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On the Sobolev distance of convex bodies

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Summary. Let $\mathscr{K}^d$ denote the cone of all convex bodies in the Euclidean space $\mathbb{E}^d$. The mapping $K \mapsto h_K$ of each body $K \in \mathscr{K}^d$ onto its support function induces a metric δ_w on $\mathscr{K}^d$ by $\delta_w(K, L) := \|h_L - h_K\|_w$ where $\|\cdot\|_w$ is the Sobolev 1-norm on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{E}^d$. We call $\delta_w(K, L)$ the Sobolev distance of K and L. The goal of our paper is to develop some fundamental properties of the Sobolev distance.

In Section 1 we derive, subsequent to basic facts, an estimate for $\delta_w(K, L)$ by the quermassintegrals W_{d-2}, W_{d-1} of K, L and the mixed volume $V(K, L, \mathbb{B}^d, \ldots, \mathbb{B}^d)$ ($\mathbb{B}^d$ = unit ball) under the assumption that the Steiner points of K and L coincide (Theorem 1). In the remaining sections we discuss the relationship between δ_w and the widely examined Hausdorff metric δ_∞. For the plane case (and only for this case) there exists a bound $\delta_\infty \leqslant C\delta_w$ with a universal constant $C > 0$. The best possible constant C is given by Theorem 2. We show that this constant is equal to the norm of the general Sobolev imbedding operator on the interval $[0, \pi]$ which was calculated by Marti [10]. Furthermore, the proof of Theorem 2 produces the smallest body $K_w \in \mathscr{K}^2$ which satisfies $\delta_\infty(K_w, \mathbb{B}^2) = C\delta_w(K_w, \mathbb{B}^2)$. We call K_w the *minimal body* of the Sobolev distance and establish a close connection between K_w and the Minkowski structure of $\mathscr{K}^2$ (Theorem 3).

It should be mentioned that Wellerding [16] applied the Sobolev distance to the problem of best approximation of a plane convex body L by the images σK of a convex body K under proper rigid motions σ of $\mathbb{E}^2$.

1. Let $\mathbb{E}^d$ ($d \geqslant 2$) be the d-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. $\mathbb{B}^d := \{u \in \mathbb{E}^d \mid |u| \leqslant 1\}$ is its unit ball with the

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d-dimensional volume

$$\omega_d := \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \qquad (\Gamma = \text{gamma function}).$$

If ω is the Lebesgue measure of the unit sphere $\mathbb{S}^{d-1} = \partial \mathbb{B}^d$ we have $\omega(\mathbb{S}^{d-1}) = d \cdot \omega_d$. By $\mathcal{K}^d$ we denote the *cone* of all convex bodies (nonempty, compact, convex point sets) in $\mathbb{E}^d$ provided with Minkowski addition and nonnegative scalar multiplication. An analytic representation of a body $K \in \mathcal{K}^d$ is given by its *support function* $H_K : \mathbb{E}^d \to \mathbb{R}$, $H_K(u) := \sup\{\langle u, v\rangle \mid v \in K\}$ which is positively homogeneous and convex. These properties imply that H_K is Lipschitz continuous and twice differentiable almost everywhere (a.e.) (see [1]). Let $\mathscr{C}(\mathbb{S}^{d-1})$ denote the real Banach space of all continuous functions $f: \mathbb{S}^{d-1} \to \mathbb{R}$ equipped with the uniform norm $\|\cdot\|_\infty$. From

$$H_{K+L} = H_K + H_L, \qquad H_{\lambda K} = \lambda H_K \quad (\lambda \geqslant 0)$$

it follows that the set of all restricted support functions $h_K = H_K \mid \mathbb{S}^{d-1}$ forms a positive cone in $\mathscr{C}(\mathbb{S}^{d-1})$. The vector space $\mathscr{V}(\mathbb{S}^{d-1})$ of all differences of support functions is dense in $\mathscr{C}(\mathbb{S}^{d-1})$.

The homogeneity of H_K yields that if H_K is differentiable at $x \in \mathbb{S}^{d-1}$, then

$$\operatorname{grad} H_K(x) = x \cdot h_K(x) + \operatorname{grad}_s h_K(x) \tag{1.1}$$

where grad_s is the gradient on $\mathbb{S}^{d-1}$. $\operatorname{grad} H_K(x)$ is the unique point of contact of K and the support hyperplane of K with normal x. By the properties of H_K (1.1) is valid a.e. on $\mathbb{S}^{d-1}$ and

$$\nabla_s(h_K, h_L) := \langle \operatorname{grad}_s h_K, \operatorname{grad}_s h_L \rangle \in \mathscr{L}^\infty(\mathbb{S}^{d-1}), \qquad K, L \in \mathcal{K}^d$$

where $\mathscr{L}^\infty(\mathbb{S}^{d-1})$ is the real vector space of all essentially bounded Lebesgue measurable functions on $\mathbb{S}^{d-1}$. Therefore, the inner product

$$(f \mid g)_w := \int_{\mathbb{S}^{d-1}} (fg + \nabla_s(f, g))\, d\omega_0, \qquad \omega_0 := \frac{1}{d \cdot \omega_d}\omega$$

is well defined on $\mathscr{V}(\mathbb{S}^{d-1})$. The *Sobolev* 1-*norm* $\|f\|_w := (f \mid f)_w^{1/2}$ on $\mathscr{V}(\mathbb{S}^{d-1})$ induces the *Sobolev distance* $\delta_w(K, L) := \|h_L - h_K\|_w$ of two convex bodies $K, L \in \mathcal{K}^d$. The pair $(\mathcal{K}^d, \delta_w)$ forms a metric space. The metric δ_w is congruence invariant and equivalent (but not uniformly equivalent) to the Hausdorff metric δ_∞ on $\mathscr{K}^d$ (compare [13]). Because of

$$\delta_w(K, L)^2 = \int_{\mathbb{S}^{d-1}} |\operatorname{grad}(H_L - H_K)(x)|^2 \, d\omega_0(x)$$

the Sobolev distance of K and L is a mean square of the Euclidean distance of related points grad $H_K(x)$ and grad $H_L(x)$ of contact. In this sense δ_w^2 can be decomposed into a normal and a tangential component by

$$\delta_w(K, L)^2 = \| h_L - h_K \|_2^2 + \int_{\mathbb{S}^{d-1}} \nabla_s (h_L - h_K) \, d\omega_0$$

where

$$(f|g) := \int_{\mathbb{S}^{d-1}} fg \, d\omega_0, \qquad \|f\|_2 := (f|f)^{1/2}, \qquad \nabla_s f := \nabla_s(f, f).$$

The normal component $\delta_2(K, L) := \|h_L - h_K\|_2$ is the usual $\mathscr{L}_2$-metric on $\mathscr{K}^d$ (see, for example, [2], [4], [8], [11], [15]). On the other hand, the Lipschitz continuity of $f \in \mathscr{V}(\mathbb{S}^{d-1})$ implies that f is a constant if $\nabla_s f$ vanishes a.e. Therefore, from $\nabla_s h_K = \nabla_s h_L$ a.e. it follows that $h_L = h_K + \text{const.}$, i.e., the parallelism of the convex bodies K and L. Hence,

$$\rho_2([K], [L]) := \left(\int_{\mathbb{S}^{d-1}} \nabla_s (h_L - h_K) \, d\omega_0 \right)^{1/2}$$

defines a metric on the set $\mathscr{P}^d := \{[K] \mid K \in \mathscr{K}^d\}$ of parallel classes of convex bodies and

$$\delta_w(K, L)^2 = \delta_2(K, L)^2 + \rho_2([K], [L])^2, \qquad K, L \in \mathscr{K}^d.$$

Furthermore, there is a close connection between the Sobolev distance and the *Steiner point* (d-th curvature centroid)

$$s(K) = d \int_{\mathbb{S}^{d-1}} x \cdot h_K(x) \, d\omega_0(x) = \int_{\mathbb{S}^{d-1}} \operatorname{grad} H_K(x) \, d\omega_0(x)$$

of a body $K \in \mathscr{K}^d$. It is not hard to see that

$$\delta_w(K + a, L)^2 = \delta_w(K - s(K), L - s(L))^2 + |a + s(K) - s(L)|^2 \tag{1.2}$$

for all $K, L \in \mathscr{K}^d$, $a \in \mathbb{E}^d$. A consequence is that the minimum of the Sobolev distance of L and all translates of K is given by $\delta_w(K - s(K), L - s(L))$. By restriction of δ_w to translates $a + \mathscr{K}_0^d$ of the subcone $\mathscr{K}_0^d := \{K \in \mathscr{K}^d \mid s(K) = 0\}$ of $\mathscr{K}^d$ we derive an upper bound for $\delta_w(K, L)$ by quermassintegrals of K, L and the mixed volume $\tilde{V}(K, L) := V(K, L, \mathbb{B}^d, \ldots, \mathbb{B}^d)$ (for the definition of V see, e.g., Leichtweiss [9]).

THEOREM 1. *Let $K, L \in \mathscr{K}^d$ and $s(K) = s(L)$. Then*

$$\delta_w(K, L)^2 \leqslant \frac{2d^2}{d+1}\left(\frac{W_{d-1}(L) - W_{d-1}(K)}{\omega_d}\right)^2 + \frac{(d-1)(2d+1)}{d+1}\,\frac{2\tilde{V}(K, L) - W_{d-2}(K) - W_{d-2}(L)}{\omega_d}.$$

Proof. The main tool of our proof is a strengthened version of Wirtinger's lemma due to Schneider [12] (Lemma 1, p. 53–54) extended from $\mathscr{C}^2(\mathbb{S}^{d-1})$ to $\mathscr{V}(\mathbb{S}^{d-1})$ by approximation. It says that if $g \in \mathscr{V}(\mathbb{S}^{d-1})$ satisfies

$$\hat{g}(0) := \int_{\mathbb{S}^{d-1}} g(x)\, d\omega_0(x) = 0 \quad \text{and} \quad \int_{\mathbb{S}^{d-1}} x \cdot g(x)\, d\omega_0(x) = 0,$$

then

$$\int_{\mathbb{S}^{d-1}} g^2\, d\omega_0 \leqslant \frac{1}{2d} \int_{\mathbb{S}^{d-1}} \nabla_s g\, d\omega_0$$

where equality holds iff g is a spherical harmonic of degree two. Applying this to

$$g := \frac{2d}{2d+1}(f - \hat{f}(0)), \qquad f := h_L - h_K,$$

we get

$$\|f\|_2^2 = \hat{f}(0)^2 + \|f - \hat{f}(0)\|_2^2 \leqslant \hat{f}(0)^2 + \frac{1}{2d+1}\left(\|f - \hat{f}(0)\|_2^2 + \int_{\mathbb{S}^{d-1}} \nabla_s f\, d\omega_0\right)$$
$$= \frac{2d}{2d+1}\hat{f}(0)^2 + \frac{1}{2d+1}\|f\|_w^2.$$

Hence,

$$\|f\|_w^2 = d\,\|f\|_2^2 - (d-1)\int_{\mathbb{S}^{d-1}} \left(f^2 - \frac{1}{d-1}\nabla_s f\right) d\omega_0$$
$$\leqslant \frac{2d^2}{2d+1}\hat{f}(0)^2 + \frac{d}{2d+1}\|f\|_w^2 - (d-1)\int_{\mathbb{S}^{d-1}} \left(f^2 - \frac{1}{d-1}\nabla_s f\right) d\omega_0,$$

which yields

$$\|f\|_w^2 \leqslant \frac{2d^2}{d+1}\hat{f}(0)^2 - \frac{(d-1)(2d+1)}{d+1}\int_{\mathbb{S}^{d-1}} \left(f^2 - \frac{1}{d-1}\nabla_s f\right) d\omega_0.$$

Well known integral representations

$$W_{d-1}(K) = \omega_d \int_{\mathbb{S}^{d-1}} h_K\, d\omega_0 = \omega_d \hat{h}_K(0), \qquad W_{d-2}(K) = \tilde{V}(K, K),$$

$$\tilde{V}(K, L) = \omega_d \int_{\mathbb{S}^{d-1}} \left(h_K h_L - \frac{1}{d-1}\nabla_s(h_K, h_L)\right) d\omega_0$$

of W_{d-1}, W_{d-2} and $\tilde{V}$ by support functions (see Heil [7], Schneider [12]) complete the proof. □

REMARK. Applying Theorem 1 to *normalized* bodies

$$\bar{K} = \frac{\omega_d}{W_{d-1}(K)}(K - s(K)), \qquad \bar{L} = \frac{\omega_d}{W_{d-1}(L)}(L - s(L))$$

one obtains a stability result for the inequality

$$\tilde{V}(K, L)^2 \geqslant W_{d-2}(K) \cdot W_{d-2}(L), \qquad K, L \in \mathscr{K}^d$$

(a special form of the Aleksandrov–Fenchel inequality) in which equality holds iff K and L are homothetic (see Schneider [12], Theorem 2 and the remarks on p. 56, and Goodey and Groemer [6], Theorem 3 and the following remark). In [12] and [6] we find results analogous to our Theorem 1 involving $\mathscr{L}_2$-metric instead of δ_w.

2. In the case of the plane ($d = 2$) each $f : \mathbb{S}^1 \to \mathbb{R}$ may be considered as a 2π-periodic function $f = f(t)$, $t \in \mathbb{R}$. Then, the Sobolev distance of $K, L \in \mathscr{K}^2$ is

$$\delta_w(K, L) = [\,\|h_L - h_K\|_2^2 + \|h'_L - h'_K\|_2^2]^{1/2}$$

where

$$\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} f(t)^2\,dt, \qquad f' := \frac{df}{dt}.$$

It is well known that on Sobolev 1-spaces over bounded intervals of $\mathbb{R}$ the Sobolev norm is stronger than the uniform norm. From Fuglede [5], for example, one derives for a 2π-periodic function f that if

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = 0,$$

then

$$\|f\|_\infty^2 \leqslant \pi \int_0^{2\pi} f'(t)^2\,dt = 2\pi^2 \|f'\|_2^2.$$

Applying this to an arbitrary $f \in \mathscr{V}(\mathbb{S}^1)$ we get

$$\|f\|_\infty \leqslant |\hat{f}(0)| + \sqrt{2}\pi\,\|f'\|_2 \leqslant \|f\|_2 + \sqrt{2}\pi\,\|f'\|_2 \leqslant \sqrt{1+2\pi^2}\,\|f\|_w.$$

Hence, we have the bound

$$\delta_\infty(K, L) \leqslant \sqrt{1+2\pi^2}\,\delta_w(K, L), \qquad K,\ L \in \mathscr{K}^2 \tag{2.1}$$

for the Hausdorff distance $\delta_\infty(K, L) := \|h_L - h_K\|_\infty$ which is the usual deviation on $\mathscr{K}^2$ [$\mathscr{K}^d$]. The following theorem gives the best possible constant in (2.1).

THEOREM 2. *Let $K, L \in \mathscr{K}^2$. Then*

$$\delta_\infty(K, L) \leqslant \sqrt{\pi \cdot \coth \pi}\,\delta_w(K, L). \tag{2.2}$$

The universal constant

$$C := \sqrt{\pi \cdot \coth \pi} = \left(1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2}\right)^{1/2} \approx 1.7757\ldots$$

is best possible.

Proof. We have to show that the infimum of the image $\delta_w(\mathscr{M}_1)$ of

$$\mathscr{M}_1 := \{(K, L) \in \mathscr{K}^2 \times \mathscr{K}^2 \mid \delta_\infty(K, L) = 1\},$$

under δ_w is equal to $1/C$. Then, the general result (2.2) follows by transition from K and L to $K/\delta_\infty(K, L)$ and $L/\delta_\infty(K, L)$. Let $(K, L) \in \mathcal{M}_1$. W.l.o.g. assume that $\operatorname{Min}_{u \in K} |v_0 - u| = |v_0 - u_0| = 1$, $u_0 \in K$, $v_0 \in L$. If σ denotes the reflection at the line through u_0 and v_0, then *Blaschke's symmetrization* $K' = (K + \sigma K)/2$, $L' = (L + \sigma L)/2$ produces

$$\delta_\infty(K', L') = 1 \quad \text{and} \quad \delta_w(K', L') \leqslant \delta_w(K, L),$$

since $\|\cdot\|_\infty$ and $\|\cdot\|_w$ are convex functionals on $\mathscr{V}(\mathbb{S}^1)$. Therefore, $\inf \delta_w(\mathcal{M}_1) = \inf \delta_w(\mathcal{M}_2)$ where

$$\mathcal{M}_2 := \{(K, L) \in \mathcal{M}_1 \mid h_L(0) - h_K(0) = 1, h_K(t) = h_K(-t), h_L(t) = h_L(-t)\}.$$

In order to compute the last infimum, we follow Marti [10] who determined the least constant C' in the Sobolev inequality $\|f\|_\infty \leqslant C' \|f\|_w$ on $[0, t_0]$ involving the absolute minimum of

$$\left.\begin{aligned} &I[y] := \int_0^{t_0} F(t, y, y')\, dt, \qquad F(t, y, y') = y^2 + y'^2 \\ &y(0) = 1,\ y(t_0) \text{ variable,} \end{aligned}\right\} \tag{2.3}$$

in the class of $\mathscr{C}^2$-functions. In the case $t_0 = \pi$ the Euler–Lagrange equation of I and the transversality condition $F_{y'} = 0$ at the line $t = \pi$ induce the boundary value problem

$$y'' - y = 0, \qquad y(0) = 1, \quad y'(\pi) = 0. \tag{2.4}$$

Since $y^2 + y'^2$ is convex as a function of both arguments, the solution

$$y_0(t) := \cosh t - \tanh \pi \cdot \sinh t = \frac{\cosh(\pi - t)}{\cosh \pi} > 0$$

of (2.4) supplies the absolute minimum of (2.3) with $t_0 = \pi$ (see Troutman [14], p. 59).

Now, define $K_w \in \mathscr{K}^2$ by 2π-periodic extension of

$$h_{K_w}(t) := 1 - \frac{1}{2} y_0(|t|), \qquad t \in [-\pi, \pi]. \tag{2.5}$$

h_{K_w} is a support function, since

$$h_{K_w}(t) + h''_{K_w}(t) = 1 - y_0(|t|) \geqslant 0, \qquad t \in [-\pi, \pi]$$

and

$$h'_{K_w}(-\pi^+) - h'_{K_w}(\pi^-) = 0, \qquad h'_{K_w}(0^+) - h'_{K_w}(0^-) > 0.$$

Because of $(2K_w, 2\mathbb{B}^2) \in \mathcal{M}_2$ we have $\delta_w(2K_w, 2\mathbb{B}^2) \leqslant \inf \delta_w(\mathcal{M}_3)$ where $\mathcal{M}_3 := \{(K, L) \in \mathcal{M}_2 \,|\, h_K, h_L$ of the class $\mathscr{C}^2\}$. A well known approximation argument (see Bonnesen/Fenchel [3], p. 36–37 and Heil [7], Lemma 4.1 and its applications) can be used to verify that $\inf \delta_w(\mathcal{M}_2) = \delta_w(2K_w, 2\mathbb{B}^2) = \min \delta_w(\mathcal{M}_1)$. Finally,

$$\delta_w(2K_w, 2\mathbb{B}^2)^2 = \frac{1}{\pi} I[y_0] = \frac{1}{\pi} \int_0^{\pi} y_0(y_0 - y_0'')\, dt + \frac{1}{\pi} y_0 y_0' \Big|_0^{\pi}$$

$$= -\frac{1}{\pi} y_0(0) y_0'(0) = \frac{1}{\pi} \tanh \pi = C^{-2} \tag{2.6}$$

and Fourier expansion of $\cosh t$ on $[-\pi, \pi]$ complete the proof. □

A further characterization of the convex body K_w can be derived if we ask for equality in (2.2) if $L = \mathbb{B}^2$ is fixed. The last proof shows that

$$\delta_\infty(K, \mathbb{B}^2) = \sqrt{\pi \cdot \coth \pi}\, \delta_w(K, \mathbb{B}^2) \neq 0$$

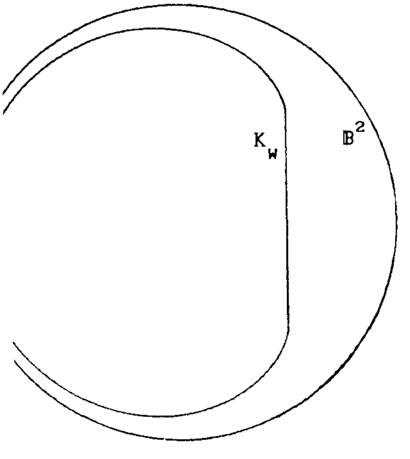

Figure 1.

holds for $K = K(\alpha)$, $h_{K(\alpha)}(t) := 1 - y_0(|t|)/\alpha$, $t \in [-\pi, \pi]$. In order to obtain convexity of the regions $K(\alpha)$, we have to choose the parameter α from the interval $[2, \infty[$. Therefore, $K_w = K(2)$ is the smallest convex body of the family $(K(\alpha))$. In view of its specific properties we call K_w the *minimal body* of the Sobolev distance δ_w.

Our next theorem shows a relationship between the algebraic structure of $\mathscr{K}^2$ and the minimal body K_w of δ_w. It arises from the question how inequality (2.2) can be improved by restriction to the subcone $\mathscr{K}_0^2$ or, even stronger, by restriction to those convex subsets of $\mathscr{K}_0^2$ which contain all bodies of a fixed perimeter $(=2W_1)$.

THEOREM 3. (*a*) *Let* $K, L \in \mathscr{K}^2$ *and* $s(K) = s(L)$. *Then*

$$\delta_\infty(K, L) \leqslant (C^2 - 1)^{1/2} \delta_w(K, L)$$

where equality holds for $K = K_w - s(K_w)$ *and* $L = \mathbb{B}^2$. *Therefore, the universal constant*

$$(C^2 - 1)^{1/2} = \left(2 \sum_{n=1}^{\infty} \frac{1}{1+n^2}\right)^{1/2} \approx 1.4673\ldots$$

is best possible.

(*b*) *Let* $K, L \in \mathscr{K}^2$, $s(K) = s(L)$ *and* $W_1(K) = W_1(L)$. *Then*

$$\delta_\infty(K, L) \leqslant (C^2 - 2)^{1/2} \delta_w(K, L)$$

where equality holds for

$$K = \bar{K}_w = \frac{\pi}{W_1(K_w)} (K_w - s(K_w)) \quad \text{and} \quad L = \mathbb{B}^2.$$

Therefore, the universal constant

$$(C^2 - 2)^{1/2} = \left(2 \sum_{n=2}^{\infty} \frac{1}{1+n^2}\right)^{1/2} \approx 1.0738\ldots$$

is best possible.

Proof. We omit the proof of (a) and turn to the very similar proof of (b).

Let $K, L \in \mathscr{K}^2$, $s(K) = s(L)$, $W_1(K) = W_1(L)$ and $\delta_\infty(K, L) = 1$. Using the invariance of the quermassintegral $W_{d-1} = W_1$ and the equivariance of the Steiner point s under Blaschke's symmetrization and the congruence invariance [equivariance] of $\delta_\infty, \delta_w, W_1, s$, respectively, we can assume that $K, L \in \mathscr{K}_0^2$ with

$$h_L(0) - h_K(0) = 1, \qquad h_K(t) = h_K(-t), \quad h_L(t) = h_L(-t).$$

Setting $\eta := (\tanh \pi)/\pi < 1/3$ we compute for the minimal body K_w

$$s(K_w) = \left(-\frac{\eta}{2}, 0\right), \qquad W_1(K_w) = \frac{\pi}{2}(2 - \eta). \tag{2.7}$$

Now, define $K', L' \in \mathscr{K}^2$ by

$$K' := (1 - 2\eta)K + 2s(K_w), \qquad L' := (1 - 2\eta)L + \eta \mathbb{B}^2.$$

From

$$h_{L'}(t) - h_{K'}(t) = \eta + \eta \cos t + (1 - 2\eta)(h_L(t) - h_K(t))$$

we deduce that $\delta_\infty(K', L') = 1$. Hence, by (2.6) and (1.2), our assumptions $s(K) = s(L)$ and $W_1(K) = W_1(L)$ imply

$$\eta = \delta_w(2K_w, 2\mathbb{B}^2)^2 \leqslant \delta_w(K', L')^2 = 2\eta^2 + (1 - 2\eta)^2 \delta_w(K, L)^2,$$

which yields

$$\delta_w(K, L)^2 \geqslant \frac{\eta}{1 - 2\eta} = \left(\frac{1}{\eta} - 2\right)^{-1} = (C^2 - 2)^{-1}. \tag{2.8}$$

By (2.7), equality occurs in (2.8) for

$$K = \frac{2}{1 - 2\eta}(K_w - s(K_w)), \qquad L = \frac{2W_1(K_w)}{(1 - 2\eta)\pi} \mathbb{B}^2. \qquad \square$$

In the plane case the Aleksandrov–Fenchel inequality reduces to $V(K, L)^2 \geqslant V(K) \cdot V(L)$ where equality holds iff K and L are homothetic. For this inequality we deduce from Theorem 1 and Theorem 3(b) a kind of unrestricted stability with respect to the Hausdorff distance.

COROLLARY 1. *Let $K, L \in \mathscr{K}^2$ with $W_1(K) \neq 0$, $W_1(L) \neq 0$, and let $\bar{K}, \bar{L}$ denote its normalizations. Then*

$$\delta_\infty(\bar{K}, \bar{L})^2 \leqslant \frac{10}{3\pi}(C^2 - 2)[V(\bar{K}, \bar{L}) - V(\bar{K})^{1/2}V(\bar{L})^{1/2}]$$

$$= \frac{10\pi}{3}(C^2 - 2)\frac{V(K, L) - V(K)^{1/2}V(L)^{1/2}}{W_1(K)W_1(L)}. \qquad \square$$

3. We finish with some remarks on the Sobolev distance in the case $d \geqslant 3$.

In order to examine the existence of an inequality $\delta_\infty \leqslant C\delta_w$, it suffices to consider bodies of revolution with axis $x_0 \in \mathbb{S}^{d-1}$. Let

$$\mathbf{H} := \{\sigma \in \mathbf{SO}(d) \mid \sigma x_0 = x_0\} \cong \mathbf{SO}(d-1),$$

then each $\mathbf{H}$-zonal function $f \in \mathscr{V}(\mathbb{S}^{d-1})$ may be considered as a function of the angle $t \in [0, \pi]$ with the axis x_0. The Sobolev norm of such f is given by

$$\|f\|_w^2 = \frac{(d-1)\omega_{d-1}}{d \cdot \omega_d}\int_0^\pi [f(t)^2 + f'(t)^2] \sin^{d-2} t \, dt,$$

which shows that there is no universal Sobolev inequality $\|\cdot\|_\infty \leqslant C\|\cdot\|_w$ on $V(\mathbb{S}^{d-1})$. The main argument is the singularity of the variational problem

$$\left.\begin{array}{l} I[y] := \displaystyle\int_0^{t_0} (y^2 + y'^2) \sin^{d-2} t \, dt = \text{Min!} \\ y(0) = 1, y(t_0) \text{ variable,} \end{array}\right\}$$

at the left boundary $t = 0$. In the case $d \geqslant 4$ one gets the non-existence of C directly from the (compact) convex cone $L(R)$, $R > 0$ with

$$h_{L(R)}(t) = \begin{cases} \cos t, & t \in [0, \varphi] \\ R \sin t, & t \in [\varphi, \pi] \end{cases}, \quad R = \cot \varphi, 0 < \varphi \leqslant \pi/2$$

and its circular face $K(R)$ in the hyperplane $\langle u, x_0 \rangle = 0$. (These are the bodies L_5 and K_5 by Vitale [15], which lead to an estimate for $\delta_\infty(K, L)$ in terms of the $\mathscr{L}_p$-distance of K, L and the diameter of $K \cup L$.) We have $\delta_\infty(K(R), L(R)) = 1$ and, setting $f_R := h_{L(R)} - h_{K(R)}$,

$$\int_0^\pi [f_R(t)^2 + f_R'(t)^2] \sin^{d-2} t \, dt = (1 + R^2)\int_0^\varphi \sin^{d-2} t \, dt = \sin^{-2} \varphi \int_0^\varphi \sin^{d-2} t \, dt.$$

Hence,

$$\lim_{R\to\infty}\int_0^{\pi}[f_R(t)^2+f'_R(t)^2]\sin^{d-2}t\,dt=\lim_{\varphi\to 0}\frac{\sin^{d-3}\varphi}{2\cos\varphi}=0\qquad(d\geqslant 4).$$

Nevertheless, we think that the Sobolev distance deserves consideration also in higher dimensions. A basic problem seems to be the determination of sharp estimates of δ_∞ by δ_w (inclusive of minimal bodies) on certain subclasses of $\mathscr{K}^d\times\mathscr{K}^d$ which, for example, arise from the demand that $\operatorname{diam}(K\cup L)\leqslant D$ ($D>0$ fixed).

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