Applications of Extensions of Additive Functions

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1. Introduction

The results of Z. Daróczy and L. Losonczi [5] on the extensions of additive functions seem to have important applications in the theory of functional equations.

We shall use the following definitions and results of Z. Daróczy and L. Losonczi.

DEFINITION 1.1. Let D be a set in the (x, y) plane and let be

$$\begin{aligned} D_x &= \{x \mid \exists y,\ (x, y) \in D\}, \\ D_y &= \{y \mid \exists x,\ (x, y) \in D\}, \\ D_{x+y} &= \{x + y \mid (x, y) \in D\}. \end{aligned}$$

The function f is additive on the set D if $f: D_f = D_x \cup D_y \cup D_{x+y} \to R$ satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{1}$$

for all $(x, y) \in D$.

DEFINITION 1.2. Let f be additive on a set D. If there exists a real function $F: R \to R$, such that

$$F(x) = f(x), \qquad x \in D_f$$

and F is additive for all $x, y \in R$, then the function F is called an extension of f.

THEOREM 1.1. *If f is additive on the set $H_r = \{(x, y) \mid x, y \geq 0,\ x + y < r\}$ or on the set $K_r = \{(x, y) \mid x^2 + y^2 < r^2\}$ where $r > 0$, then f has one and only one extension F.*

DEFINITION 1.3. Let f be additive on a set D. If there exists a real function $G: R \to R$ and a point $(u, v) \in D$, such that G is additive for all $x, y \in R$ and

$$\begin{aligned} f(x) - f(u) - f(v) &= G(x) - G(u) - G(v), && x \in D_{x+y} \\ f(x) - f(u) &= G(x) - G(u), && x \in D_x \\ f(x) - f(v) &= G(x) - G(v), && x \in D_y, \end{aligned}$$

then the function G is called a quasi-extension of f.

It is easy to see, that if f has a quasi-extension, then in definition 1.3. we can put

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an arbitrary point $(a, b) \in D$ instead of the point $(u, v) \in D$. In the case $(0, 0) \in D$ the notion of quasi-extension does not differ from the notion of the extension, since $f(0)=0$, $G(0)=0$.

THEOREM 1.2. *If f is additive on an open connected domain, then f has one and only one quasi-extension G.*

Using this theorem we can give the general solution of (1) on an open connected domain D, for the general form of G (or F) may be given in terms of a Hamel basis.

In this paper we shall deal with the following problems.

PROBLEM A. Let $f:(a, b) \to R$ be a real function satisfying the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \qquad x, y \in (a, b). \tag{2}$$

We wish to find the general solution of (2).

PROBLEM B. Let $f:(0, 1) \to R$ be a real function, which satisfies Hosszú's functional equation (see [2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [13], [14])

$$f(x + y - xy) + f(xy) = f(x) + f(y) \tag{3}$$

for all $x, y \in (0, 1)$. What is the general solution of (3) on the interval $(0, 1)$?

PROBLEM C. Let $f:[0, 1] \to R$ be a real function, which satisfies the functional equation (3) for all $x, y \in [0, 1]$. We will determine the general solution of (3) on the interval $[0, 1]$.

2. Problem A

LEMMA 1. *If the function $f:(a, b) \to R$ satisfies (2) for all $x, y \in (a, b)$, then the function*

$$A(x) = f\left(x + \frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right), \qquad A:\left(-\frac{b-a}{2}, \frac{b-a}{2}\right) \to R \tag{4}$$

satisfies the functional equation (1) for all $(x, y) \in D_1$, where

$$D_1 = \left\{(x, y) \mid x, y \in \left(-\frac{b-a}{2}, \frac{b-a}{2}\right), -x - \frac{b-a}{2} < y < -x + \frac{b-a}{2}\right\}.$$

Proof. Putting

$$x \to x + \frac{a+b}{2}, \qquad y \to y + \frac{a+b}{2}$$

in (2), we have

$$2f\left(\frac{x+y}{2}+\frac{a+b}{2}\right)=f\left(x+\frac{a+b}{2}\right)+f\left(y+\frac{a+b}{2}\right), \qquad x, y\in\left(-\frac{b-a}{2}, \frac{b-a}{2}\right). \tag{5}$$

Setting $y=0$ in (5) we obtain the identity

$$2f\left(\frac{x}{2}+\frac{a+b}{2}\right)=f\left(x+\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)$$

and putting this into (5) we get

$$f\left(x+y+\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)=f\left(x+\frac{a+b}{2}\right)+f\left(y+\frac{a+b}{2}\right),$$

where

$$x, y\in\left(-\frac{b-a}{2}, \frac{b-a}{2}\right)$$

and

$$a<x+y+\frac{a+b}{2}<b.$$

From this last equation we obtain, by (4), that the function

$$A:\left(-\frac{b-a}{2}, \frac{b-a}{2}\right)\to R$$

is additive for all $(x, y)\in D_1$ (see fig. 1.).

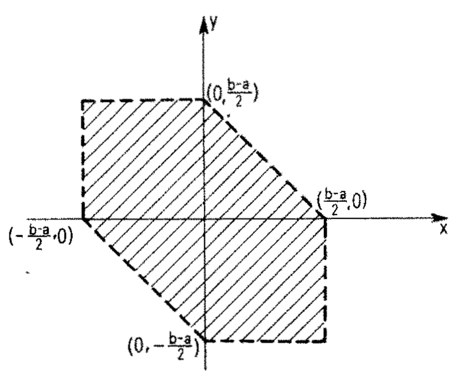

Fig. 1

THEOREM 1. *If* $f:(a, b)\to R$ *is a solution of equation* (2) *in the interval* (a, b), *then f has the form*

$$f(x)=\bar{A}\left(x-\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right),\qquad x\in(a, b), \tag{6}$$

where $\bar{A}$ *is the quasi extension (or extension) of the function A defined by* (4).

Proof. By Lemma 1 the function

$$A:\left(-\frac{b-a}{2},\frac{b-a}{2}\right)\to R,$$

defined by (4), is additive on the domain D_1. Thus by Theorem 1.2, A has one and only one quasi-extension $\bar{A}$ (now an extension) for which $A=\bar{A}$ on $(-(b-a)/2, (b-a)/2)$. Then (6) follows from (4). The general form of $\bar{A}$ may be given by Hamel basis.

It is easy to see that (6) satisfies (2). Therefore (6) is the general solution of (2) in the interval (a, b).

COROLLARY 1. *If* $f:(0, 1)\to R$ *is a solution of* (2) *in the interval* $(0, 1)$, *then f has the form*

$$f(x)=\bar{A}\left(x-\tfrac{1}{2}\right)+f\left(\tfrac{1}{2}\right),\qquad x\in(0, 1), \tag{7}$$

where $\bar{A}$ *is the quasi extension of the function A defined by* (4), *in which we now put* $a=0, b=1$.

3. Problem B

LEMMA 2. *If the function* $f:(0, 1)\to R$ *satisfies* (3) *for all* $x, y\in(0, 1)$, *then the function*

$$A(t)=f\left(t+\tfrac{1}{2}\right)-f\left(\tfrac{1}{2}\right),\qquad A:\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\to R \tag{8}$$

satisfies the functional equation (1) *for all* $(t, s)\in D$, *where*

$$D=\left\{(t, s)\mid -\tfrac{1}{2}<t<0, \tfrac{1}{2}+\frac{1}{2t-1}<s<\tfrac{1}{2}+t\right\}.$$

Proof. The function

$$F(x, y)=f(x)+f(y)-f(xy) \tag{9}$$

satisfies the equation

$$F(xy, z)+F(x, y)=F(x, yz)+F(y, z) \tag{10}$$

for all $x, y, z\in(0, 1)$. On the other hand we have

$$F(x, y)=f(x+y-xy).$$

Putting this into (10) we obtain the equation

$$f(xy+z-xyz)+f(x+y-xy)=f(x+yz-xyz)+f(y+z-yz) \quad (11)$$

for all $x, y, z \in (0, 1)$.

By the substitution

$$\left.\begin{aligned} t+\tfrac{1}{2} &= xy+z-xyz \\ s+\tfrac{1}{2} &= x+y-xy \\ \tfrac{1}{2} &= y+z-yz \end{aligned}\right\} \quad (12)$$

we obtain from (11) the functional equation

$$f(t+\tfrac{1}{2})+f(s+\tfrac{1}{2})=f(t+s+\tfrac{1}{2})+f(\tfrac{1}{2}),$$

which implies (1) for A on the domain (t, s) defined by (12). We shall show that this set is the set D defined in our Lemma 2..

By the substitution

$$z=\frac{\frac{1}{2}-y}{1-y}\,(0<y<\tfrac{1}{2})$$

we get from (12) the transformation

$$\left.\begin{aligned} t &= -\frac{y(1-x)}{2(1-y)} \\ s &= x+y-xy-\tfrac{1}{2} \end{aligned}\right\} ,\quad 0<y<\tfrac{1}{2},\quad 0<x<1. \quad (12')$$

The (x, y) domain of the transformation (12′) is the open rectangle $(0, 1)\ x\ (0, \frac{1}{2})$. The transformation (12′) is regular in this open rectangle, because the Jacobian of (12′) is

$$J=\begin{vmatrix} \dfrac{\partial t}{\partial x} & \dfrac{\partial t}{\partial y} \\ \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{y}{2(1-y)} & \dfrac{x-1}{2(1-y)^2} \\ 1-y & 1-x \end{vmatrix} = \frac{(1-x)(1+y)}{2(1-y)}>0.$$

It is well known that under a regular transformation the image of an open connected domain is again an open connected domain (see [15]).

The points (t, s) form an open connected domain in the plane. Now we are going to determine this domain.

The Jacobian J is positive on the closed rectangle $[0, x_0]\ x\ [0, \frac{1}{2}]$ for every $0<x_0<1$. The image of this closed rectangle is a closed domain and the boundary of this domain is the image of the boundary of the closed rectangle.

(a) *The image of the boundary* $x=0$ is the curve

$$\begin{cases} t = -\dfrac{y}{2(1-y)} \\ s = y - \frac{1}{2} \end{cases}, \quad 0 \leqslant y \leqslant \tfrac{1}{2},$$

i.e. it is the Hyperbola

$$s = \tfrac{1}{2} + \frac{1}{2t-1}, \quad (-\tfrac{1}{2} \leqslant t \leqslant 0).$$

(b) *The image of the boundary* $y=0$ is the line

$$\begin{cases} t = 0 \\ s = x - \frac{1}{2} \end{cases}, \quad (0 \leqslant x \leqslant x_0).$$

(c) *The image of the boundary* $y=\frac{1}{2}$ is the line

$$\begin{cases} t = \dfrac{x-1}{2} \\ s = \frac{1}{2}x \end{cases}, \quad (0 \leqslant x \leqslant x_0),$$

i.e. the line

$$s = t + \tfrac{1}{2}, \quad \left(-\tfrac{1}{2} \leqslant t \leqslant \frac{x_0 - 1}{2}\right).$$

(d) *The image of the boundary* $x=x_0$ is the Hyperbola

$$\begin{cases} t = -\dfrac{y(1-x_0)}{2(1-y)} \\ s = x_0(1-y) + y - \frac{1}{2} \end{cases}, \quad (0 < y < \tfrac{1}{2}, \quad 0 < x_0 < 1),$$

which can be written in the form

$$s = x_0 - \tfrac{1}{2} + \frac{2(1-x_0)\,t}{2t + x_0 - 1}, \quad -\tfrac{1}{2}(1-x_0) \leqslant t \leqslant 0.$$

If $x_0 \to 1$ then we get for the boundary of the image of the open rectangle $(0, 1)\,x\,(0, \frac{1}{2})$ the curves

$$s = \tfrac{1}{2} + \frac{1}{2t-1}, (-\tfrac{1}{2} \leqslant t \leqslant 0); \quad \left\{\begin{matrix} t = 0 \\ s = x - \frac{1}{2} \end{matrix}\right\}, (0 \leqslant x \leqslant 1); \quad s = t + \tfrac{1}{2}$$

(see fig. 2.).

Thus the image of the open rectangle $(0, 1)\,x\,(0, \frac{1}{2})$ is the set D.

Now it is easy to prove the following

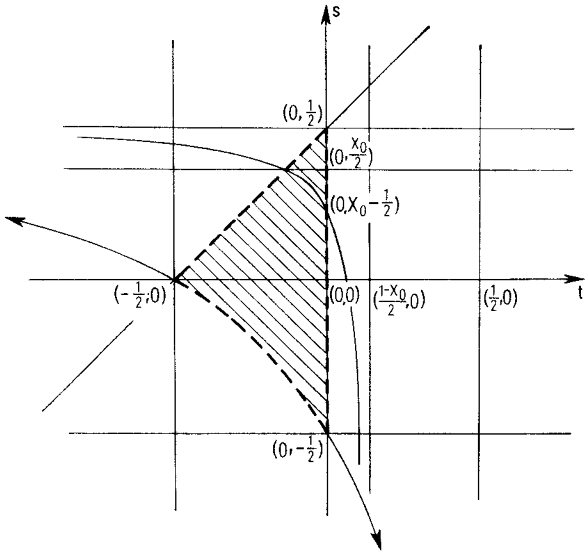

Fig. 2

THEOREM 2. *If* $f:(0, 1)\to R$ *is a solution of the functional equation* (3) *in the interval* (0.1), *then f has the form*

$$f(x)=\bar{A}(x-\tfrac{1}{2})+f(\tfrac{1}{2}),\qquad x\in(0,1),\tag{13}$$

where $\bar{A}$ *is the quasi-extension of the function A defined by* (8).

Proof. By Lemma 2 the function $A:(-\frac{1}{2},\frac{1}{2})\to R$ defined in (8) is additive on the domain D. So, by theorem 1.2, A has one and only one quasi extension $\bar{A}$ with $\bar{A}(x)=A(x)$ for all $x\in(-\frac{1}{2},\frac{1}{2})$. Thus from (8) it follows that f has the form (13).

The general form of $\bar{A}$ can be given by Hamel basis. It is easy to see that (13) indeed satisfies (3), and so (13) is the general solution of (3) on (0, 1).

From theorems 1 and 2 we can deduce the following

COROLLARY 2. *Let* $f:(0,1)\to R$. *Then the functional equation* (3) *and the Jensen equation* (2) *are equivalent on* (0, 1).

4. Problem C

THEOREM 3. *If* $f:[0, 1]\to R$ *is a solution of the functional equation* (3) *on the interval* [0, 1], *then f has the form*

$$f(x)=\begin{cases}a & ,\quad x=0\\ \bar{A}(x-\frac{1}{2})+f(\frac{1}{2}), & x\in(0,1)\\ b & ,\quad x=1\end{cases}$$

where $\bar{A}$ *is the quasi-extension of the function A defined by* (8).

Proof. If $x, y \in (0, 1)$ then xy and $x+y-xy \in (0, 1)$. On the other hand if $x=0$ or $x=1$ or $y=0$ or $y=1$ then (3) is reduced to identities. The value of f on $(0, 1)$ does not depend on $f(0)$ and $f(1)$ and vice versa.

From Theorem 2 we obtain for f the form (13) in the open interval $(0, 1)$. From the above remark it follows that $f(0)=a$ and $f(1)=b$ are arbitrary.

5. Remarks

1) From Theorem 2 it follows that under certain regularity conditions (f is continuous, Lebesgue-integrable (see [3]) or measurable (see [6]) in the interval $(0, 1)$) the general solution of (3) has the form

$$f(x) = Ax + B, \qquad x \in (0, 1),$$

where A and B are arbitrary constants.

2) From Theorem 3 it follows that the general Lebesgue-integrable or measurable solution of (3) in the interval $[0, 1]$ has the form

$$f(x) = \begin{cases} a \quad , & x = 0 \\ Ax + B, & x \in (0, 1) \\ b \quad , & x = 1 \end{cases}$$

where a, A, B, b are arbitrary constants.

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