Improved Bounds for the Disk-packing Constant

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In a previous paper [3], to which we refer the reader for definitions and notations, we presented an algorithm for computing the disk-packing constant S. Because of the relatively slow convergence of the process, the practical consequence of the method was to show that

$$1.272441 < S < 1.350000. \tag{1}$$

A lower bound of 1.28467 which we had obtained by an earlier ad hoc method [2] was thus superior to that obtained by the systematic algorithm. Here, we will give an improvement of certain inequalities used in [3], which combined with the basic algorithm of [3] allow us to show that

$$1.300197 < S < 1.314534. \tag{2}$$

This result lends considerable weight to the heuristic estimate $S \approx 1.306951$, obtained by Melzak [6].

The improvement is a result of some new inequalities involving the disk-packing function $M(a, b, c; t)$. Our principal new result is that M is a strictly convex function of (a, b, c). This, combined with the fact that M is a symmetric function of these three variables and an auxiliary lemma, allows us to show that

$$\left(a + \frac{b+c}{2}\right)^{-t} M(0, 1, 1; t) \leqslant M(a, b, c; t) \leqslant \tfrac{1}{2}((a+b)^{-t} + c^{-t})\, M(0, 1, 1; t). \tag{3}$$

These inequalities replace similar (but weaker) results used in [3] and give the improvement from (1) to (2).

1. Preliminaries

We use the notation of [2] and [3] for the most part. Let $T(a, b, c)$ be the region bounded by three mutually tangent circles of curvatures a, b, c, where $a, b, c \geqslant 0$, and at most one of a, b, c equals zero. (We do not insist, as in [3], that $a \leqslant b \leqslant c$). Let $\{r_n\}$ be the sequence of radii of the disks in a simple osculatory packing of $T(a, b, c)$, and define, for real t,

$$M(a, b, c; t) = \sum_{n=1}^{\infty} r_n^t. \tag{4}$$

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We shall omit the variable t whenever convenient. The disk-packing constant S is defined by

$$\left.\begin{aligned} S &= \inf\{t : M(a, b, c; t) < \infty\} \\ &= \sup\{t : M(a, b, c; t) = \infty\}, \end{aligned}\right\} \qquad (5)$$

and is independent of the curvatures a, b, c (See [7]).

The osculatory packing of $T(a, b, c)$ can be described in the following way (as in [2], or [4]): In $T(a, b, c)$ inscribe a circle with curvature s. In the figure so formed there are three curvilinear triangles called the first generation of triangles. In each of these we inscribe a circle, the first generation of circles. Continuing in this way, we inscribe, at the nth step, 3^n circles, (the n th generation) in 3^n curvilinear triangles. The nth generation can be indexed by the set $G_n = \{1, 2, 3\}^n$, and if G_0 consists of a single element which is a vector with no components, then each disk in the packing can be indexed by a vector $\alpha \in \bigcup_{n=0}^{\infty} G_n = G$. It is shown in [2] that if $\alpha = (i_1, \ldots, i_n) \in G_n$ and if $a(\alpha)$, $b(\alpha)$, $c(\alpha)$, $s(\alpha)$ denote the curvatures of the sides and inscribed circle, respectively, of the α-th triangle, then

$$(a(\alpha), b(\alpha), c(\alpha), s(\alpha)) = (a, b, c, s)\, P_{i_1} \ldots P_{i_n}, \qquad (6)$$

where

$$s = a + b + c + 2(ab + bc + ca)^{1/2}, \qquad (7)$$

and P_1, P_2, P_3 are matrices given by

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}. \qquad (8)$$

For the purposes of this paper, it will be convenient to introduce the variable $d = d(a, b, c) = (ab + bc + ca)^{1/2}$, so $d(\alpha) = d(a(\alpha), b(\alpha), c(\alpha))$, ($d$ is the curvature of the circle circumscribed to $T(a, b, c)$). Then $s(\alpha) = a(\alpha) + b(\alpha) + c(\alpha) + 2d(\alpha)$ and it is clear that (cf. [4], p.286)

$$(a(\alpha), b(\alpha), c(\alpha), d(\alpha)) = (a, b, c, d)\, S_{i_1} S_{i_2} \ldots S_{i_n} \qquad (9)$$

where

$$S_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}. \qquad (10)$$

For our purposes, the important fact to note is that the S_i are non-negative matrices with non-zero row sums, and thus

$$\left.\begin{aligned} s(\alpha) &= (a(\alpha), b(\alpha), c(\alpha), d(\alpha))\,(1,1,1,2)^T \\ &= (a, b, c, d)\, S_{i_1} \ldots S_{i_n} (1,1,1,2)^T \\ &= (a, b, c, d)\,(w_1(\alpha), w_2(\alpha), w_3(\alpha), w_4(\alpha))^T, \\ &= w_1(\alpha)\, a + w_2(\alpha)\, b + w_3(\alpha)\, c + w_4(\alpha)\, d, \end{aligned}\right\} \tag{11}$$

where the quantities $w_i(\alpha)$ are positive.

Thus we have

$$M(a, b, c; t) = \sum_{\alpha \in G} (w_1(\alpha)\, a + w_2(\alpha)\, b + w_3(\alpha)\, c + w_4(\alpha)\, d)^{-t}. \tag{12}$$

For the purpose of obtaining upper bounds on S, we need inequalities for sums of $s(\alpha)^{-t}$ over finite subsets $G' \subset G$. We shall let M' be a generic symbol for sums of the form

$$M'(a, b, c; t) = \sum_{\alpha \in G'} s(\alpha)^{-t}, \tag{13}$$

and we assume that G' is such that $M'(a, b, c; t)$ is symmetric in the three variables (a, b, c). The necessary and sufficient condition for this is that if $\alpha \in G'$ then the vectors obtained by replacing the first component of α by any of 1, 2 or 3 are also in G'. Since G is a union of finite subsets, any inequality for M' implies a similar inequality for M, but M' is finite for all real t, so a discussion of cases of equality in the inequalities can avoid certain trivialities.

2. Main Results concerning the disk-Packing Function

We begin with a result whose proof uses inversion.

LEMMA 1. *Let $M'(a, b, c; t)$ be defined as in* (13). *Then*

$$M'(a, b, c; t) \leqslant \tfrac{1}{2}(b^{-t} + c^{-t})\, M'(0, 1, 1; t) \tag{14}$$

for any real t. There is equality if and only if $b = c$ and $a = 0$.

Proof. It is clear from (13) and (11) that M' decreases in all variables and is homogeneous of degree $-t$, (see also [7]), and hence we have

$$M'(a, b, c) \leqslant M'(0, b, c) = b^{-t} M'(0, 1, c/b). \tag{15}$$

We may assume that $b \leqslant c$ since both sides of (14) are symmetric in b and c. If $b = c$, (14) reduces to (15), so we suppose $\gamma = b/c < 1$. We wish to invert $T(0, 1, 1)$ into

$T(0, 1, \gamma^{-1})$. Let the radius of the inverting circle be ϱ. Then we can choose coordinates with origin 0 at the centre of the inverting circle. Let C_1, C_2, C_3 be circles of radii γ, 1, 1 respectively, tangent to the real axis at the points $\varrho - 2\sqrt{\gamma}$, ϱ, $\varrho + 2$ respectively, so C_2 is tangent to C_1 and C_3. If we choose ϱ so $\varrho^2/(\varrho+2)^2 = \gamma$, then C_3 inverts into C_1, and C_2 and the real axis are left invariant. Call this inversion I.

The disks in the packing of $T(0, 1, 1)$, other than those with centres on $x = \varrho + 1$, occur in pairs of equal radii with centres at points $(\varrho + \varepsilon, y)$ and $(\varrho + 2 - \varepsilon, y)$, say, obtained by reflection across the line $x = \varrho + 1$. For a circle of radius r in the packing of $T(0, 1, 1)$, let r' be the radius of the circle obtained by applying I, and let r'' be the radius of the circle obtained by first reflecting across $x = \varrho + 1$, and then applying I. If the circle of radius r has centre $(\varrho + \varepsilon, y)$, then

$$r' = \varrho^2 \left[(\varrho + \varepsilon)^2 + y^2 - r^2 \right]^{-1} < \varrho^2 r (\varrho + \varepsilon)^{-2}, \quad \text{since} \quad y > r \tag{16}$$

and likewise

$$r'' < \varrho^2 r (\varrho + 2 - \varepsilon)^{-2}. \tag{17}$$

Thus, we have

$$\left.\begin{aligned} (r')^t + (r'')^t &< \varrho^{2t} r^t \left((\varrho + \varepsilon)^{-2t} + (\varrho + 2 - \varepsilon)^{-2t} \right) \\ &\leqslant \varrho^{2t} r^t \left(\varrho^{-2t} + (\varrho + 2)^{-2t} \right) = r^t (1 + \gamma^t). \end{aligned}\right\} \tag{18}$$

The second inequality results from the fact that the function of ε involved is decreasing for $0 < \varepsilon < 1$, and increasing for $1 < \varepsilon < 2$, hence has its maximum at 0 and 2.

Thus, using an obvious notation,

$$\left.\begin{aligned} 2M'(0, 1, \gamma^{-1}) &= \sum_{\alpha \in G'} (r'(\alpha))^t + \sum_{\alpha \in G} (r''(\alpha))^t \\ &\leqslant (1 + \gamma^t) \sum_{\alpha \in G'} r(\alpha)^t = (1 + \gamma^t)\, M'(0, 1, 1), \end{aligned}\right\} \tag{19}$$

which, combined with (15), proves (14). Strict equality holds in (15), for finite M, if $a > 0$, and in (18) if $\gamma < 1$.

THEOREM 1. *Let $M'(a, b, c; t)$ be defined as in* (13), *and let $t > 0$. Then M' is a strictly convex function of (a, b, c) in the region*

$$R = \{(a, b, c) : a, b, c \geqslant 0 \quad \text{and} \quad ab + bc + ca > 0\}.$$

Proof. Let $p_i = (a_i, b_i, c_i)$, $i = 1, 2, 3$ be points in R, with $p_1 \neq p_2$, such that $p_3 = up_1 + (1 - u) p_2$ for some u with $0 < u < 1$. Let $M_i' = M'(a_i, b_i, c_i; t)$. There are two cases to consider: (i) $p_2 = \lambda p_1$ for some $\lambda > 0$, (*ii*) $p_2 \neq \lambda p_1$ for any $\lambda > 0$.

In case (i),

$$M_3' = (u + (1-u)\lambda)^{-t} M_1' < uM_1' + (1-u)\lambda^{-t} M_1' = uM_1' + (1-u) M_2'.$$

For case (*ii*), we use equation (11), denoting $w_j(\alpha)$ by w_j for $j = 1, 2, 3, 4$. For $i = 1, 2, 3$, let

$$s_i(\alpha) = w_1 a_i + w_2 b_i + w_3 c_i + w_4 d(a_i, b_i, c_i). \tag{20}$$

Then, using the fact that $d(a, b, c) = \sqrt{ab + bc + ca}$ is a concave function of (a, b, c), (see [1], p.35 or [5]), we have

$$\left.\begin{aligned} &us_1(\alpha) + (1-u) s_2(\alpha) \\ &\quad = w_1 a_3 + w_2 b_3 + w_3 c_3 + w_4 \{ud(a_1, b_1, c_1) + (1-u) d(a_2, b_2, c_2)\} \\ &\quad < w_1 a_3 + w_2 b_3 + w_3 c_3 + w_4 d(a_3, b_3, c_3) = s_3(\alpha). \end{aligned}\right\} \tag{21}$$

(Strict inequality holds since $p_2 \neq \lambda p_1$). By Hölder's inequality (with exponents $-1/t$ and $1/(1+t)$), we have, from (21),

$$\left.\begin{aligned} &us_1(\alpha)^{-t} + (1-u) s_2(\alpha)^{-t} \\ &\qquad \geqslant (u + (1-u))^{t+1} (us_1(\alpha) + (1-u) s_2(\alpha))^{-t} > s_3(\alpha)^{-t}. \end{aligned}\right\} \tag{22}$$

Summing over $\alpha \in G'$, we have

$$M_3' < uM_1' + (1-u) M_2',$$

which proves the strict convexity of M' for $t > 0$.

Remark. It should be noted that we have made no use of special properties of the w_i other than $w_i \geqslant 0$.

COROLLARY 1. *Let $(a, b, c) \in R$, and suppose $(A, B, C) \in R$ is a linear combination, with non-negative coefficients, of the six points $(a, b, c), \ldots, (c, b, a)$ obtained by permuting a, b, c in all possible ways. Let $t > 0$. Then*

$$(A + B + C)^t M'(A, B, C) \leqslant (a + b + c)^t M'(a, b, c) \tag{23}$$

with strict inequality unless (A, B, C) is proportional to a permutation of (a, b, c).

Proof. This follows immediately from the facts that $M'(a, b, c)$ is a homogeneous function of degree $-t$ in (a, b, c), is symmetric in (a, b, c), and is stricly convex.

COROLLARY 2. *If* $(a, b, c) \in R$ *and* $t > 0$, *then*

$$M'(a, b, c) \leqslant \tfrac{1}{2}((a+b)^{-t} + c^{-t})\, M'(0, 1, 1). \tag{25}$$

Furthermore, if a, b, c *are equal to the side lengths of a triangle then*

$$M'(a, b, c) \leqslant \left(\frac{a+b+c}{2}\right)^{-t} M'(0, 1, 1). \tag{26}$$

If $a \leqslant b \leqslant c$, *strict inequality holds in* (25) *and* (26) *unless* $a = 0$ and $b = c$.

Proof. The point (a, b, c) is a convex combination of $(0, a+b, c)$ and $(a+b, 0, c)$. Hence Corollary 1 implies

$$M'(a, b, c) \leqslant M'(0, a+b, c). \tag{27}$$

The result (25) thus follows from Lemma 1.

If a, b, c are equal to the side lengths of a triangle, then $a = x + y$, $b = y + z$, $c = z + x$ where x, y and z are non-negative. Hence (26) is a special case of Corollary 1, since

$$(a, b, c) = x(1, 0, 1) + y(1, 1, 0) + z(0, 1, 1).$$

Remark. The inequality (25) is most effective when a, b, c are ordered so that $a \leqslant b \leqslant c$. The inequality (26) is better than (25) whenever it applies; however, in our applications, a, b and c always satisfy $a + b < c$.

COROLLARY 3. *If* $(a, b, c) \in R$, *and* $t > 0$, *then*

$$M'(a, b, c) \geqslant \left(a + \frac{b+c}{2}\right)^{-t} M'(0, 1, 1). \tag{28}$$

Strict inequality holds unless $a = 0$ *and* $b = c$.

Proof. By Corollary 1,

$$M'(a, b, c) \geqslant M'\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) \tag{29}$$

and, by Lemma 1(b) of [3], if $B \leqslant C$, then

$$M'(A, B, C) \geqslant (A + C)^{-t} M'(0, 1, 1). \tag{30}$$

Combining (29) with (30) yields (28).

One can bypass [3] by giving an alternate proof of Lemma 1(b) as follows: use induction on n to show $w_1(\alpha) \leqslant w_2(\alpha) + w_3(\alpha)$ for $\alpha \in G_n$.

Then, since $B \leqslant C$,

$$w_1 A + w_2 B + w_3 C + w_4 (AB + BC + CA)^{1/2} \leqslant (A + C)(w_2 + w_3 + w_4),$$

from which (30) follows immediately.

Remark. Corollary 1 gives many amusing inequalities as special cases. For example, for any $(a, b, c) \in R$,

$$\left(\frac{a+b+c}{3}\right)^{-t} M'(1, 1, 1) \leqslant M'(a, b, c). \tag{31}$$

3. The Disk-Packing Constant

As was shown in [3], inequalities such as (25) and (28) can be used to generate lower and upper bounds for S which converge to S. We refer the reader to [3] for a precise description of the process used. Briefly, suppose one has an inequality of the form

$$M'(a, b, c, t) \leqslant F(a, b, c; t)\, M'(0, 1, 1; t) \tag{32}$$

or

$$F(a, b, c; t)\, M'(0, 1, 1; t) \leqslant M'(a, b, c; t). \tag{33}$$

Then, for any number $\kappa > 0$, there is a constructively defined function $f(\kappa, F; t)$ for which the equation $f(\kappa, F; t) = 1$ has a unique solution $t = \nu(\kappa, F)$, and $1 < \nu(\kappa, F) < 2$. If F satisfies (32), then $S \leqslant \nu(\kappa, F)$ and if F satisfies (33), then $\nu(\kappa, F) \leqslant S$. If in addition, one assumes that

$$(a + c)^{-t} \leqslant F(a, b, c; t) \leqslant b^{-t}, \tag{34}$$

then

$$|S - \nu(\kappa, F)| < (\log 10)/(\log \kappa). \tag{35}$$

Computations have been carried out, based on the inequalities (14), (25) and (28). The inequality (26) is not useful here since it is not valid for the appropriate triples (a, b, c). The values of κ are the same as those used in [3], and the reason for these choices is given there. The computations were performed on an IBM 360/67 computer. The longest computation was that of $\nu(841, F)$ with F as in (28), which used 560 seconds of C.P.U. time.

The results are shown below in comparison with the results obtained in [3]. The rows headed S, A indicate the parameters for the least squares fit of a curve $S + A(\log \kappa)^{-1}$ to $\nu(K, F)$, and the row RMS gives the root mean square deviation of this approximation. It is clear from this that the rate of convergence is still very nearly $1/(\log \kappa)$, but the better inequalities (25) and (28) have greatly improved the starting values. Note, for example, that the second entry in the second column, which took 5 seconds of computation, is far better than the fourth entry in the fourth column which took 128 seconds of computation on an IBM 360/75.

Table of $\nu(\kappa, F)$ for various κ and F

$F(a, b, c; t)$	$\left(\frac{a+b+c}{2}\right)^{-t}$	$\frac{1}{2}((a+b)^{-t}+c^{-t}$	$\frac{1}{2}(b^{-t}+c^{-t})$	b^{-t}	$(a+c)^{-t}$
type of bound	lower	upper	upper	upper	lower
$\kappa =$ 4	1.282599	1.345722	1.384291	1.571658	1.191561
25	1.295224	1.322910	1.338060	1.410266	1.246116
144	1.298681	1.317054	1.326648	1.373234	1.263876
841	1.300197	1.314534	1.321906	1.357603	1.272441
S	1.304812	1.306138	1.304773	1.297644	1.291789
A	0.030796	0.054765	0.109870	0.377927	0.139798
RMS	4.05×10^{-5}	1.91×10^{-4}	6.14×10^{-4}	3.16×10^{-3}	1.36×10^{-3}

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