).

4. In this part of the paper we shall prove the following analogue of Javor's theorem (which will be used also in the proof of our main result).

LEMMA. *Assume that X is a complex linear topological Hausdorff space. If a continuous function $f: X \to \mathbb{C}$ is a solution of equation* (1) *such that*

$$f(X) \setminus \mathbb{R} \neq 0 \tag{17}$$

then

$$f(x) = F(x + N), \qquad x \in X, \tag{18}$$

where N is a closed linear subspace of the space X of co-dimension one and $F: X/N \to \mathbb{C}$ is a continuous solution of equation (1).

Proof. Let us begin with recalling the following three basic properties of the solution f:

(i) $f(0) = 1$;

(ii) if $x, y \in X$ and $f(x) = f(y) \neq 0$, then $x - y$ is a period of the function f;

(iii) a vector $x \in X$ is a period of the function f iff $f(x) = 1$.

(For a proof cf. e.g. [1; Section 2.5.1].)

Accept now notation (9). It follows from properties (i) and (iii) that N is a subgroup of the group $(X, +)$ and, in order to prove that it is a linear subspace of the space X, it is enough to show that $\mathbb{R} \cdot N \subset N$ and $\alpha_0 N \subset N$ for some $\alpha_0 \in \mathbb{C} \setminus \mathbb{R}$. To this end fix an $x \in N$ and consider the function $\phi: \mathbb{C} \to \mathbb{C}$ defined by the formula

$$\phi(\alpha) = f(\alpha x).$$

It is easy to observe that ϕ is a continuous solution of equation (1) and

$$\phi(1) = f(x) = 1.$$

Hence and from the Proposition we infer that $\phi|_{\mathbb{R}} = 1$ which means that $\alpha x \in N$ for every $\alpha \in \mathbb{R}$, or $\mathbb{R} \cdot N \subset N$. Moreover, it follows from properties (ii) and (iii) that

$$f(X) \cdot N \subset N$$

and this, jointly with our assumption (17), proves that $\alpha_0 N \subset N$ for some $\alpha_0 \in \mathbb{C} \setminus \mathbb{R}$. Consequently, N is a closed linear subspace of the space X.

Further we argue as P. Javor did in his paper [6]. On account of property (iii), formula (18) defines a function $F: X/N \to \mathbb{C}$, which is of course continuous (as the function f is continuous) and fulfills equation (1). It remains to show that

$$\dim X/N = 1.$$

Let us observe that $\dim X/N \geqslant 1$, as in the opposite case the function f would be constant, and therefore real-valued, contrary to our assumption (17). Suppose that $\dim X/N > 1$ and let $\mathscr{E}$ be a two-dimensional subspace of the space X/N. Consider also the set $\mathscr{U}$ defined as the image of the set $X \setminus f^{-1}(\{0\})$ under the quotient map. It follows from property (i) and from the fact that the quotient map is open that $\mathscr{U} \cap \mathscr{E}$ is a neighbourhood of the origin in the space $\mathscr{E}$. On the other hand the function $F|_{\mathscr{U}}$ is one-to-one which follows from properties (ii) and (iii). Hence the function $F|_{\mathscr{U} \cap \mathscr{E}}$ maps continuously and in a one-to-one manner a neighbourhood of the origin in two-dimensional complex space $\mathscr{E}$ into the complex space $\mathbb{C}$, which is of course impossible (cf. in particular [8; Theorem 1.21]). This contradiction concludes the proof of the lemma.

5. Passing to the proof of the theorem, assume that X is a complex linear topological Hausdorff space and let $f: X \to \mathbb{C}$ be a continuous solution of equation (1). If $f(X) \subset \mathbb{R}$ then the theorem of N. Brillouët and J. Dhombres gives the representation

$$f = \phi_0 \circ x_0^*,$$

where $\phi_0: \mathbb{R} \to \mathbb{R}$ is a continuous solution of equation (1) and $x_0^*: X \to \mathbb{R}$ is a continuous and real-linear functional on the space X. Define the functions $\phi: \mathbb{C} \to \mathbb{C}$ and $x^*: X \to \mathbb{C}$ by the formulas

$$\phi(\alpha) = \phi_0(\operatorname{Re} \alpha), \qquad x^*(x) = x_0^*(x) - ix_0^*(ix).$$

Of course ϕ is a continuous solution of equation (1), x^* is a continuous (complex-) linear functional on the space X and

$$\phi(x^*x) = \phi_0(x_0^*x) = f(x)$$

for every $x \in X$.

Assume now that condition (17) is fulfilled. Then, as follows from the lemma, f has the form (18), where N is a closed linear subspace of the space X of co-dimension one and $F: X/N \to \mathbb{C}$ is a continuous solution of equation (1). Let Y be a one-dimensional subspace of the space X such that

$$X = Y \oplus N$$

and fix an $x_0 \in Y \setminus \{0\}$. Defining the functions $\phi: \mathbb{C} \to \mathbb{C}$ and $x^*: X \to \mathbb{C}$ by the formulas

$$\phi(\alpha) = F(\alpha x_0 + N), \qquad x^*(\alpha x_0 + n) = \alpha$$

for every $\alpha \in \mathbb{C}$ and $n \in N$, we see that ϕ is a continuous solution of equation

(1), x^* is a continuous linear functional on the space X and, for every $\alpha \in \mathbb{C}$ and $n \in N$,

$$f(\alpha x_0 + n) = F(\alpha x_0 + n + N) = F(\alpha x_0 + N) = \phi(\alpha) = \phi[x^*(\alpha x_0 + n)],$$

which concludes the proof.

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