Distributive lattices with an additional unary operation

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Let $\mathcal{K}$ denote the class of algebras of the form $\langle L; \wedge, \vee, 0, 1, f\rangle$ where $\wedge$, $\vee$, 0, 1 behave as in distributive lattices with 0 and 1 and f is a unary operation satisfying

$$f(x \wedge y) = f(x) \vee f(y), \qquad f(x \vee y) = f(x) \wedge f(y), \tag{1}$$

$$f(0) = 1, \qquad f(1) = 0. \tag{2}$$

For $p \geq 1$ and $q \geq 0$ denote by $\mathcal{K}_{p,q}$ the equational subclass of $\mathcal{K}$ obtained by adjoining the equation

$$f^q(x) = f^{2p+q}(x). \tag{3}$$

The class $\mathcal{K}_{1,0}$ is the class of DeMorgan algebras and the subclass of $\mathcal{K}_{1,0}$ given by the equation $x \wedge f(x) = 0$ is the class of Boolean algebras. Stone algebras can be shown to be the subclass of $\mathcal{K}_{1,1}$ defined by $x \wedge f(x) = 0$.

In this paper algebraic consequences of equations (1), (2) and (3) are investigated. A key result is that principal congruence relations of an algebra L in $\mathcal{K}$ can be expressed as the join of principal congruence relations of the underlying distributive lattice of L. This is used to show that the congruence extension property holds for $\mathcal{K}$. This leads to a characterization of the standard semigroup of operators of $\mathcal{K}$ and $\mathcal{K}_{p,q}$. Equation (3) is shown to imply a number of finitary conditions. In particular, $\mathcal{K}_{p,q}$ is locally finite and has only a finite number of subdirectly irreducibles, all of which are finite.

Received May 15, 1974

Regular Polyhedra—Old and New

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In elementary geometry a *polyhedron* is traditionally defined as a family of polygons (called *faces* of the polyhedron) such that each edge of one of the faces is an edge of just one other face, and satisfying a few additional conditions (usually "connectedness" of the family of polygons, restrictions on the subfamilies that share a vertex, and—in case of infinite polyhedra—a local finiteness condition). Equally accepted is the analogous definition of a *polygon* as a "connected" (and locally finite) family of segments (the *edges* of the polygon) such that each edge shares each of its vertices with just one other edge.

It is surprising that the two definitions seem never to have been considered simultaneously, and that investigators of polyhedra have always treated polygons as "patches" of the plane or some other 2-manifold. This lack of consistency is particularly striking in the theory of regular polyhedra in which non-convex figures have been considered for many centuries; moreover, the families of the Petrie polygons of the regular polyhedra lead naturally to (regular) "polyhedra" in the "new" sense. The main aim of the present paper is to start a systematic investigation of regular polyhedra free from this inconsistency.

Among the many possible definitions of regularity we adopt a very elegant, useful and restrictive one (see, for example, H. S. M. Coxeter, *Regular Complex Polytopes*, Cambridge Univ. Press 1974): A polygon or a polyhedron is *regular* if its symmetries act transitively on its *flags*. (A vertex and an edge incident with it form a flag in a polygon, while a vertex, an edge and a face, all mutually incident, form a flag in a polyhedron.)

It is well known that the *regular polygons* in the Euclidean 3-space E^3 are of one of the following types: (i) convex polygons; (ii) star polygons; (iii) the apeirogon; (iv) zig-zag polygons; (v) skew polygons (prismatic and antiprismatic); (vi) helical polygons. (All except types (v) and (vi) are planar; types (iii), (iv) and (vi) are infinite.)

Traditionally recognized regular polyhedra in E^3 are: (a) the 5 Platonic polyhedra; (b) the 4 Kepler-Poinsot polyhedra; (c) the 3 regular tessellations of the plane; (d) the 3 infinite "skew" regular polyhedra of Petrie-Coxeter. To these we add the following "new" classes of regular polyhedra in E^3:

(e) Finite regular polyhedra with skew polygons as faces (9 types).

(f) Infinite regular polyhedra with skew polygons as faces (3 one-parameter families of types, and 3 other types).

(g) Regular polyhedra with zig-zag polygons as faces (6 one-parameter families of types).

(h) Regular polyhedra with helical polygons as faces (3 one-parameter families of types and 5 types not in those families).

While it is easily proved that no other types exist in class (e), the completeness of the list of known polyhedra in classes (f), (g) and (h) is only conjectured.

Polyhedra of classes (e) and (f) (or at least objects closely related to them) have been mentioned in the literature (see, for example, A. H. Schoen, Abstracts 658-30 and 68T-D6, Notices Amer. Math. Soc. *15* (1968), pp. 727 and 929; M.

Burt, Spatial Arrangement and polyhedra with curved surfaces and their architectural applications, M. Sc. thesis, Technion, Haifa 1966). On the other hand, the 1-skeleta of some of the polyhedra in class (h) seem to be of interest in structural chemistry (A. F. Wells, The geometric basis of crystal chemistry, I, II, Acta Cryst. 7 (1954), 535–554).

Received September 9, 1975.