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Probabilistic convergence structures

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Summary. In this paper we try to argue that it is necessary to replace the topological convergence structure of Menger spaces with an appropriate probabilistic concept of convergence.

DEFINITION. Let X be a set and $\mathscr{X}$ be the set of all nets on X; for every $\alpha \in \mathscr{X}$, let $\mathscr{X}_\alpha$ be the set of all subnets of α and $\mathscr{X}_\alpha^*$ be the set of all generalized subnets of α. Let $p: \mathscr{X} \times X \to I = [0, 1]$ be a mapping and $T: I \times I \to I$ be a t-function. We say that p is a *probabilistic convergence function* (p.c.f.) relating to T if the following conditions are satisfied:

(P1) $p(\alpha, x) = 1$, for every $x \in X$ and $\alpha \in \mathscr{X}$ with $\alpha \subseteq \{x\}$.

(P2) $p(\alpha, x) \leqslant p(\beta, x)$, for every $x \in X$, $\alpha \in \mathscr{X}$ and $\beta \in \mathscr{X}_\alpha$.

(P3) $p(\alpha, x) \geqslant \inf\{\sup\{p(\gamma, x): \gamma \in \mathscr{X}_\beta^*\}: \beta \in \mathscr{X}_\alpha\}$, for every $x \in X$ and $\alpha \in \mathscr{X}$.

(P4) For every $x \in X$, $\alpha \in \mathscr{X}$, $(\beta_d)_{d \in D} \subseteq \mathscr{X}$, where $\alpha: D \to X$, $\beta_d: E_d \to X$, $d \in D$, we have

$$p(\delta, x) \geqslant T(p(\alpha, x), \inf_{d \in D} p(\beta_d, \alpha(d))),$$

where

$$\delta: D \times \prod_{d \in D} E_d \to X, \delta((d, (e_c)_{c \in D})) = \beta_d(e_d).$$

We may interpret $p(\alpha, x)$ as "the probability that the net α converges to x". If $\mathscr{C} = \{(\alpha, x) \in \mathscr{X} \times X: p(\alpha, x) = 1\}$ then, from (P1)–(P4), it follows that $\mathscr{C}$ is a convergence class of a topological space. In a Menger space (X, F, T), $\mathscr{C}$ is the convergence class of the uniformity defined by Schweizer and Sklar.

Now, let (θ, T) be a Frank probabilistic topological structure on X and $p_\theta: \mathscr{X} \times X \to I$ be the mapping defined by $p_\theta(\alpha, x) = \sup\{\lambda \in I: \alpha$ converges to x in the closure space $(X, \theta(\cdot, \lambda))\}$. Then p_θ is a p.c.f. relating to T. Conversely, for every p.c.f. relating to T, p, there exists exactly one probabilistic topological structure (θ, T) such that $p_\theta = p$. Therefore, a probabilistic convergence function is a probabilistic variant of a topological convergence class. In Menger spaces we generalize some results of Schweizer and Sklar concerning the "continuity" of a probabilistic metric. In the context of Sherwood's E-spaces we obtain classical results in probability theory as special cases of more general results.

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1. Introduction

The notion of probabilistic metric spaces was introduced by Menger in 1942 (see [9]); an adequate uniformity for these spaces was defined in 1960 by Schweizer and Sklar [11]. The results that have been obtained afterwards (see, for example, [1], [3], [10], [13]) use the convergence structure induced by this uniformity (with exceptions, see [15] and [16]).

Probabilistic metric structures are introduced for modelling those situations in which we have only probabilistic information about the distances between points of the space. Such being the situation, it stands to reason that, in this framework, the topological properties ought also to bear a probabilistic stamp.

In a metric space (X, σ), the net α converges to $x \in X$ if and only if the net of real numbers $\sigma(\alpha, x)$ converges to zero. But, in a probabilistic metric space (X, F, T), for every $x, y \in X$ and $a > 0$, we know just "the probability that $\sigma(x, y) < a$" $= F_{xy}(a)$. Hence we cannot simply answer "yes" or "no" to the question: "Does $\sigma(\alpha, x)$ converge to 0?". The probabilistic answer to this question is the appropriate one. This is a first sign that the problem of convergence in probabilistic metric spaces is not well founded.

A second argument for the previous remark is the following: Schweizer, Sklar and Thorp [12] prove that, if the t-norm T is left-continuous at $(1, 1)$, then the uniformity introduced in [11] is metrizable; in particular, for Sherwood's E-spaces (see [14]), the convergence in these metrizable spaces is convergence in probability (this fact was noticed by Drossos in [3]). Therefore, for the study of convergence in this uniform space it is very comfortable to use a metric which induces the uniform structure of the space, instead of the probabilistic metric.

In this paper we will argue that it is necessary to replace the topological convergence structure of Menger spaces by an adequate probabilistic concept: *the probabilistic convergence structure*.

Section 2 of this paper is preliminary. In Section 3 we introduce the general concept of a probabilistic convergence structure. In the theory of probabilistic topological structures there are two equivalent methods. A. The first is a pattern of those situations in which we know the probability that a certain topological condition holds (for example: Menger's statistical metrics [9], the function c of Frank [8] and the dilatation function of the author [5]); B. Another way is to consider a family $\{\varphi_\lambda : \lambda \in [0, 1]\}$ of topological structures as model of the ideal topological structure φ, in the following sense. If the condition C is satisfied for φ_λ then the probability of the event "C is satisfied for φ" is not smaller than λ (this idea was used by Frank for the definition of probabilistic topological spaces [8], by the author for the introduction of probabilistic proximities and uniformities [4], and for the equivalent forms of type B of Menger's metrics—the probabilistic pseudometrics [6]).

The A-forms for probabilistic convergence structures are *probabilistic convergence functions* and the B-forms are *probabilistic convergence classes*.

We give a characterization of Frank's probabilistic topological structures (see [8]) using probabilistic convergence structures. This result is a probabilistic variant of the characterization of topological structures by means of convergence structures (see [2; 35.A 18]).

Section 4 is concerned with sequential probabilistic convergence structures.

In Section 5, in the particular case of Menger spaces, we obtain two results concerning the "continuity" of a probabilistic metric. These results generalize Theorems 8.1 and 8.2 of [11].

For the E-spaces of Sherwood [14], we obtain in Section 6 a generalization of some well-known results in probability theory. We remark that these results are given without proofs in [7].

2. Preliminaries

Let X be a set and $\theta: \mathscr{P}(X) \to \mathscr{P}(X)$. This θ is a closure operation in the sense of Čech on X if (1) $\theta(\phi) = \phi$; (2) $A \subseteq \theta(A)$, for each $A \subseteq X$; (3) $\theta(A \cup B) = \theta(A) \cup \theta(B)$, for each $A, B \subseteq X$ (see [2, 14 A.1]).

Let I be the closed unit interval and T be a t-function (a left-continuous and nondecreasing mapping from $I \times I$ into I with $T(0, \lambda) = T(\lambda, 0) = 0$ for every $\lambda \in I$). A probabilistic topological structure (p.t.s.) on X is a pair (θ, T), where $\theta: \mathscr{P}(X) \times I \to \mathscr{P}(X)$ is a mapping satisfying

(T1) $\theta(\cdot, \lambda)$ is a closure operation in the sense of Čech on X, for every $\lambda \in I$,
(T2) $\theta(\cdot, 0)$ is the indiscrete closure on X,
(T3) $\theta(\cdot, \lambda) \subseteq \theta(\cdot, \mu)$ if $\mu \leqslant \lambda$, and
(T4) $\theta(\theta(\cdot, \mu), \lambda) \subseteq \theta(\cdot, T(\lambda, \mu))$, for every $\lambda, \mu \in I$.

If (θ, T) is a p.t.s. on X, then (X, θ, T) is a probabilistic topological space. The notion of a probabilistic topological space was introduced by Frank [8] and he gave the following probabilistic interpretation. If $x \in \theta(A, \lambda)$ then the probability of the event "x is in the closure of A" is not smaller than λ.

Let (θ, T) be a p.t.s. on X; θ is left-continuous if

(T5) $\theta(\cdot, \lambda) = \bigcap_{\mu < \lambda} \theta(\cdot, \mu)$, for every $\lambda \in (0, 1]$.

It should be noted that the condition (T3) is a weakened form of the condition (T5). Conversely, we easily get the following:

LEMMA 2.1. *Let (θ, T) be a* p.t.s. *on X and let $\theta_{lc}: \mathscr{P}(X) \times I \to \mathscr{P}(X)$ defined by $\theta_{lc}(\cdot, 0) = \theta(\cdot, 0)$ and $\theta_{lc}(\cdot, \lambda) = \bigcap_{\mu < \lambda} \theta(\cdot, \mu)$, for every $\lambda \in (0, 1]$. Then (θ_{lc}, T) is the finest left-continuous p.t.s. on X coarser than (θ, T). ((θ', T) is coarser than (θ'', T) iff $\theta'(\cdot, \lambda) \supseteq \theta''(\cdot, \lambda)$, for every $\lambda \in I$). We say that θ_{lc} is the left-continuous modification of θ; we remark that θ is left-continous iff $\theta = \theta_{lc}$.*

The following result is concerned with the description of p.t.s. in terms of neighbourhoods; the proof is analogous to the proof of the similar theorem for topological structures and we omit it.

PROPOSITION 2.2. *Let (θ, T) be a p.t.s. on X and, for every $\lambda \in I, x \in X$, let $\mathscr{V}_\lambda(x)$ be the neighbourhood system of x in the closure space $(X, \theta(\cdot, \lambda))$; then, for every $x \in X$, the family $\{\mathscr{V}_\lambda(x): \lambda \in I\}$ satisfies the following:*

(N1) *For every $\lambda \in I, \mathscr{V}_\lambda(x)$ is a neighbourhood system of x for a closure operation on X (i.e. (a) if $V \in \mathscr{V}_\lambda(x)$ and $W \supseteq V$ then $W \in \mathscr{V}_\lambda(x)$, (b) for every $V \in \mathscr{V}_\lambda(x), x \in V$, and (c) if $V, W \in \mathscr{V}_\lambda(x)$ then $V \cap W \in \mathscr{V}_\lambda(x)$).*

(N2) $\mathscr{V}_0(x) = \{X\}$.

(N3) $\mathscr{V}_\mu(x) \subseteq \mathscr{V}_\lambda(x)$ *if* $\mu \leqslant \lambda$.

(N4) *For every $V \in \mathscr{V}_{T(\lambda, \mu)}(x)$ there exists $W \in \mathscr{V}_\lambda(x)$ such that, for every $y \in W$, $V \in \mathscr{V}_\mu(y)$.*

On the other hand, if, for each $x \in X$, the family $\{\mathscr{V}_\lambda(x): \lambda \in I\}$ satisfies the conditions (N1)–(N4), then there exists exactly one p.t.s. (θ, T) on X such that, for every $x \in X$ and $\lambda \in I$, $\mathscr{V}_\lambda(x)$ is the neighbourhood system of x in the closure space $(X, \theta(\cdot, \lambda))$. We remark that θ is left-continous iff

(N5) $\mathscr{V}_\lambda(x) = \bigcup_{\mu < \lambda} \mathscr{V}_\mu(x)$, for every $\lambda \in (0, 1]$ and $x \in X$.

Let $\mathscr{U} = \{\mathscr{U}_\lambda : \lambda \in I\}$ be a family of filters on $X \times X$ and let T be a t-function; the pair $(\mathscr{U}, T)$ is called a probabilistic uniform structure on X (see [4, Definition 5.2.]), if the following conditions hold:

(U1) $\mathscr{U}_\lambda$ is a semiuniformity on X, for every $\lambda \in I$ (i.e. $\mathscr{U}_\lambda$ is a filter in $X \times X$, for every $U \in \mathscr{U}_\lambda$ and $x \in X$ $(x, x) \in U$ and $U^{-1} \in \mathscr{U}_\lambda$ for every $U \in \mathscr{U}_\lambda$).

(U2) $\mathscr{U}_0 = \{X \times X\}$.

(U3) $\mathscr{U}_\mu \subseteq \mathscr{U}_\lambda$ if $\mu \leqslant \lambda$.

(U4) For every $U \in \mathscr{U}_{T(\lambda,\mu)}$ there exists $V \in \mathscr{U}_\lambda$, $W \in \mathscr{U}_\mu$ such that $V \circ W \subseteq U$.

A left-continuous probabilistic uniformity on X is a probabilistic uniformity $(\mathscr{U}, T)$ for which

(U5) $\mathscr{U}_\lambda = \bigcup_{\mu < \lambda} \mathscr{U}_\mu$, for every $\lambda \in (0, 1]$.

Let $(\mathscr{U}, T)$ be a probabilistic uniformity on X and, for every $\lambda \in I$, let $\theta(\cdot, \lambda)$ be the closure induced by $\mathscr{U}_\lambda$. Then (θ, T) is a p.t.s. on X (see [4, Proposition 5.2]). We say that (θ, T) is the p.t.s. induced by $(\mathscr{U}, T)$. If $(\mathscr{U}, T)$ is a left-continuous probabilistic uniformity then (θ, T) is left-continuous also.

In this paper every t-norm (see [11]) is left-continuous on $I \times I$. Let (F, T) be a Menger metric on X (see [11]); for every $a > 0, \mu \in [0, 1)$ we define $U_{a,\mu} = \{(x, y): F_{xy}(a) > \mu\}$. Then, for every $\lambda \in (0, 1]$, $\mathscr{B}_\lambda = \{U_{a,\mu}: a > 0, \mu < \lambda\}$ is a base for a semiuniformity $\mathscr{U}_\lambda$ on X; let $\mathscr{U} = \{\mathscr{U}_\lambda: \lambda \in I\}$, where $\mathscr{U}_0 = \{X \times X\}$. Then $(\mathscr{U}, T)$ is a left-continuous probabilistic uniformity on X (see [4, Proposition 6.1]). We say that $(\mathscr{U}, T)$ is induced by (F, T). We remark that $\mathscr{U}_1$ is the uniformity defined by Schweizer and Sklar on (X, F, T) in [11].

3. Probabilistic convergence structures

Let X be a set, $(A, <)$ and $(B, \leqslant)$ be two directed sets and $\alpha: A \to X$, $\beta: B \to X$ be two nets. We say that β is a *subnet* of α if there exists a one-to-one mapping $f: B \to A$ such that

a) $\beta = \alpha \circ f$,
b) for every $a \in A$ there exists $b \in B$ such that $a < f(b)$,
c) $b_1 \leqslant b_2$ implies $f(b_1) < f(b_2)$;

β is a *generalized subnet* of α (see [2, 15 B.17]) if

a*) $\beta = \alpha \circ f$,
b*) for every $a \in A$ there exists $b \in B$ such that $f(c) > a$, for every $c \in B$ with $c \geqslant b$.

Let $\mathscr{X}$ be the set of all nets on X; for every $\alpha \in \mathscr{X}$, let $\mathscr{X}_\alpha$ be the set of all subnets of α and $\mathscr{X}^*_\alpha$ be the set of all generalized subnets of α. Obviously, $\alpha \in \mathscr{X}_\alpha \subseteq \mathscr{X}^*_\alpha$, for every $\alpha \in \mathscr{X}$. The notation $\alpha \subseteq Y$ means that the range of the net α is in the set $Y \subseteq X$.

Let $(D, \leqslant)$ be a directed set and, for every $d \in D$, let $(E_d, \leqslant_d)$ also be a directed set, and let $\beta_d: E_d \to X$ be nets. Then $D \times \prod_{d \in D} E_d$, endowed with the product order $(\leqslant \times \prod_{d \in D} \leqslant_d)$, is a directed set and $\delta: D \times \prod_{d \in D} E_d \to X$, $\delta((d, (e_c)_{c \in D})) = \beta_d(e_d)$ is a net; we say that δ is *the combination net of the nets* $(\beta_d)_{d \in D}$.

DEFINITION A. Let $T: I \times I \to I$ be a t-function and let $p: \mathscr{X} \times X \to I$ be a mapping; p is a *probabilistic convergence function on* X relating to T if the following conditions are satisfied:

(P1) $p(\alpha, x) = 1$ for every $x \in X$ and $\alpha \in \mathscr{X}$ with $\alpha \subseteq \{x\}$,
(P2) $p(\alpha, x) \leqslant \inf\{p(\beta, x): \beta \in \mathscr{X}_\alpha\}$ for every $\alpha \in \mathscr{X}$, $x \in X$,

(P3) $p(\alpha, x) \geqslant \inf\{\sup\{p(\gamma, x): \gamma \in \mathscr{X}^*_\beta\}: \beta \in \mathscr{X}_\alpha\}$, for every $\alpha \in \mathscr{X}$, $x \in X$,

(P4) For every $x \in X$, $\alpha \in \mathscr{X}$, $(\beta_d)_{d \in D} \subseteq \mathscr{X}$, where $\alpha: D \to X$, $\beta_d: E_d \to X$, $d \in D$, we have $p(\delta, x) \geqslant T(p(\alpha, x), \inf_{d \in D} p(\beta_d, \alpha(d)))$, where δ is the combination net of the nets $(\beta_d)_{d \in D}$.

REMARK 3.1. We may interpret $p(\alpha, x)$ as: "the probability that the net α converges to x"; hence a probabilistic convergence function estimates, for every net $\alpha \in \mathscr{X}$ and for every $x \in X$, the degree of convergence of α to x. From (P1), every constant net converges with probability one. From (P2) it follows that, for every subnet β of a net α, the probability for β to be convergent is not smaller than the probability that α is convergent. The condition (P3) is equivalent to the following:

(P3′) If for every subnet β of α there exists a generalized subnet γ of β such that $\lambda < p(\gamma, x)$ then $\lambda \leqslant p(\alpha, x)$.

We remark that, if for every subnet β of α there exists a generalized subnet γ of β such that $p(\gamma, x) = 1$, then $p(\alpha, x) = 1$. Finally, the condition (P4) is equivalent with

(P4′) If $\lambda < p(\alpha, x)$ and, for every $d \in D$, $\mu < p(\beta_d, \alpha(d))$, then $T(\lambda, \mu) < p(\delta, x)$.

From (P4) it follows that if $p(\alpha, x) = 1$ and $p(\beta_d, \alpha(d)) = 1$, for every $d \in D$, then $p(\delta, x) = 1$ (with the supplementary condition $T(1, 1) = 1$). So, if $\mathscr{C} = \{(\alpha, x) \in \mathscr{X} \times X: p(\alpha, x) = 1\}$, then from (P1)–(P4) it follows that:

(a) $(\alpha, x) \in \mathscr{C}$, for every $x \in X$ and $\alpha \in \mathscr{X}$ with $\alpha \subseteq \{x\}$.

(b) If $(\alpha, x) \in \mathscr{C}$ then $(\beta, x) \in \mathscr{C}$, for every $\beta \in \mathscr{X}_\alpha$.

(c) If $(\alpha, x) \notin \mathscr{C}$ then there exists a $\beta \in \mathscr{X}_\alpha$ such that, for every $\gamma \in \mathscr{X}^*_\beta$, $(\gamma, x) \notin \mathscr{C}$.

(d) If $(\alpha, x) \in \mathscr{C}$ and $(\beta_d, \alpha(d)) \in \mathscr{C}$, for every $d \in D$, then $(\delta, x) \in \mathscr{C}$ (the condition on iterated limits).

These conditions are necessary and sufficient for a convergence relation to be a convergence class of a topological space (see [2, 35 A.18]).

In Theorems 3.4 and 3.5 we give a characterization of p.t.s. using probabilistic convergence functions. Therefore, a probabilistic convergence function is a probabilistic variant of a topological convergence class.

We must remark that the axioms of Definition A are independent of any probabilistic determination, hence this interpretation for p is not essential.

We recall that a *convergence class* is a set $\mathscr{C} \subseteq \mathscr{X} \times X$ which satisfies the conditions (a), (b), (c) from Remark 3.1 and the following:

(d′) Let $(D, \leqslant)$ be a directed set and $(\beta_d)_{d \in D} \subseteq \mathscr{X}$, where $\beta_d: E_d \to X$, for every $d \in D$. If $(\beta_d, x) \in \mathscr{C}$, for every $d \in D$, then $(\delta, x) \in \mathscr{C}$, where δ is the combination net of the nets $(\beta_d)_{d \in D}$ (the condition of diagonalization).

These conditions are necessary and sufficient for a convergence relation to be a convergence class of a closure space (see [2, 35 A.17]).

DEFINITION B. Let $\mathscr{C} = \{\mathscr{C}_\lambda : \lambda \in I\}$, where $\mathscr{C}_\lambda \subseteq \mathscr{X} \times X$, for every $\lambda \in I$, and let T be a t-function; $\mathscr{C}$ is a *probabilistic convergence class on* X relating to T if

(C1) $\mathscr{C}_\lambda$ is a convergence class for every $\lambda \in I$,

(C2) $\mathscr{C}_0 = \mathscr{X} \times X$,

(C3) $\mathscr{C}_\lambda = \bigcap_{\mu < \lambda} \mathscr{C}_\mu$ for every $\lambda \in (0, 1]$,

(C4) For every $x \in X$, $\alpha \in \mathscr{X}$, $(\beta_d)_{d \in D} \subseteq \mathscr{X}$, where $\alpha: D \to X$, $\beta_d: E_d \to X$, $d \in D$, if $(\alpha, x) \in \mathscr{C}_\lambda$ and $(\beta_d, \alpha(d)) \in \mathscr{C}_\mu$ for every $d \in D$, then $(\delta, x) \in \mathscr{C}_{T(\lambda,\mu)}$, where δ is the combination net of the nets $(\beta_d)_{d \in D}$.

REMARKS 3.2. (1) The condition of diagonalization, (d′), is a consequence of (a) and (C4). Hence (C1) can be replaced by (a), (b) and (c). (2) It follows from (C3) that if $\mu \leqslant \lambda$ then $\mathscr{C}_\lambda \subseteq \mathscr{C}_\mu$. (3) The interpretation of $(\alpha, x) \in \mathscr{C}_\lambda$ is: "the probability that the net α converges to x is not smaller than λ" (see also Remark 3.1).

In the following theorem we show that Definitions A and B are equivalent.

THEOREM 3.3. *The mapping* $\Phi: p \mapsto \mathscr{C}^p$, *where* $\mathscr{C}^p = \{\mathscr{C}^p_\lambda : \lambda \in I\}$, $\mathscr{C}^p_\lambda = \{(\alpha, x) \in \mathscr{X} \times X: p(\alpha, x) \geqslant \lambda\}$, *for every* $\lambda \in I$, *is a bijection between the set of all probabilistic convergence functions and the set of all probabilistic convergence classes relating to the same t-function T.*

Proof. Property (C4) is a consequence of (P4), because T is nondecreasing. If $x \in X, \alpha \in \mathscr{X}$ and $\alpha \subseteq \{x\}$ then, from (P1), $p(\alpha, x) = 1 \geqslant \lambda$, for every $\lambda \in I$, hence $(\alpha, x) \in \mathscr{C}^p_\lambda$. Let now $\alpha \in \mathscr{X}, \beta \in \mathscr{X}_\alpha$ and $(\alpha, x) \in \mathscr{C}^p_\lambda$; then, from (P2), $p(\beta, x) \geqslant p(\alpha, x) \geqslant \lambda$, hence $(\beta, x) \in \mathscr{C}^p_\lambda$. We suppose that, for every $\beta \in \mathscr{X}_\alpha$, there exists $\gamma \in \mathscr{X}^*_\alpha$ such that $(\gamma, x) \in \mathscr{C}^p_\lambda$. Then $p(\gamma, x) \geqslant \lambda$, hence $\sup_{\gamma \in \mathscr{X}^*_\beta} p(\gamma, x) \geqslant \lambda$, for every $\beta \in \mathscr{X}_\alpha$. From (P3), $p(\alpha, x) \geqslant \lambda$, therefore $(\alpha, x) \in \mathscr{C}^p_\lambda$. Then (C1) follows from Remark 3.2(1). It follows directly from the definition that the family $\mathscr{C}^p$ satisfies conditions (C2) and (C3). Therefore, if p is a probabilistic convergence function, then $\Phi(p) = \mathscr{C}^p$ is a probabilistic convergence class relating to the same t-function T. If $\Phi(p_1) = \Phi(p_2)$ then, for every $\lambda \in I$, $p_1(\alpha, x) \geqslant \lambda$ iff $p_2(\alpha, x) \geqslant \lambda$, hence iff $p_1 = p_2$. So, Φ is one-to-one. Let $\mathscr{C}$ be a probabilistic convergence class relating to T, where $\mathscr{C} = \{\mathscr{C}_\lambda : \lambda \in I\}$; we define $p: \mathscr{X} \times X \to I$, letting $p(\alpha, x) = \sup\{\lambda \in I: (\alpha, x) \in \mathscr{C}_\lambda\}$. We remark that, for every $\alpha \in \mathscr{X}, x \in X$, $(\alpha, x) \in \mathscr{C}_0$; hence there exists $\lambda \in I$ such that $(\alpha, x) \in \mathscr{C}_\lambda$. The properties (P1), (P2) and (P3) follow from (C1) and (C3). It remains to establish (P4). Let $x \in X, \alpha \in \mathscr{X}, (\beta_d)_{d \in D} \subseteq \mathscr{X}$, where $\alpha: D \to X$, $\beta_d: E_d \to X$, for every $d \in D$ and let δ be the combination net of the nets $(\beta_d)_{d \in D}$. If $u = p(\alpha, x) = 0$, or $v = \inf_{d \in D} p(\beta_d, \alpha(d)) = 0$ then $T(u, v) = 0$ and (P4) is obvious. Let $u > 0$ and $v > 0$. From the left-continuity of T at (u, v), for every $v < T(u, v)$ there exist $\lambda < u$ and $\mu < v$ such that $v < T(\lambda, \mu)$. From the definition of p we have $(\alpha, x) \in \mathscr{C}_\lambda$ and $(\beta_d, \alpha(d)) \in \mathscr{C}_\mu$, for every $d \in D$. Then, from (C4), $(\delta, x) \in \mathscr{C}_{T(\lambda,\mu)}$. Hence $p(\delta, x) \geqslant T(\lambda, \mu) > v$ and this for every $v < T(u, v)$. It follows that (P4) is satisfied. Hence p is a probabilistic convergence function on X. Let $\Phi(p) = \mathscr{C}^p = \{\mathscr{C}^p_\lambda : \lambda \in I\}$ and let $\lambda \in I$. For every $(\alpha, x) \in \mathscr{C}_\lambda$ we have $p(\alpha, x) \geqslant \lambda$, hence $(\alpha, x) \in \mathscr{C}^p_\lambda$, so that $\mathscr{C}_\lambda \subseteq \mathscr{C}^p_\lambda$. If $(\alpha, x) \in \mathscr{C}^p_\lambda$ then $p(\alpha, x) \geqslant \lambda$; hence, for every $\mu < \lambda$, $p(\alpha, x) > \mu$ and, from the definition of p, $(\alpha, x) \in \mathscr{C}_\mu$. Therefore, $(\alpha, x) \in \bigcap_{\mu<\lambda} \mathscr{C}_\mu = \mathscr{C}_\lambda$, so that $\mathscr{C}^p_\lambda \subseteq \mathscr{C}_\lambda$. It follows that $\Phi(p) = \mathscr{C}$, hence Φ is onto.

Now, we say that we have a *probabilistic convergence structure* on a set X if we have a probabilistic convergence function on X or a probabilistic convergence class on X.

The following two theorems are concerned with questions related to the definition of a probabilistic topological structure by specifying the degree of convergence of α to x, for every net α and for every $x \in X$. So, from Theorem 3.4, we conclude that with every left-continuous p.t.s. on X we can associate a probabilistic convergence class, and from Theorem 3.5 it follows that this association is a bijection.

Let (θ, T) be a p.t.s. on X; for every $\lambda \in I$ let $\alpha \xrightarrow{\lambda} x$ denote the situation in which the net α converges to x in the closure space $(X, \theta(\cdot, \lambda))$.

THEOREM 3.4. *Let (θ, T) be a left-continuous p.t.s. on X; for every $\lambda \in I$ let $\mathscr{C}^\theta_\lambda = \{(\alpha, x) \in \mathscr{X} \times X : \alpha \xrightarrow{\lambda} x\}$ and let $\mathscr{C}^\theta = \{\mathscr{C}^\theta_\lambda : \lambda \in I\}$. Then $\mathscr{C}^\theta$ is a probabilistic convergence class on X relating to T.*

Proof. From definition, $\mathscr{C}^\theta_\lambda$ is the convergence class of $\theta(\cdot, \lambda)$, for every $\lambda \in I$ (see [2, 35 A.1]). (C2) is a consequence of (T2). From (T3) we have that $\mathscr{C}^\theta_\lambda \subseteq \mathscr{C}^\theta_\mu$, for every $\mu < \lambda$, hence $\mathscr{C}^\theta_\lambda \subseteq \bigcap_{\mu<\lambda} \mathscr{C}^\theta_\mu$. If $(\alpha, x) \notin \mathscr{C}^\theta_\lambda$, where $\alpha : D \to X$, then $\alpha \stackrel{\lambda}{\nrightarrow} x$, hence there exists $V \in \mathscr{V}_\lambda(x)$ such that, for every $d \in D$, there is $e_d \geqslant d$ such that $\alpha(e_d) \in X - V$. But, from (N5), there exists $\mu < \lambda$ such that $V \in \mathscr{V}_\mu(x)$ (see Proposition 2.2); therefore $\alpha \stackrel{\mu}{\nrightarrow} x$. Hence $(\alpha, x) \notin \bigcap_{\mu<\lambda} \mathscr{C}^\theta_\mu$. This proves (C3). Now let $x \in X$, $\alpha \in \mathscr{X}$; $(\beta_d)_{d \in D} \subseteq \mathscr{X}$, where $\alpha : D \to X$ and for every $d \in D$, $\beta_d : E_d \to X$; we suppose that $(\alpha, x) \in \mathscr{C}^\theta_\lambda$ and $(\beta_d, \alpha(d)) \in \mathscr{C}^\theta_\mu$, for every $d \in D$. For every $V \in \mathscr{V}_{T(\lambda,\mu)}(x)$ there exists $W \in \mathscr{V}_\lambda(x)$ such that, for every $y \in W$, $V \in \mathscr{V}_\mu(y)$ (see Proposition 2.2 (N4)). Since $\alpha \xrightarrow{\lambda} x$, there exists a $d_0 \in D$ such that, for every $d \geqslant d_0$, $\alpha(d) \in W$, so that $V \in \mathscr{V}_\mu(\alpha(d))$. For every $d \in D$, $\beta_d \xrightarrow{\mu} \alpha(d)$; hence there exists $e^0_d \in E_d$ such that, for every $e_d \geqslant e^0_d$, $e_d \in E_d$, $\beta_d(e_d) \in V$. For every $d \in D$ with $d \ngeqslant d_0$, we denote by e^0_d some element of E_d and let $c_0 = (d_0, (e^0_d)_{d \in D}) \in D \times \prod_{d \in D} E_d$. Now, for every $c = (d, (e_u)_{u \in D}) \in D \times \prod_{d \in D} E_d$ with $c \geqslant c_0$, we have

$d \geqslant d_0$ and $e_u \geqslant e_u^0$, for every $u \in D$; hence $\delta(c) = \beta_d(e_d) \in V$ (because $e_d \geqslant e_d^0$), where δ is the combination net of $(\beta_d)_{d \in D}$. Therefore, $\delta \xrightarrow{T(\lambda, \mu)} x$, so that, $(\delta, x) \in \mathscr{C}^\theta_{T(\lambda, \mu)}$.

We say that $\mathscr{C}^\theta$ is the probabilistic convergence class of the p.t.s. (θ, T).

THEOREM 3.5. *Let T be a t-function which satisfies the condition $T(1, \lambda) = \lambda$, for every $\lambda \in I$. The mapping $\Psi: \theta \rightsquigarrow \mathscr{C}^\theta$ is a bijection between the set of all left-continuous p.t.s. (θ, T) on a set X and the set of all probabilistic convergence classes on X relating to T.*

Proof. We must prove that, for every probabilistic convergence class $\mathscr{C}$ relating to T, there exists exactly one left-continuous p.t.s. (θ, T) such that $\mathscr{C} = \mathscr{C}^\theta$. Let $\mathscr{C}$ be a probabilistic convergence class on X relating to $T, \mathscr{C} = \{\mathscr{C}_\lambda : \lambda \in I\}$. For every $\lambda \in I$, $\mathscr{C}_\lambda$ is a convergence class, hence we define $\theta: \mathscr{P}(X) \times I \to \mathscr{P}(X)$ by $\theta(A, \lambda) = \{x \in X$: there exist $\alpha \subseteq A, \alpha \in \mathscr{X}$ with $(\alpha, x) \in \mathscr{C}_\lambda\}$ and we say that $\theta(\cdot, \lambda)$ is a closure operation and $\mathscr{C}^\theta_\lambda = \mathscr{C}_\lambda$, for every $\lambda \in I$ (see [2, 35 A.6]). It remains to establish for θ (T2)–(T5). (T2) is a consequence of (C2), and (T3) is a consequence of (C3). It follows that $\theta(A, \lambda) \subseteq \bigcap_{\mu < \lambda} \theta(A, \mu) = \theta_{\mathrm{lc}}(A, \lambda)$, for every $A \subseteq X$ and $\lambda \in (0, 1]$. Let $x \in \theta_{\mathrm{lc}}(A, \lambda)$; from Lemma 2.1., $\theta_{\mathrm{lc}}(\cdot, \lambda)$ is a closure operation; hence there exists an $\alpha \in \mathscr{X}, \alpha \subseteq A$, such that α converges to x in $(X, \theta_{\mathrm{lc}}(\cdot, \lambda))$. For every $\beta \in \mathscr{X}_\alpha, \beta: D \to X$ converges to x in $(X, \theta_{\mathrm{lc}}(\cdot, \lambda))$. For every $d \in D$, let $X_d = \{\beta(e): d \leqslant e\} \subseteq X, D_d = \{e: d \leqslant e\}$ and let $\gamma = \beta / D_d \in \mathscr{X}_\beta$ (where β / D_d is the restriction of β to D_d); then γ converges to x in $(X, \theta_{\mathrm{lc}}(\cdot, \lambda))$ and $\gamma \subseteq X_d$. Therefore, $x \in \theta_{\mathrm{lc}}(X_d, \lambda) = \bigcap_{\mu < \lambda} \theta(X_d, \mu)$. It follows that, for every $\mu < \lambda$, there exists a $\gamma^\mu_d \in \mathscr{X}, \gamma^\mu_d \subseteq X_d$, such that $(\gamma^\mu_d, x) \in \mathscr{C}_\mu$. Let $\gamma^\mu_d: E^\mu_d \to X$ and let δ^μ be the combination net of the nets $(\gamma^\mu_d)_{d \in D}$. Hence $\delta^\mu: D \times \prod_{u \in D} E^\mu_u \to X, \delta^\mu(d, (e_u)_{u \in D}) = \gamma^\mu_d(e_d)$. For every $d \in D$ and $(e_u)_{u \in D} \in \prod_{u \in D} E_u$, there is an $e \in D$ such that $d \leqslant e$ and $\beta(e) = \gamma^\mu_d(e_d)$ (because $\gamma^\mu_d(e_d) \in X_d$). Then we define a mapping $f: D \times \prod_{d \in D} E^\mu_d \to D$, letting $f(d, (e_u)_{u \in D}) = e$. It follows that $\delta^\mu = \beta \circ f$ and, for every $d \in D$ and $(e_u)_{u \in D} \in \prod_{u \in D} E^\mu_u$, there exists a $c = (d, (e_u)_{u \in D}) \in D \times \prod_{u \in D} E^\mu_u$ such that, for every $c' = (d', (e'_u)_{u \in D}) \geqslant (d, (e_u)_{u \in D}) = c$, we have that $d' \geqslant d$ and $e'_u \geqslant e_u$, for every $u \in D$, hence $f(c') = f(d', (e'_u)_{u \in D}) = e' \geqslant d' \geqslant d$. It follows that $\delta^\mu \in \mathscr{X}^*_\beta$. Let $\alpha' \in \mathscr{X}, \alpha': D \to X\, \alpha'(d) = x$, for every $d \in D$. Then $(\alpha', x) \in \mathscr{C}_1$ and $(\gamma^\mu_d, \alpha'(d)) = (\gamma^\mu_d, x) \in \mathscr{C}_\mu$, for every $d \in D$. From (C4), we have $(\delta^\mu, x) \in \mathscr{C}_{T(1,\mu)} = \mathscr{C}_\mu$. Hence, for every $\mu < \lambda$ and for every $\beta \in \mathscr{X}_\alpha$, there exists $\delta^\mu \in \mathscr{X}^*_\beta$ such that $(\delta^\mu, x) \in \mathscr{C}_\mu$. From (C1) (see also Remark 3.2(1)), we have that, for every $\mu < \lambda, (\alpha, x) \in \mathscr{C}_\mu$; hence $(\alpha, x) \in \bigcap_{\mu < \lambda} \mathscr{C}_\mu = \mathscr{C}_\lambda$ (from (C3)). It follows that $x \in (A, \lambda)$, hence we have proved (T5). Now, let $x \in \theta(\theta(A, \mu), \lambda)$; there exists $\alpha \subseteq \theta(A, \mu), \alpha: D \to X$, such that $(\alpha, x) \in \mathscr{C}_\lambda$. For every $d \in D$, $\alpha(d) \in \theta(A, \mu)$; hence there exists $\beta_d: E_d \to X, \beta_d \subseteq A$ such that $(\beta_d, \alpha(d)) \in \mathscr{C}_\mu$. Let δ be the combination net of the nets $(\beta_d)_{d\in D}$. From (C4), $(\delta, x) \in \mathscr{C}_{T(\lambda,\mu)}$ and $\delta \subseteq A$. Hence $x \in \theta(A, T(\lambda, \mu))$, so that we have (T4). Therefore (θ, T) is a left-continuous p.t.s. and $\Psi(\theta) = \mathscr{C}^\theta = \mathscr{C}$. If (θ_1, T) is another left-continuous p.t.s. on X with $\Psi(\theta_1) = \mathscr{C}$, then $\mathscr{C}^{\theta_1} = \mathscr{C}^\theta$, hence, for every $\lambda \in I$, $\mathscr{C}^{\theta_1}_\lambda = \mathscr{C}^\theta_\lambda$. Therefore, for every $A \subseteq X$, $\lambda \in I$, $x \in \theta_1(A, \lambda)$ if and only if there exists $\alpha \in \mathscr{X}$, $\alpha \subseteq A$ such that α converges to x in $(X, \theta_1(\cdot, \lambda))$ and this holds if and only if $(\alpha, x) \in \mathscr{C}^{\theta_1}_\lambda = \mathscr{C}^\theta_\lambda = \mathscr{C}_\lambda$, hence, if and only if $x \in \theta(A, \lambda)$.

REMARKS 3.6. (1) The mapping $\Phi^{-1} \circ \Psi: \theta \rightsquigarrow p_\theta$ defined by $p_\theta(\alpha, x) = \sup\{\lambda \in I: \alpha \overset{\lambda}{\to} x\}$, for every $\alpha \in \mathscr{X}$, $x \in X$, is a bijection too.

(2) Let (θ, T) be a left-continuous p.t.s. on X, where T is a t-function with $T(1, \lambda) = \lambda$, for every $\lambda \in I$, and let $\Psi(\theta) = \mathscr{C}^\theta = \{\mathscr{C}^\theta_\lambda: \lambda \in I\}$ and $\Phi^{-1}(\mathscr{C}^\theta) = p_\theta$; then $\alpha \overset{\lambda}{\to} x$ iff $(\alpha, x) \in \mathscr{C}^\theta_\lambda$, and this is so iff $p_\theta(\alpha, x) \geqslant \lambda$, for every $\lambda \in I$, $\alpha \in \mathscr{X}$, $x \in X$. If follows that $x \in \theta(A, \lambda)$ iff there exists $\alpha \subseteq A$ with $p_\theta(\alpha, x) \geqslant \lambda$, so that $c(x, A) = \sup_{\alpha \subseteq A} p_\theta(\alpha, x)$, where $c: X \times \mathscr{P}(X) \to I$, $c(x, A) = \sup\{\lambda: x \in \theta(A, \lambda)\}$ is the function defined by Frank [8]. Indeed, it is obvious that $c(x, A) \leqslant \sup_{\alpha \subseteq A} p_\theta(\alpha, x)$, and for every $\alpha \subseteq A$, $\alpha: D \to X$, from $p_\theta(\alpha, x) \geqslant \lambda = p_\theta(\alpha, x)$, we have that $\alpha \overset{\lambda}{\to} x$ hence $x \in \theta(\alpha(D), \lambda) \subseteq \theta(A, \lambda)$, so that, $c(x, A) \geqslant \lambda = p_\theta(\alpha, x)$. The significance of this equality is the following: the probability that x is in the closure of A is equal to the least upper bound of the probability that α to converges to x, when $\alpha \in \mathscr{X}$, $\alpha \subseteq A$.

EXAMPLES 3.7. By Theorem 3.4, we can associate to every left-continuous p.t.s. on X a probabilistic convergence class and, from Theorem 3.3, a probabilistic convergence function.

(a) Let $(X, \mathscr{T})$ be a topological space and let $(\bar{\ })$ be the $\mathscr{T}$-closure on this space. We define $\theta: \mathscr{P}(X) \times I \to \mathscr{P}(X)$ letting $\theta(A, \lambda) = \bar{A}$ if $\lambda > 0$ and let $\theta(\cdot, 0)$ be the indiscrete closure on X. Then (θ, T) is a left-continuous p.t.s. on X, where $T(\lambda, \mu) = 1$ if $\lambda \neq 0$ and $\mu \neq 0$ and $T(\lambda, \mu) = 0$ if $\lambda = 0$ or $\mu = 0$ (see [8]). Let $\mathscr{C}^\theta = \{\mathscr{C}^\theta_\lambda\}_{\lambda \in I}$ and p_θ be the associate probabilistic convergence class and the probabilistic convergence function. Then, for every $\lambda \neq 0$, $\mathscr{C}^\theta_\lambda$ is the convergence class of $(\bar{\ })$, $\mathscr{C}^\theta_0 = \mathscr{X} \times X$, and $p_\theta(\alpha, x) = 1$ if α converges to x in $(X, \mathscr{T})$ and $p_\theta(\alpha, x) = 0$ if α does not converge to x in $(X, \mathscr{T})$.

(b) Let (F, T) be a Menger metric on X, where T is a left-continuous t-norm, and let (θ, T) be the left-continuous p.t.s. induced by this Menger metric ((θ, T) is induced by the probabilistic uniformity induced by (F, T)). Then, for every $\lambda \in I$:

(1) $(\alpha, x) \in \mathscr{C}^\theta_\lambda$ iff, for every $a > 0$, $\underline{\lim}_d F_{\alpha(d)x}(a) \geqslant \lambda$, and

(2) $p_\theta(\alpha, x) = \lim_{a \to 0} \underline{\lim}_d F_{\alpha(d)x}(a)$,

for every $\alpha \in \mathscr{X}, x \in X$, where $\underline{\lim}_d F_{\alpha(d)x}(a)$ is the lower limit of the real net $(F_{\alpha(d)x}(a))_{d \in D}$.

Proof. Let $(\mathscr{U}, T)$ be the probabilistic uniformity induced by (F, T), where $\mathscr{U} = \{\mathscr{U}_\lambda\}_{\lambda \in I}$. We know that, for every $\lambda \in (0, 1]$, $\mathscr{B}_\lambda = \{U_{a,\mu}: a > 0, \mu < \lambda\}$ is a base for the semiuniformity $\mathscr{U}_\lambda$, where $U_{a,\mu} = \{(x, y): F_{xy}(a) > \mu\}$. Let $\alpha: D \to X$ be a net; then $\alpha \xrightarrow{\lambda} x$ iff for every $a > 0$ and $\mu < \lambda$ there exists $d_0 \in D$ such that, for every $d \geqslant d_0$, we have $F_{\alpha(d)x}(a) > \mu$. Hence $\alpha \xrightarrow{\lambda} x$ iff for every $a > 0$ and $\mu < \lambda$ $\sup_{d_0 \in D} \inf_{d \geqslant d_0} F_{\alpha(d)x}(a) = \underline{\lim}_d F_{\alpha(d)x}(a) > \mu$. The proof of (1) follows from the remark that $(\alpha, x) \in \mathscr{C}_\lambda^\theta$ iff $\alpha \xrightarrow{\lambda} x$ (see Remark 3.6(2)). Now, $p_\theta(\alpha, x) = \sup\{\lambda \in I: \alpha \xrightarrow{\lambda} x\}$ (see Remark 3.6(1)), hence, from (1), $p_\theta(\alpha, x) = \sup\{\lambda: \inf_{a>0} \underline{\lim}_d F_{\alpha(d)x}(a) \geqslant \lambda\} = \lim_{a \to 0} \underline{\lim}_d F_{\alpha(d)x}(a)$ because $F_{xy}(\cdot)$ is a nondecreasing mapping).

4. Sequential probabilistic convergence structures

Let us denote by N the set of natural numbers and let $\mathscr{X}_0$ be the set of all sequences on X; for every $\alpha \in \mathscr{X}_0$, let $\mathscr{X}_\alpha$ be the set of all subsequences of α. We remark that $\mathscr{X}_\alpha^* = \mathscr{X}_\alpha$ in the case of sequences.

DEFINITION A_0. Let T be a t-function and let $p: \mathscr{X}_0 \times X \to I$ be a mapping; p is a sequential probabilistic convergence function (s.p.c.f.) on X relating to T if the following conditions are satisfied:

($P1_0$) $p(\alpha, x) = 1$, for every $x \in X$ and $\alpha \in \mathscr{X}_0$ with $\alpha \subseteq \{x\}$,

($P2_0$) $p(\alpha, x) \leqslant \inf\{p(\beta, x): \beta \in \mathscr{X}_\alpha\}$, for every $\alpha \in \mathscr{X}_0, x \in X$,

($P3_0$) $p(\alpha, x) \geqslant \inf\{\sup\{p(\gamma, x): \gamma \in \mathscr{X}_\beta\}: \beta \in \mathscr{X}_\alpha\}$, for every $\alpha \in \mathscr{X}_0, x \in X$,

($P4_0$) For every $x \in X, \alpha \in \mathscr{X}_0$, $(\beta^n)_{n \in N} \subseteq \mathscr{X}_0$ there exists a strictly increasing sequence of natural numbers $(k_n)_{n \in N}$ such that $p(\delta, x) \geqslant T(p(\alpha, x), \inf_{n \in N} p(\beta^n, \alpha(n)))$, where $\delta \in \mathscr{X}_0$, $\delta(n) = \beta^n(k_n)$, for every $n \in N$.

DEFINITION B_0. Let $\mathscr{C} = \{\mathscr{C}_\lambda: \lambda \in I\}$ where $\mathscr{C}_\lambda \subseteq \mathscr{X}_0 \times X$, for every $\lambda \in I$, and let T be a t-function; $\mathscr{C}$ is a sequential probabilistic convergence class (s.p.c.c.) on X relating to T if:

($C1_0$) $\mathscr{C}_\lambda$ is a sequential relation satisfying the following:

(a) $(\alpha, x) \in \mathscr{C}_\lambda$, for every $x \in X$ and $\alpha \in \mathscr{X}_0$ with $\alpha \subseteq \{x\}$,

(b) if $(\alpha, x) \in \mathscr{C}_\lambda$ and $\beta \in \mathscr{X}_\alpha$ then $(\beta, x) \in \mathscr{C}_\lambda$,

(c) if $(\alpha, x) \notin \mathscr{C}_\lambda$ then there exists $\beta \in \mathscr{X}_\alpha$ such that, for every $\gamma \in \mathscr{X}_\beta$, $(\gamma, x) \notin \mathscr{C}_\lambda$, (C2$_0$) $\mathscr{C}_0 = \mathscr{X}_0 \times X$,

(C3$_0$) $\mathscr{C}_\lambda = \bigcap_{\mu < \lambda} \mathscr{C}_\mu$, for every $\lambda \in (0, 1]$,

(C4$_0$) For every $x \in X$, $\alpha \in \mathscr{X}_0$, $(\beta^n)_{n \in N} \subseteq \mathscr{X}_0$ with $(\alpha, x) \in \mathscr{C}_\lambda$ and $(\beta^n, \alpha(n)) \in \mathscr{C}_\mu$, for every $n \in N$, there exists a strictly increasing sequence of natural numbers $(k_n)_{n \in N}$ such that $(\delta, x) \in \mathscr{C}_{T(\lambda,\mu)}$, where $\delta(n) = \beta^n(k_n)$, for every $n \in N$.

Definitions A_0 and B_0 are equivalent too; indeed the mapping $\mathscr{C} \to p$, where $p: \mathscr{X}_0 \times X \to I$, $p(\alpha, x) = \sup\{\lambda \in I: (\alpha, x) \in \mathscr{C}_\lambda\}$, is a bijection between the set of all s.p.c.c. and the set of all s.p.c.f. on X. The proof is analogous to the proof of Theorem 3.3. and we omit it.

We say that a p.t.s. (θ, T) satisfies the first axiom of countability (or that (θ, T) is of a countable local character) if, for every $\lambda \in I$ and for every $x \in X$, there exists a countable local base at x in the closure space $(X, \theta(\cdot, \lambda))$.

We now give the sequential variant of Theorem 3.4.

THEOREM 4.1. *Let (θ, T) be a left-continuous p.t.s. on X satisfying the first axiom of countability; for every $\lambda \in I$, let $\mathscr{C}^\theta_\lambda = \{(\alpha, x) \in \mathscr{X}_0 \times X: \alpha \xrightarrow{\lambda} x\}$ and let $\mathscr{C}^\theta = \{\mathscr{C}^\theta_\lambda: \lambda \in I\}$. Then $\mathscr{C}^\theta$ is a s.p.c.c. on X relating to T.*

Proof. For every $\lambda \in I$, $\mathscr{C}^\theta_\lambda$ is the sequential convergence class of the space $(X, \theta(\cdot, \lambda))$. Hence $\mathscr{C}^\theta$ satisfies (C1$_0$) (see [2, 35 B.2 and 35 B.6]). For (C2$_0$) and (C3$_0$) we have the same proof as in Theorem 3.4. Now, let $x \in X$, $\alpha \in \mathscr{X}_0$, $(\beta^n)_{n \in N} \subseteq \mathscr{X}_0$ and $\lambda, \mu \in I$ with $(\alpha, x) \in \mathscr{C}^\theta_\lambda$ and $(\beta^n, \alpha(n)) \in \mathscr{C}^\theta_\mu$, for every $n \in N$. Because $\alpha \xrightarrow{\lambda} x$, there exists a monotone local base $\{V_n\}_{n \in N}$ for $\mathscr{V}_\lambda(x)$ with $\alpha(n) \in V_n$ for each $n \in N$ (see [2, Exercise 5 (Section 15)]). Let $\{W_n\}_{n \in N}$ be a monotone local base for $\mathscr{V}_{T(\lambda,\mu)}(x)$ with $W_0 = X$. Let us denote by $\mathrm{Int}_\lambda A = X - \theta(X - A, \lambda)$ the interior of A in $(X, \theta(\cdot, \lambda))$. We remark that, for every $n \in N$, there exists a $k \in N$ such that $V_n \subseteq \mathrm{Int}_\mu W_k$ (at least $k = 0$) and if $V_n \subseteq \mathrm{Int}_\mu W_k$ then, for every $i \leqslant k$, $V_n \subseteq \mathrm{Int}_\mu W_i$. Now, for every $n \in N$, let $U'_n = W_n$, if $V_n \subseteq \bigcap_{n=1}^\infty \mathrm{Int}_\mu W_k$ and $U'_n = W_k$, if $V_n \subseteq \mathrm{Int}_\mu W_k$ and $V_n \nsubseteq \mathrm{Int}_\mu W_{k+1}$ and let $U_n = \bigcap_{k=1}^n U'_k$. Clearly, $U_n \in \mathscr{V}_{T(\lambda,\mu)}(x)$ and $V_n \subseteq \mathrm{Int}_\mu U_n$, for every $n \in N$. For every $U \in \mathscr{V}_{T(\lambda,\mu)}(x)$, there exists $W_p \subseteq U$; from $W_p \in \mathscr{V}_{T(\lambda,\mu)}(x)$ and the condition (N4) of Proposition 2.2, there exists $V \in \mathscr{V}_\lambda(x)$ such that, for every $y \in V$, $W_p \in \mathscr{V}_\mu(y)$. Hence $V \subseteq \mathrm{Int}_\mu W_p$. Because $\{V_n\}_{n \in N}$ is a local base for $\mathscr{V}_\lambda(x)$, there exists $q \in N$ such that $V_q \subseteq V$, hence $V_q \subseteq \mathrm{Int}_\mu W_p$. It follows that $U_q \subseteq U'_q \subseteq W_p \subseteq U$. Hence $\{U_n\}_{n \in N}$ is a monotone local base for $\mathscr{V}_{T(\lambda,\mu)}(x)$ and $V_n \subseteq \mathrm{Int}_\mu U_n$, for every $n \in N$. For every $n \in N$, $\alpha(n) \in V_n \subseteq \mathrm{Int}_\mu U_n$; hence $U_n \in \mathscr{V}_\mu(\alpha(n))$. But $\beta^n \xrightarrow{\mu} \alpha(n)$, so that there exists $k_n \in N$ such that $\beta^n(k_n) \in U_n$. Clearly, we can choose $(k_n)_{n \in N}$ to be a strictly increasing sequence. If $\delta(n) = \beta^n(k_n)$, for every $n \in N$, then $\delta \xrightarrow{T(\lambda,\mu)} x$; hence $(\delta, x) \in \mathscr{C}^\theta_{T(\lambda,\mu)}$.

Conversely, we have

THEOREM 4.2. *If $\mathscr{C}$ is a s.p.c.c. on X relating to the t-function T, where $\mathscr{C} = \{\mathscr{C}_\lambda\}_{\lambda \in I}$, and we define $\theta: \mathscr{P}(X) \times I \to \mathscr{P}(X)$ letting $\theta(A, \lambda) = \{x \in X$: there exists $\alpha \in \mathscr{X}_0, \alpha \subseteq A$ with $(\alpha, x) \in \mathscr{C}_\lambda\}$, for every $A \subseteq X$ and $\lambda \in I$, then (θ, T) is a p.t.s. on X and $\mathscr{C}_\lambda \subseteq \mathscr{C}_\lambda^\theta$, for every $\lambda \in I$ ($\mathscr{C}_\lambda^\theta$ is the sequential convergence class of $(X, \theta(\cdot\,, \lambda))$).*

Proof. For every $\lambda \in I$, $\theta(\cdot\,, \lambda)$ is a closure on X. (T2) is a consequence of $(C2_0)$ and (T3) is a consequence of $(C3_0)$. For every λ, $\mu \in I$, $A \subseteq X$, $A \neq \emptyset$, and $x \in \theta(\theta(A, \mu), \lambda)$ there exists $\alpha \in \mathscr{X}_0, \alpha \subseteq \theta(A, \mu)$ such that $(\alpha, x) \in \mathscr{C}_\lambda$. For every $n \in N$, $\alpha(n) \in \theta(A, \mu)$; hence there exists a $\beta^n \in \mathscr{X}_0, \beta^n \subseteq A$, such that $(\beta^n, \alpha(n)) \in \mathscr{C}_\mu$. By $(C4_0)$ there exists a strictly increasing sequence of natural numbers $(k_n)_{n \in N}$ such that $(\delta, x) \in \mathscr{C}_{T(\lambda,\mu)}$, where $\delta(n) = \beta^n(k_n)$, for every $n \in N$. Moreover, $\delta \subseteq A$, so that $x \in \theta(A, T(\lambda, \mu))$ and we have (T4). Now let $\lambda \in I$ and let $\mathscr{C}_\lambda^\theta$ be the sequential convergence class of the space $(X, \theta(\cdot\,, \lambda))$. If $(\alpha, x) \notin \mathscr{C}_\lambda^\theta$ then there exist $V \in \mathscr{V}_\lambda(x)$ and $\beta \in \mathscr{X}_\alpha$ such that $\beta \subseteq X - V$. From $x \notin \theta(X - V, \lambda)$, we have that $(\beta, x) \notin \mathscr{C}_\lambda$ and from $(C1_0)$ (b), $(\alpha, x) \notin \mathscr{C}_\lambda$.

REMARK 4.3. If $\mathscr{C}_\lambda$ is a sequential structure, single-valued at constant sequences, then $(X, \theta(\cdot\,, \lambda))$ is a semi-separated space and $\mathscr{C}_\lambda = \mathscr{C}_\lambda^\theta$ (see [2, 35 B.9]). The condition that $\mathscr{C}_\lambda$ be single-valued at each constant sequence means that, for every $x, y \in X$, if $\alpha \in \mathscr{X}_0, \alpha \subseteq \{x\}$ and $\alpha \xrightarrow{\lambda} y$ then $x = y$. In the case of Menger space (X, F, T) this means that, if $F_{xy}(0+) \geqslant \lambda$, then $x = y$ (see Example 3.7(b)); in particular, if X is an E-space (see Section 6) then this condition means that, if $\mu(\{t \in \Omega$: $x(t) = y(t)\}) \geqslant \lambda$, then $x = y$, where $(\Omega, \mathscr{K}, \mu)$ is a probability space and X is the quotient space of the space of random variables on Ω relating to the equality μ-almost everywhere.

5. Probabilistic convergence structures of Menger spaces

Let (X, F, T) be a Menger space under the left-continuous t-norm T. Then, for every $\lambda \in (0, 1]$, $\mathscr{B}_\lambda^0 = \{U_{1/n,\lambda - 1/n}: n \in N^*\}$ is a countable base for the semiuniformity $\mathscr{U}_\lambda$ (see 2.). Hence, the p.t.s. (θ, T) induced by $(\mathscr{U}, T)$ satisfies the first axiom of countability. Now let $\mathscr{C}$ be the s.p.c.c. relating to T ($\mathscr{C} = \{\mathscr{C}_\lambda: \lambda \in I\}$) and p be the s.p.c.f. on X induced by (θ, T) (see Theorem 4.1). Then, for every $\lambda \in I$,

(1) $(\alpha, x) \in \mathscr{C}_\lambda$ iff, for every $a > 0$, $\underline{\lim}_n F_{\alpha(n)x}(a) \geqslant \lambda$, and

(2) $p(\alpha, x) = \lim_{a \to 0} \underline{\lim}_n F_{\alpha(n)x}(a)$, for every $\alpha \in \mathscr{X}_0$, $x \in X$, (see Example 3.7(b)).

We remark that $p(\alpha, x) = 1$ iff $(\alpha, x) \in \mathscr{C}_1$, hence iff $\alpha \xrightarrow{1} x$; but this is equivalent to the convergence of α to x in the uniformity defined by Schweizer and Sklar [11]. Therefore, $\alpha \xrightarrow{1} x$ iff, for every $a > 0$, $\lim_n F_{\alpha(n)x}(a) = 1$.

In the following we give two theorems concerning "the continuity" of a probabilistic metric.

THEOREM 5.1. *Let (X, F, T) be a Menger space and let p be the s.p.c.f. on this space; then, for every $\alpha, \beta \in \mathscr{X}_0$, $x, y \in X$ and $a > 0$:*

$$\underline{\lim}_n F_{\alpha(n)\beta(n)}(a) \geqslant T(F_{xy}(a), T(p(\alpha, x), p(\beta, y))).$$

Proof. For every $\eta > 0$ there exist $v < T(p(\alpha, x), p(\beta, y))$ and $\varepsilon > 0$ such that $T(F_{xy}(a), T(p(\alpha, x), p(\beta, y))) < T(F_{xy}(a) - \varepsilon, v) + \eta$ (T is left-continuous at the point $(F_{xy}(a), T(p(\alpha, x), p(\beta, y)))$). Because T is left-continuous at $(p(\alpha, x), p(\beta, y))$, there exist $\lambda < p(\alpha, x)$ and $\mu < p(\beta, y)$ such that $v < T(\lambda, \mu)$. Because F_{xy} is left-continuous at $a > 0$ there exists $\delta > 0$ such that $F_{xy}(a) - F_{xy}(a - 2\delta) < \varepsilon$. Now, from $\lambda < p(\alpha, x)$ and $\mu < p(\beta, y)$, there exists $n_0 \in N$ such that, for every $n \geqslant n_0$, $F_{\alpha(n)x}(\delta) > \lambda$ and $F_{\beta(n)y}(\delta) > \mu$. Then for every $n \geqslant n_0$ we have

$$\begin{aligned} F_{\alpha(n)\beta(n)}(a) &\geqslant T(F_{\alpha(n)y}(a - \delta), F_{y\beta(n)}(\delta)) \geqslant T(T(F_{\alpha(n)x}(\delta), F_{xy}(a - 2\delta)), F_{\beta(n)y}(\delta)) \\ &= T(F_{xy}(a - 2\delta), T(F_{\alpha(n)x}(\delta), F_{\beta(n)y}(\delta))) \geqslant T(F_{xy}(a) - \varepsilon, T(\lambda, \mu)) \\ &\geqslant T(F_{xy}(a) - \varepsilon, v) > T(F_{xy}(a), T(p(\alpha, x), p(\beta, y))) - \eta. \end{aligned}$$

Hence $\underline{\lim}_n F_{\alpha(n)\beta(n)}(a) \geqslant T(F_{xy}(a), T(p(\alpha, x), p(\beta, y))) - \eta$ for every $\eta > 0$ and so we have the desired proof.

REMARK 5.2. If $p(\alpha, x) = p(\beta, y) = 1$ (hence if $\alpha \xrightarrow{1} x$ and $\beta \xrightarrow{1} y$) then we have $\underline{\lim}_n F_{\alpha(n)\beta(n)}(a) \geqslant F_{xy}(a)$, for every $a > 0$. Therefore, Theorem 5.1 is a generalization of Theorem 8.1 of [11].

COROLLARY 5.3. *For every $\alpha \in \mathscr{X}_0$, $x, y \in X$ and $a > 0$ we have*

$$\underline{\lim}_n F_{\alpha(n)y}(a) \geqslant T(F_{xy}(a), p(\alpha, x)).$$

Proof. We remark that, if $\beta \subseteq \{y\}$, then $p(\beta, y) = 1$.

THEOREM 5.4. *Let (X, F, T_m) be a Menger space, where $T_m : I \times I \to I$ is defined by $T_m(\lambda, \mu) = \max(\lambda + \mu - 1, 0)$, for every $\lambda, \mu \in I$; then for every $\alpha, \beta \in \mathscr{X}_0$, $x, y \in X$ we have*

$$T_m(\overline{\lim_n} F_{\alpha(n)\beta(n)}(a), T_m(p(\alpha, x), p(\beta, y))) \leqslant F_{xy}(a),$$

at all continuity points $a > 0$ of $F_{xy}(\cdot)$.

Proof. Let $\eta > 0$; from the continuity of $F_{xy}(\cdot)$ at a, there exists $\delta > 0$ such that $F_{xy}(a + 2\delta) - F_{xy}(a) < \eta/3$. From the left-continuity of T_m at $(F_{xy}(a), T_m(p(\alpha, x), p(\beta, y)))$, there exists $v < T_m(p(\alpha, x), p(\beta, y))$ such that

$$T_m(F_{xy}(a), T_m(p(\alpha, x), p(\beta, y))) < T_m(F_{xy}(a), v) + \eta/3. \tag{1}$$

Because T_m is left-continuous at $(p(\alpha, x), p(\beta, y))$, there exist $\lambda < p(\alpha, x)$ and $\mu < p(\beta, y)$ such that $v < T_m(\lambda, \mu)$; hence there exists an $n_0 \in N$ such that, for every $n \geqslant n_0$, $F_{\alpha(n)x}(\delta) > \lambda$ and $F_{\beta(n)y}(\delta) > \mu$. Now we have for every $n \geqslant n_0$

$$\begin{aligned} F_{xy}(a) + \eta/3 > F_{xy}(a + 2\delta) &\geqslant T_m(F_{x\beta(n)}(a + \delta), F_{\beta(n)y}(\delta)) \\ &\geqslant T_m(T_m(F_{\alpha(n)\beta(n)}(a), F_{\alpha(n)x}(\delta)), F_{\beta(n)y}(\delta)) \\ &= T_m(F_{\alpha(n)\beta(n)}(a), T_m(F_{\alpha(n)x}(\delta), F_{\beta(n)y}(\delta))) \\ &\geqslant T_m(F_{\alpha(n)\beta(n)}(a), T_m(\lambda, \mu)) > T_m(F_{\alpha(n)\beta(n)}(a), v) \\ &\geqslant F_{\alpha(n)\beta(n)}(a) + v - 1. \end{aligned} \tag{2}$$

(a) If $F_{xy}(a) + T_m(p(\alpha, x), p(\beta, y)) \geqslant 1$ then, from (1),

$$\begin{aligned} 0 &\leqslant F_{xy}(a) + T_m(p(\alpha, x), p(\beta, y)) - 1 < T_m(F_{xy}(a), v) + \eta/3 \\ &= F_{xy}(a) + v - 1 + \eta/3; \end{aligned}$$

hence

$$v - 1 > T_m(p(\alpha, x), p(\beta, y)) - \eta/3 - 1. \tag{3}$$

Thus, from (2) and (3), $F_{xy}(a) + \eta/3 > F_{\alpha(n)\beta(n)}(a) + T_m(p(\alpha, x), p(\beta, y)) - \eta/3 - 1$ or $F_{\alpha(n)\beta(n)}(a) + T_m(p(\alpha, x), p(\beta, y)) - 1 < F_{xy}(a) + 2\eta/3$ for every $n \geqslant n_0$ and we obtain the proof of the theorem.

(b) If $F_{xy}(a) + T_m(p(\alpha, x), p(\beta, y)) < 1$ then for $\lambda = p(\alpha, x) - \eta/3$ and $\mu = p(\beta, y) - \eta/3$ there exists an $n_1 \in N$ such that, for every $n \geqslant n_1$, $F_{\alpha(n)x}(\delta) > \lambda$ and $F_{\beta(n)y}(\delta) > \mu$. So, similarly to (2), we obtain

$$\begin{aligned} F_{xy}(a) + \eta/3 \geqslant T_m(F_{\alpha(n)\beta(n)}(a), T_m(\lambda, \mu)) &\geqslant F_{\alpha(n)\beta(n)}(a) + \lambda + \mu - 2 \\ &= F_{\alpha(n)\beta(n)}(a) + p(\alpha, x) + p(\beta, y) - 2\eta/3 - 2, \end{aligned}$$

for every $n \geqslant n_1$ and this gives the desired proof.

REMARK 5.5. If $p(\alpha, x) = p(\beta, y) = 1$ then we have, from Corollary 5.3 and Theorem 5.4, $\overline{\lim}_n F_{\alpha(n)\beta(n)}(a) \leqslant F_{xy}(a) \leqslant \underline{\lim}_n F_{\alpha(n)\beta(n)}(a)$, hence we obtain the continuity of $F_{..}(a)$ at (x, y). Therefore Theorem 5.4 generalizes Theorem 8.2 of [11].

COROLLARY 5.6. *For every $\alpha \in \mathscr{X}_0$, $x, y \in X$ and for all continuity points $a > 0$ of $F_{xy}(\cdot)$, we have*

$$\overline{\lim}_n F_{\alpha(n)y}(a) + p(\alpha, x) - 1 \leqslant F_{xy}(a).$$

REMARK 5.7. From Corollaries 5.3 and 5.6 we obtain $p(\alpha, x) - 1 \leqslant \underline{\lim}_n F_{\alpha(n)y}(a) - F_{xy}(a) \leqslant \overline{\lim}_n F_{\alpha(n)y}(a) - F_{xy}(a) \leqslant 1 - p(\alpha, x)$, for every $x, y \in X$, $\alpha \in \mathscr{X}_0$ and for all continuity points $a > 0$ of $F_{xy}(\cdot)$. Hence $F_{\alpha(.)y}(a)$ converges to $F_{xy}(a)$ faster than α to x, and this for every sequence α and for every $x \in X$.

6. Probabilistic convergence structure of E-spaces

Let $(\Omega, \mathscr{K}, \mu)$ be a space with a probability measure μ on $\mathscr{K}$, a σ-algebra of subsets of Ω. Let (Y, d) be a separable metric space and let $\mathscr{M}$ be the family of all $\mathscr{K} - \mathscr{B}_d$-measurable mappings of Ω in Y ($\mathscr{B}_d$ is the σ-algebra of the Borel subsets of (Y, d)). For every $x, y \in \mathscr{M}$ and $a > 0$ let us write $A_{xy}(a) = \{\omega \in \Omega \colon d(x(\omega), y(\omega)) < a\}$. Because (Y, d) is a separable space, it follows that $A_{xy}(a) \in \mathscr{K}$ for every $x, y \in \mathscr{M}$ and $a > 0$.

Let $\doteq$ be the equality μ-almost everywhere and let X be the quotient space $\mathscr{M}/\doteq$. One can easily check that, if $x' \doteq y'$ and $x'' \doteq y''$, then $\mu(A_{x'y'}(a)) = \mu(A_{x''y''}(a))$, for every $a > 0$. Hence we can define a mapping $F\colon X \times X \to \Delta$, letting $F_{xy}(a) = \mu(A_{xy}(a))$, for every $a > 0$, and $F_{xy}(a) = 0$, for every $a \leqslant 0$, where Δ is the set of the distribution functions. (X, F, T_m) is a Menger space relating to the t-norm T_m (T_m has been defined in Theorem 5.4). This Menger space is called the E-space relating to (Y, d) (see [14]).

Let $(\mathscr{U}, T_m)$ be the probabilistic uniformity induced by (F, T_m). We recall that $\mathscr{U} = \{\mathscr{U}_\lambda\}_{\lambda \in I}$, where, for every $\lambda \in (0, 1]$, $\mathscr{B}_\lambda = \{U_{a,\gamma} \colon a > 0, \gamma < \lambda\}$ is a base for the semiuniformity $\mathscr{U}_\lambda$ and $\mathscr{U}_0 = \{X \times X\}$, and $U_{a,\gamma} = \{(x, y) \in X \times X \colon F_{xy}(a) > \gamma\}$ for every $a > 0$ and $\gamma \in [0, 1)$. For every $\alpha \in \mathscr{X}_0$ and $x \in X$, $\alpha \xrightarrow{\lambda} x$ (α is λ-Cauchy) means that α converges to x in $(X, \mathscr{U}_\lambda)$ (α is a Cauchy sequence in $(X, \mathscr{U}_\lambda)$). We recall that $\alpha \xrightarrow{\lambda} x$ iff $p(\alpha, x) = \lim_{a \to 0} \underline{\lim}_n F_{\alpha(n)x}(a) \geqslant \lambda$ (see Remark 3.6 (2)). Similarly, we can prove that α is λ-Cauchy iff $\lim_{a \to 0} \underline{\lim}_{m,n} F_{\alpha(m)\alpha(n)}(a) \geqslant \lambda$. Let $c\colon \mathscr{X}_0 \to I$ be a mapping defined by $c(\alpha) = \sup\{\lambda \in I \colon \alpha \text{ is } \lambda\text{-Cauchy}\} = \lim_{a \to 0} \underline{\lim}_{m,n} F_{\alpha(m)\alpha(n)}(a)$. Then $c(\alpha)$ gives "the probability for α to be a Cauchy sequence".

We remark that $\alpha \xrightarrow{1} x$ iff α converges in probability to x, and α is 1-Cauchy iff α is Cauchy in probability. Therefore, if (Y, d) is a complete metric space, then $\mathscr{U}_1$ is a complete uniformity.

PROPOSITION 6.1. *For every* $\alpha = (x_n)_{n \in N} \in \mathscr{X}_0$ *and* $x \in X$ *we have*

(i) $\mu(\{\omega \in \Omega\colon x_n(\omega) \to x(\omega)\}) \leqslant \sup\{\lambda \in I\colon \alpha \xrightarrow{\lambda} x\}$,

(ii) $\mu\{\omega \in \Omega\colon (x_n(\omega))_{n \in N}$ *is d-Cauchy*$\}) \leqslant \sup\{\lambda \in I\colon \alpha$ *is* λ*-Cauchy*$\}$.

Proof. (i) Let $A_{x_n x}(a) = \{\omega \in \Omega\colon d(x_n(\omega), x(\omega)) < a\}$. We remark that

$$A = \{\omega \in \Omega\colon x_n(\omega) \to x(\omega)\} = \bigcap_{a > 0} \bigcup_{n \in N} \bigcap_{m \geqslant n} A_{x_m x}(a).$$

Therefore

$$\mu(A) = \mu\left(\bigcap_{a > 0} \underline{\lim}_n A_{x_n x}(a)\right) = \lim_{a \to 0} \mu\left(\underline{\lim}_n A_{x_n x}(a)\right),$$

because $\underline{\lim}_n A_{x_n x}(a_1) \subseteq \underline{\lim}_n A_{x_n x}(a_2)$, for every $a_1 \leqslant a_2$. Hence

$$\mu(A) \leqslant \lim_{a \to 0} \underline{\lim}_n \mu(A_{x_n x}(a)) = \lim_{a \to 0} \underline{\lim}_n F_{x_n x}(a) = p(\alpha, x) = \sup\{\lambda\colon \alpha \xrightarrow{\lambda} x\}$$

(see Remark 3.6 (1)). The proof is similar for (ii).

REMARK 6.2. If $\alpha = (x_n)_{n \in N} \in \mathscr{X}_0$ converges almost everywhere (a.e.) to x (α is Cauchy a.e.) then $\alpha \xrightarrow{1} x$ (α is 1-Cauchy); hence α converges in probability to x (α is Cauchy in probability) and so we rediscover a classical result.

In the following lemma we give some properties of the mappings p and c.

LEMMA 6.3. *Let* $\alpha \in \mathscr{X}_0$ *and let* $\mathscr{X}_\alpha$ *be the set of all subsequences of* α *and* $x, y \in X$. *Then*

(a) *(i)* $|p(\alpha, x) - p(\alpha, y)| \leqslant \mu(\{\omega\colon x(\omega) \neq y(\omega)\})$ *and*

(ii) $T_m(p(\alpha, x), p(\alpha, y)) \leqslant \mu(\{\omega\colon x(\omega) = y(\omega)\})$.

(b) *(i)* $T_m(p(\alpha, x), p(\alpha, x)) \leqslant c(\alpha)$ *and*

(ii) $T_m(p(\beta, x), c(\alpha)) \leqslant p(\alpha, x)$, *for every* $\beta \in \mathscr{X}_\alpha$.

Proof. Let $\alpha = (x_n)_{n \in N}$. (a) (i) For every $n \in N$ and $a > 0$, $\{\omega\colon x(\omega) = y(\omega)\} \cap A_{x_n x}(a) \subseteq A_{x_n y}(a)$. Therefore, $\mu(\{\omega\colon x(\omega) = y(\omega)\}) + \mu(A_{x_n x}(a)) - 1 \leqslant \mu(A_{x_n y}(a))$;

hence $\mu(\{\omega: x(\omega) = y(\omega)\}) + p(\alpha, x) - 1 \leqslant p(\alpha, y)$, and so $p(\alpha, x) - p(\alpha, y) \leqslant \mu(\{\omega: x(\omega) \neq y(\omega)\})$. (ii) For every $n \in N$ and $a > 0$, we have $A_{x_n x}(a/2) \cap A_{x_n y}(a/2) \subseteq A_{xy}(a)$, so that $\underline{\lim}_n \mu(A_{x_n x}(a/2)) + \underline{\lim}_n \mu(A_{x_n y}(a/2)) - 1 \leqslant \underline{\lim}_n [\mu(A_{x_n x}(a/2)) + \mu(A_{x_n y}(a/2)) - 1] \leqslant \underline{\lim}_n \mu[A_{x_n x}(a/2) \cap A_{x_n y}(a/2)] \leqslant \mu(A_{xy}(a))$. When $a \to 0$, we obtain $T_m(p(\alpha, x), p(\alpha, y)) \leqslant \mu(\{\omega: x(\omega) = y(\omega)\})$. (b) (i) For every $n, m \in N$ and $a > 0$, we have $A_{x_n x}(a/2) \cap A_{x_m x}(a/2) \subseteq A_{x_m x_n}(a)$. Proceeding as above, we obtain the result. (ii) Let $\beta = (x_{k_n})_{n \in N}$ be a subsequence of α; for every $n \in N$ and $a > 0$ we have $A_{x_{k_n} x}(a/2) \cap A_{x_{k_n} x_m}(a/2) \subseteq A_{x_m x}(a)$. Therefore,

$$\underline{\lim}_{n} \mu(A_{x_{k_n} x}(a/2)) + \underline{\lim}_{m,n} \mu(A_{x_m x_n}(a/2)) - 1 \leqslant \underline{\lim}_{n} \mu(A_{x_{k_n} x}(a/2))$$

$$+ \underline{\lim}_{m,n} \mu(A_{x_{k_n} x_m}(a/2)) - 1 \leqslant \underline{\lim}_{n} \mu(A_{x_n x}(a)).$$

The proof follows immediately by letting $a \to 0$.

REMARKS 6.4. From (a) we obtain

(i) If $x \doteq y$ then $p(\alpha, x) = p(\alpha, y)$, for every $\alpha \in \mathscr{X}$,

(ii) If α converges in probability to x and y then $x \doteq y$, and

(iii) If $p(\alpha, x) = 0$ then $p(\alpha, y) \leqslant \mu(\{\omega: x(\omega) \neq y(\omega)\})$,
If $p(\alpha, x) = 1$ then $p(\alpha, y) \leqslant \mu(\{\omega: x(\omega) = y(\omega)\})$.

From (b) we have

(i) If α converges in probability to x then α is Cauchy in probability, and

(ii) If β converges in probability to x and α is Cauchy in probability then α converges in probability to x.

REMARK 6.5. For every $(x_n)_{n \in N} \in \mathscr{X}_0$ and $a > 0$, we have

$$\underline{\lim}_{m,n} \mu(A_{x_m x_n}(a)) \leqslant \mu\left(\overline{\lim_{\substack{m,n \\ m \neq n}}} A_{x_m x_n}(a)\right),$$

hence

$$c((x_n)_{n \in N}) \leqslant \lim_{a \to 0} \mu\left(\overline{\lim_{\substack{m,n \\ m \neq n}}} A_{x_m x_n}(a)\right).$$

Here

$$\overline{\lim_{\substack{m,n \\ m \neq n}}} A_{x_m x_n}(a) = \bigcap_{p=1}^{\infty} \bigcup_{m > n \geqslant p} A_{x_m x_n}(a).$$

The following theorem is a generalization of a well-known result of F. Riesz.

THEOREM 6.6. *Let (X, F, T_m) be an E-space relating to the separable complete metric space (Y, d). Let $(x_n)_{n\in N} \in \mathscr{X}_0$ be a sequence which satisfies the condition*:

$$c((x_n)_{n\in N}) = \lim_{a\to 0} \mu\left(\overline{\lim_{\substack{m,n\\ m\neq n}}} A_{x_m x_n}(a)\right). \qquad (*)$$

Then there exist a subsequence $(x_{k_n})_{n\in N}$ of $(x_n)_{n\in N}$ and $x \in X$ such that

$$c((x_n)_{n\in N}) \leqslant \mu(\{\omega: x_{k_n}(\omega) \to x(\omega)\}).$$

Proof. Let

$$L = c((x_n)_{n\in N}) = \lim_{a\to 0} \underline{\lim}_{m,n}\, \mu(A_{x_m x_n}(a))$$

and, from $(*)$,

$$L = \lim_{a\to 0} \mu\left(\overline{\lim_{\substack{m,n\\ m\neq n}}} A_{x_m x_n}(a)\right).$$

For every $a > 0$ there exists a $b_a \leqslant a$ such that:

$$(1)\ \ L - a < \sup_n \inf_{p,q\geqslant n} \mu(A_{x_p x_q}(b_a)) \leqslant \inf_n \mu\left(\bigcup_{p>q\geqslant n} A_{x_p x_q}(b_a)\right) < L + a.$$

Now there exists an $n(a) \in N$ such that, for every $p > q \geqslant n(a)$,

$$(2)\ \ L - a < \mu(A_{x_p x_q}(b_a)) \leqslant \mu\left(\bigcup_{p>q\geqslant n(a)} A_{x_p x_q}(b_a)\right) < L + a.$$

Therefore, for every integer $k \geqslant 1$ there exist a $b(k) \leqslant 1/2^k$ and an integer $N(k) \in N$ such that, for all $q \geqslant N(k)$ and $p > q$, the inequalities

$$(3)\ \ L - 1/2^{k-1} < \mu(A_{x_p x_q}(b(k))) \leqslant \mu\left(\bigcup_{p>q\geqslant N(k)} A_{x_p x_q}(b(k))\right) < L + 1/2^{k-1}$$

hold. Let

$$k_1 = N(1), k_2 = \max(k_1 + 1, N(2)), \ldots, k_n = \max(k_{n-1} + 1, N(n)), \ldots$$

Clearly, $k_n \uparrow +\infty$ as $n \to \infty$, so that $(x_{k_n})_{n\in N}$ is a subsequence of $(x_n)_{n\in N}$. Now we show that $(x_{k_n})_{n\in N}$ has the required property. For convenience we write $A_n = A_{x_{k_{n+1}}x_{k_n}}(b(n))$, $B_n = \bigcup_{p>q\geqslant k_n} A_{x_p x_q}(b(n))$, for every $n \in N$. Because $k_{n+1} > k_n \geqslant N(n)$, we have

(4) $L - 1/2^{n-1} < \mu(A_n) \leqslant \mu(B_n) < L + 1/2^{n-1}$.

So, from (4) we obtain

(5) $A_n \subseteq B_n$ and $\mu(B_n - A_n) = \mu(B_n) - \mu(A_n) < 1/2^{n-2}$,

for every $n \geqslant 2$, and

(6) $B_{n+1} \subseteq B_n$

for every $n \geqslant 1$ and, if $B_0 = \bigcap_{n=1}^{\infty} B_n$, then $\mu(B_0) = L$. Now, let $A_0 = \underline{\lim}_n A_n$; hence $A_0 \subseteq B_0$ and, from (5),

$$\mu(B_0 - A_0) = \lim_k \mu\left(\bigcup_{n\geqslant k} (B_0 - A_n)\right) \leqslant \lim_k \sum_{n=k}^{\infty} \mu(B_0 - A_n)$$
$$\leqslant \lim_k \sum_{n=k}^{\infty} \mu(B_n - A_n) \leqslant \lim_k \sum_{n=k}^{\infty} 1/2^{n-2} = 0.$$

It follows from (6) that $\mu(A_0) = L$. For every $\omega \in A_0$, there exists $n_0 \in N$ such that, for every $n \geqslant n_0$, $d(x_{k_{n+1}}(\omega), x_{k_n}(\omega)) < b(n)$. Let $a > 0$ and choose $n_1 \in N$ such that $n_1 \geqslant n_0$ and $1/2^{n_1-2} < a$. Then, for every $m \geqslant n \geqslant n_1$, we have

$$d(x_{k_n}(\omega), x_{k_m}(\omega)) \leqslant d(x_{k_n}(\omega), x_{k_{n+1}}(\omega)) + \cdots + d(x_{k_{m-1}}(\omega), x_{k_m}(\omega))$$
$$< b(n) + \cdots + b(m-1) \leqslant \sum_{j=n_1}^{\infty} 1/2^{j-1} = 1/2^{n_1-2} < a.$$

Hence the sequence $(x_{k_n}(\omega))_{n\in N}$ is Cauchy in the complete metric space (Y, d). Let us consider the function $x: \Omega \to Y$, defined by $x(\omega) = \lim_n x_{k_n}(\omega)$, if $\omega \in A_0$ and $x(\omega) = y_0$, if $\omega \in \Omega - A_0$, where y_0 is a fixed point in Y. Then $x \in X$, and we have $\mu(\{\omega \in \Omega: x_{k_n}(\omega) \to x(\omega)\}) \geqslant \mu(A_0) = L = c((x_n)_{n\in N}$.

REMARK 6.7. We know that $(x_n)_{n\in N}$ is Cauchy in probability iff $(x_n)_{n\in N}$ is 1-Cauchy. Hence, if $(x_n)_{n\in N}$ is Cauchy in probability then (*) holds (see Remark 6.5), and, from the previous theorem, we obtain that there exists a subsequence $(x_{k_n})_{n\in N}$ of $(x_n)_{n\in N}$ which converges almost everywhere.

COROLLARY 6.8. *Let $\alpha = (x_n)_{n \in N}$ be a sequence which satisfies the condition* (*) *of Theorem* 6.6. *Then there exists an $x \in X$ such that*

$$T_m(p(\alpha, x), p(\alpha, x)) \leqslant c(\alpha) \quad \text{and} \quad T_m(c(\alpha), c(\alpha)) \leqslant p(\alpha, x). \tag{**}$$

Proof. From Theorem 6.6, there exist a subsequence $\beta = (x_{k_n})_{n \in N}$ of α and $x \in X$ such that $c(\alpha) \leqslant \mu(\{\omega : x_{k_n}(\omega) \to x(\omega)\})$. From Proposition 6.1(i), $c(\alpha) \leqslant p(\beta, x)$, and, from Lemma 6.3(b)(ii), $T_m(c(\alpha), c(\alpha)) \leqslant T_m(p(\beta, x), c(\alpha)) \leqslant p(\alpha, x)$. The first condition of (**) is Lemma 6.3(b)(i).

Therefore (*) implies (**); the following example shows that the converse is false.

EXAMPLE 6.9. Let t be a real number in [0, 1], and let $b \geqslant 2$ be an integer; then t has a unique b-adic expansion, that is, an expansion of the form $t = \sum_{k=1}^{\infty} a_k(t)/b^k$, where the "digits" $a_k(t)$ are integers with $0 \leqslant a_k(t) < b$, for every $k \geqslant 1$, and also $a_k(t) < b - 1$ for infinitely many k. We define $x_n : [0, 1] \to R$ by $x_n(t) = a_n(t)$. Then $([0, 1], \mathscr{B}, \mu)$ is a space with a probability (μ is the Lebesgue measure on [0, 1]), and $\alpha = (x_n)_{n \in N}$ is a sequence in the induced E-space relating to the separable complete metric space R with the usual metric. For every $n < m$ and $0 < a \leqslant 1$, we have

$$\begin{aligned} A_{x_n x_m}(a) &= \{t : |x_n(t) - x_m(t)| < a\} = \{t : x_n(t) = x_m(t)\} \\ &= \bigcup_{k=0}^{b-1} \{t : a_n(t) = k = a_m(t)\}. \end{aligned}$$

But

$$\begin{aligned} \{t : a_n(t) = k = a_m(t)\} &= \{t : a_n(t) = k\} \cap \{t : a_m(t) = k\} \\ &= \bigcup_{\substack{a_i' \in \{0,1,\ldots,b-1\} \\ i = 1,2,\ldots,n-1}} [0, a_1'a_2' \ldots a_{n-1}'k; 0, a_1'a_2' \ldots a_{n-1}'k + 1) \\ &\cap \bigcup_{\substack{a_j'' \in \{0,1,\ldots,b-1\} \\ j = 1,2,\ldots,m-1}} [0, a_1''a_2'' \ldots a_{m-1}''k; 0, a_1''a_2'' \ldots a_{m-1}''k + 1) \\ &= \bigcup_{\substack{a_i' \in \{0,1,\ldots,b-1\},\ i = 1,2,\ldots,n-1 \\ a_j'' \in \{0,1,\ldots,b-1\},\ j = 1,2,\ldots,m-1}} \{[0, a_1' \ldots a_{n-1}'k; 0, a_1' \ldots a_{n-1}'k + 1) \\ &\cap [0, a_1'' \ldots a_{m-1}''k; 0, a_1'' \ldots a_{m-1}''k + 1)\} \\ &= \bigcup_{\substack{a_i' \in \{0,1,\ldots,b-1\} \\ i = 1,2,\ldots,n-1}} \bigcup_{\substack{a_j'' \in \{0,1,\ldots,b-1\} \\ j = n+1,\ldots,m-1}} [0, a_1' \ldots a_{n-1}'ka_{n+1}'' \ldots a_{m-1}''k; \\ &\quad 0, a_1' \ldots a_{n-1}'ka_{n+1}'' \ldots a_{m-1}''k + 1). \end{aligned}$$

Therefore $\mu(\{t: a_n(t) = k = a_m(t)\}) = b^{n-1} \cdot b^{m-n-1} \cdot 1/b^m = 1/b^2$, so that $\mu(A_{x_m x_n}(a)) = b \cdot 1/b^2 = 1/b$. We obtain that

$$c(\alpha) = \lim_{a \to 0} \underline{\lim}_{m,n} \mu(A_{x_m x_n}(a)) = 1/b.$$

Now, let $\beta = (x_{k_n})_{n \in N}$ be a subsequence of α; then $(x_{k_n}(t))_{n \in N}$ is convergent if and only if there exists $n_0 \in N$ such that, for every $n \geqslant n_0$, $x_{k_n}(t) = x_{k_{n_0}}(t)$. Let $A_n^i = \{t: a_n(t) = i\}$, for every $n \in N$ and $i = 0, 1, \ldots, b-1$; now, $(x_{k_n}(t))_{n \in N}$ converges iff there exist $i \in \{0, 1, \ldots, b-1\}$ and $n_0 \in N$ such that, for every $n \geqslant n_0$, $a_{k_n}(t) = i$, or $t \in A_{k_n}^i$. Therefore

$$\mu(\{t: (x_{k_n}(t))_{n \in N} \text{ converges}\}) = \mu\left(\bigcup_{i=0}^{b-1} \bigcup_{n \in N} \bigcap_{p \geqslant n} A_{k_p}^i\right) = \sum_{i=0}^{b-1} \lim_n \mu\left(\bigcap_{p \geqslant n} A_{k_p}^i\right)$$

$$\leqslant \sum_{i=0}^{b-1} \lim_n \mu(A_{k_n}^i \cap \ldots \cap A_{k_{2n}}^i)$$

$$= \sum_{i=0}^{b-1} \lim_n \mu(\{t: a_{k_n}(t) = a_{k_{n+1}}(t) = \cdots = a_{k_{2n}}(t) = i\}).$$

But

$$A = \{t: a_{k_n}(t) = a_{k_{n+1}}(t) = \cdots = a_{k_{2n}}(t) = i\}$$

$$= \{t: t = 0, a_1 \ldots a_{k_n - 1} i a_{k_n + 1} \ldots a_{k_{2n} - 1} i a_{k_{2n} + 1} \ldots\};$$

hence $\mu(A) = b^{k_n - 1} \cdot b^{k_{n+1} - k_n - 1} \ldots b^{k_{2n} - k_{2n-1} - 1} \cdot 1/b^{k_{2n}} = 1/b^{n+1}$, so that $\mu(\{t: (x_{k_n}(t))_{n \in N} \text{ converges}\}) = 0$. It follows that α does not satisfy (*) (see Theorem 6.6.).

Now, let $x^i: [0, 1] \to \{0, 1, \ldots, b-1\}$ defined by $x^i(t) = i$, for every $t \in [0, 1]$ and $i \in \{0, 1, \ldots, b-1\}$. For every $0 < a \leqslant 1$, we have $A_{x_n x^i}(a) = \{t: a_n(t) = i\}$ so that $p(\alpha, x^i) = \underline{\lim}_n \mu(A_n^i) = 1/b$. Therefore $T_m(c(\alpha), c(\alpha)) = T_m(1/b, 1/b) = 0 < 1/b = p(\alpha, x^i)$, hence α satisfies (**).

We remark that, for $i \neq j$, $x^i(t) \neq x^j(t)$, for every $t \in [0, 1]$ and $p(\alpha, x^i) = p(\alpha, x^j)$.

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REFERENCES

[1] ANTHONY, J. M., SHERWOOD, H. and TAYLOR, M., *Sequential convergence and probabilistic metric spaces*. Rev. Roumaine Math. Pures Appl. *19* (1974), 1177–1187.
[2] CECH, E., *Topological spaces*. Czechoslovak Academy of Sciences, Prague, 1966.
[3] DROSSOS, C. A., *Stochastic Menger spaces and convergence in probability*. Rev. Roumaine Math. Pures Appl. *22* (1977), 1069–1076.
[4] FLORESCU, L. C., *Structures syntopogènes probabilistes*. Publ. Math. Debrecen *28* (1981), 15–24.
[5] FLORESCU, L. C., *Uniformities with a dilatation function*. An. Ştiint. Univ. "Al. I. Cuza" Iaşi, Sect. I a Mat. (N.S.) *27* (1981), 48–53.
[6] FLORESCU, L. C., *Probabilistic pseudometrics*. Mathematica (Cluj) *24* (1982), 21–29.
[7] FLORESCU, L. C., *Convergence and completeness in probabilistic spaces* (Romanian). Proc. Nat. Conf. on Geometry and Topology, P. Neamt, June 1983. Univ. Al. I. Cuza, Iaşi, 1984, pp. 243–247.
[8] FRANK, M. J., *Probabilistic topological spaces*. J. Math. Anal. Appl. *34* (1971), 67–81.
[9] MENGER, K., *Statistical metrics*. Proc. Nat. Acad. Sci. U.S.A. *28* (1942), 535–537.
[10] MUSHTARI, L. D. KH., *On the completion of random metric spaces*. Kazan. Gos. Univ. Učen. Zap. *127* (1967), 109–119.
[11] SCHWEIZER, B. and SKLAR, A., *Statistical metric spaces*. Pacific J. Math. *10* (1960), 313–334.
[12] SCHWEIZER, B., SKLAR, A. and THORP, E., *The metrization of statistical metric spaces*. Pacific J. Math. *10* (1960), 673–675.
[13] SHERWOOD, H., *On the completion of probabilistic metric spaces*. Z. Wahrsch. verw. Geb. *6* (1966), 62–64.
[14] SHERWOOD, H., *On E-spaces and their relation to other classes of probabilistic metric spaces*. J. London Math. Soc. *44* (1969), 441–448.
[15] TARDIFF, R. M., *Topologies for probabilistic metric spaces*. Pacific J. Math. *65* (1976), 233–251.
[16] THORP, E., A *Generalized topologies for statistical metric spaces*. Fund. Math. *51* (1962), 9–21.

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