Aequationes Mathematicae 46 (1993) 91–98
University of Waterloo

0001-9054/93/010091-08 $1.50 + 0.20/0

On the definition of a probabilistic normed space

C. ALSINA, B. SCHWEIZER AND A. SKLAR

Summary. In this paper we give a new definition of a probabilistic normed space. This definition, which is based on a characterization of normed spaces by means of a betweenness relation, includes the earlier definition of A. N. Šerstnev as a special case and leads naturally to the definition of the principal class of probabilistic normed spaces, the Menger spaces.

The notion of a probabilistic normed space, in which the values of the norms are probability distribution functions rather than numbers, is a natural generalization of that of an ordinary normed linear space. Experience has shown us, however, that the realization of such a generalization is not as straightforward as it may seem at first sight. In this paper we present a new definition of a probabilistic normed space. We regard this definition as both natural and fruitful. It includes the earlier definition of A. N. Šerstnev as a special case, leads naturally to the definition of the principal class of probabilistic normed spaces, the Menger spaces, and is compatible with various possible definitions of a probabilistic inner product space (which will be the subject of a subsequent paper). It is based on the probabilistic generalization of a characterization of (ordinary) normed spaces by means of a betweenness relation (see Theorem 1) and relies on the tools we have fashioned in the course of our development of the theory of probabilistic metric spaces.

Probabilistic metric spaces were introduced by K. Menger in 1942 [1]. Subsequent refinements (see [2, Chap. 1]) have led to the definition of a probabilistic metric (or PM) space as a triple $(S, \mathscr{F}, \tau)$, where S is a set, $\mathscr{F}$ is a mapping from $S \times S$ into a space Δ^+ of distribution functions and τ is a triangle function (see below). Denoting the value of $\mathscr{F}$ at the pair (p, q) by F_{pq}, the following conditions are assumed to hold for all p, q, r in S:

AMS (1991) subject classification: Primary 46B99, 54E99. Secondary 60B11.

Manuscript received August 5, 1991 and, in final form, November 11, 1991.

(M1) $F_{pq} = \varepsilon_0$ if and only if $p = q$.

(M2) $F_{pq} = F_{qp}$.

(M3) $F_{pr} \geqslant \tau(F_{pq}, F_{qr})$.

Specifically, Δ^+ is the set of all probability distribution functions that are left-continuous on $\mathbb{R} = (-\infty, +\infty)$, 0 on $[-\infty, 0]$, and possibly discontinuous (defective) at $+\infty$. For any $a \geqslant 0$, ε_a is the distribution function given by

$$\varepsilon_a(x) = \begin{cases} 0, & x \leqslant a, \\ 1, & x > a. \end{cases} \tag{1}$$

In particular, under the usual pointwise ordering of functions, ε_0 is the maximal element of Δ^+. A triangle function is a binary operation on Δ^+ that is commutative, associative, nondecreasing in each place, and has ε_0 as identity. Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+.

Typical (continuous) triangle functions are convolution and the operations τ_T and τ_{T^*}, which are, respectively, given by

$$\tau_T(F, G)(x) = \sup\{T(F(u), G(v)) \mid u + v = x\}, \tag{2}$$

$$\tau_{T^*}(F, G)(x) = \inf\{T^*(F(u), G(v)) \mid u + v = x\}, \tag{3}$$

for all F, G in Δ^+ and all x in $\mathbb{R}$ [2, Secs. 7.2 and 7.3]. In (2), T is a continuous t-norm, i.e., a continuous binary operation on [0, 1] that is commutative, associative, nondecreasing in each place, and has 1 as identity; in (3), T^* is a continuous t-conorm, i.e., a binary operation on [0, 1] which is related to a continuous t-norm T by

$$T^*(x, y) = 1 - T(1 - x, 1 - y). \tag{4}$$

It follows without difficulty from (1)–(4) that

$$\tau_T(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b} = \tau_{T^*}(\varepsilon_a, \varepsilon_b) \tag{5}$$

for any continuous t-norm T, any continuous t-conorm T^* and any $a, b \geqslant 0$.

Probabilistic normed (PN) spaces were first defined by A. N. Šerstnev in 1962 [3, 4]. A PN space in the sense of Šerstnev is a triple $(S, \mathcal{N}, \tau)$, where S is a real linear space, τ is a continuous triangle function, and $\mathcal{N}$ is a mapping from S into

Δ^+, such that – writing N_p for $\mathcal{N}(p)$ – for all p, q in S,

(N1) $N_p = \varepsilon_0$ if and only if $p = \theta$,

(N3) $N_{p+q} \geqslant \tau(N_p, N_q)$,

(Š) $N_{\lambda p}(x) = N_p(x/|\lambda|)$ for all λ and x in $\mathbb{R}$.

Here θ is the null vector in S, and we adopt the convention that $N_p(x/0) = \varepsilon_0(x)$.
Note that ($\check{S}$) implies

(N2) $N_{-p} = N_p$,

and that, if (N1), (N2) and (N3) hold and $\mathscr{F}: S \times S \to \Delta^+$ is defined via $F_{pq} = N_{p-q}$, then $(S, \mathscr{F}, \tau)$ is a PM-space.

Over the years, the theory of PM-spaces has undergone a substantial development. In contrast, since its initial application by Šerstnev to problems of best approximation [5, 6], there has been little real progress in the theory of PN-spaces. The principal reason for this is the fact that condition (Š) seems to be too strong. It implies, for example, that every one-dimensional subspace of a PN-space is a simple PM-space [2, Sec. 8.4]. More generally, it imposes a structure theory which is too similar to the theory of ordinary normed linear spaces; and it also has the drawback that we have never been able to formulate a reasonable definition of a probabilistic inner product space that is naturally compatible with (Š). Consideration of these matters has led us to a new and more general definition of a probabilistic normed space. It is based on an alternate definition of an ordinary normed linear space which is contained in the following:

THEOREM 1. *Let V be a real linear space and f a mapping from V into $\mathbb{R}^+ = [0, \infty)$. Then $f(\lambda p) = |\lambda| f(p)$ for all p in V and all λ in $\mathbb{R}$ (whence, in particular, $f(\theta) = 0$) if and only if*

(n2) $f(-p) = f(p)$,

and

(n4) $f(p) = f(\alpha p) + f((1-\alpha)p)$,

for all p in V and all α in $[0, 1]$. Thus the pair (V, f) is a normed linear space if and only if (n2), (n4) *and the conditions*

(n1) $f(p) \neq 0$ *if* $p \neq \theta$, *and*

(n3) $f(p+q) \leqslant f(p) + f(q)$ *for all* p, q *in* V,

are satisfied.

Proof. The necessity of (n2) and (n4) is trivial. As for their sufficiency, we start with the fact that, for any p in V, (n4) yields

$$f(p) = f(0p) + f(1p) = f(\theta) + f(p),$$

whence $f(\theta) = 0$. Consequently,

$$f(0p) = f(\theta) = 0 = 0f(p),$$

for all p in V. We now proceed by induction: If there is a non-negative integer n such that $f(np) = nf(p)$ for all p in V then we have, using (n4), for all p in V

$$\begin{aligned} f((n+1)p) &= f\left(\frac{n}{n+1}(n+1)p\right) + f\left(\frac{1}{n+1}(n+1)p\right) \\ &= f(np) + f(p) = (n+1)f(p). \end{aligned}$$

Hence $f(np) = nf(p)$ for all p in V and all non-negative integers n. A standard argument now yields the result that $f(rp) = rf(p)$ for all p in V and all non-negative rational numbers r. Next, if $0 \leqslant \lambda < \nu$ then, for any p in V,

$$\begin{aligned} f(\nu p) &= f\left(\frac{\lambda}{\nu}\nu p\right) + f\left(\frac{\nu-\lambda}{\nu}\nu p\right) \\ &= f(\lambda p) + f((\nu-\lambda)p) \geqslant f(\lambda p). \end{aligned}$$

Thus, for a fixed p in V, the expression $f(\lambda p)$ is nondecreasing in λ for $\lambda \geqslant 0$. Consequently, for any p in V, $f(\lambda p) = \lambda f(p)$ for all $\lambda \geqslant 0$, irrational as well as rational. Finally application of (n2) yields $f(\lambda p) = |\lambda| f(p)$ for all p in V and all λ in $\mathbb{R}$, and the proof is complete.

Note that Theorem 1 states that, if $f(\theta) = 0$, then the condition $f(\lambda p) = |\lambda| f(p)$ can be replaced by (n2) and a betweenness condition, namely the requirement, equivalent to (n4), that points on the linear segment joining θ and p are metrically between θ and p (see [2, Sec. 3.3]).

We now apply Theorem 1 to obtain:

LEMMA 1. *Suppose the pair* $(S, \mathcal{N})$ *satisfies* (N1) *and* (N2). *Then* $(S, \mathcal{N})$ *satisfies* (Š) *if and only if*

$$N_p = \tau_M(N_{\alpha p}, N_{(1-\alpha)p}) \tag{6}$$

for all p *in* S *and all* α *in* $[0, 1]$, *where* M *is the t-norm given by* $M(x, y) = \min(x, y)$ *for all* x, y *in* $[0, 1]$.

Proof. For any F in Δ^+, let $F^\wedge$ denote the left-continuous quasi-inverse of F, i.e., the function defined for all t in $[0, 1]$ by

$$F^\wedge(t) = \sup\{x \mid F(x) < t\}. \tag{7}$$

It is known that, for any F, G, H in Δ^+, $H = \tau_M(F, G)$ if and only if $H^\wedge = F^\wedge + G^\wedge$ [2, Sec. 7.7]. Thus (6) holds if and only if

$$N_p^\wedge = N_{\alpha p}^\wedge + N_{(1-\alpha)p}^\wedge \tag{8}$$

for all p in S and all α in $[0, 1]$.

Now suppose that $(S, \mathcal{N})$ satisfies (N1), (N2) and (Š). Then, for any p in S, any α in $(0, 1)$, and any x in $\mathbb{R}$,

$$N_{\alpha p}(x) = N_p(x/\alpha) \quad \text{and} \quad N_{(1-\alpha)p}(x) = N_p(x/(1-\alpha)).$$

It follows from (7) that

$$N_{\alpha p}^\wedge = \alpha N_p^\wedge \quad \text{and} \quad N_{(1-\alpha)p}^\wedge = (1-\alpha)N_p^\wedge,$$

whence (8) holds for α in $(0, 1)$. Since (8) holds automatically for $\alpha = 0$ and $\alpha = 1$ (by virtue of (N1)), (8) holds for all α in $[0, 1]$, whence (6) holds.

In the other direction, if (N1), (N2) and (6) hold then it follows from (8) that the function $f_t: S \to \mathbb{R}^+$ defined for a fixed t in $[0, 1]$ by $f_t(p) = N_p^\wedge(t)$ satisfies conditions (n2) and (n4) of Theorem 1. Therefore, for all λ in $\mathbb{R}$ and all t in $[0, 1]$,

$$N_{\lambda p}^\wedge(t) = f_t(\lambda p) = |\lambda| f_t(p) = |\lambda| N_p^\wedge(t),$$

whence $N_{\lambda p}^\wedge = |\lambda| N_p^\wedge$, which is equivalent to (Š) and completes the proof.

An immediate consequence of Lemma 1 is:

THEOREM 2. *If the triple* $(S, \mathcal{N}, \tau)$ *satisfies* (N1), (N2) *and* (N3), *then* $(S, \mathcal{N}, \tau)$ *is a PN space in the sense of Šerstnev if and only if* (6) *holds for all* p *in* S *and all* α *in* $[0, 1]$.

COROLLARY. *If* $(S, \mathcal{N}, \tau)$ *is a PN space in the sense of Šerstnev then*

$$\tau(N_{\alpha p}, N_{(1-\alpha)p}) \leqslant N_p = \tau_M(N_{\alpha p}, N_{(1-\alpha)p}) \tag{9}$$

for all p *in* S *and all* α *in* $[0, 1]$.

The bounds in (9) are related to certain betweenness relations in PM spaces (see [2, Chap. 14]). This observation, together with the fact that Theorem 1 shows that ordinary normed spaces can be defined in terms of betweenness relations motivates the following:

DEFINITION 1. A *probabilistic normed space* (briefly, a *PN space*) is a quadruple $(S, \mathcal{N}, \tau, \tau^*)$, where S is a real linear space, τ and τ^* are continuous triangle functions, and $\mathcal{N}$ is a mapping from S into Δ^+ such that, for all p, q in S, the following conditions hold:

(N1) $N_p = \varepsilon_0$ if and only if $p = \theta$,

(N2) $N_{-p} = N_p$,

(N3) $N_{p+q} \geqslant \tau(N_p, N_q)$,

(N4) $N_p \leqslant \tau^*(N_{\alpha p}, N_{(1-\alpha)p})$, for all α in $[0, 1]$.

Note that (N3) and (N4) together yield

$$\tau(N_{\alpha p}, N_{(1-\alpha)p}) \leqslant N_p \leqslant \tau^*(N_{\alpha p}, N_{(1-\alpha)p}) \tag{10}$$

for all p in S and all α in $[0, 1]$.

If $\tau^* = \tau_M$ and equality holds in (N4) then, by Lemma 1, we have a PM space in the sense of Šerstnev, and conversely. Note also that if $(S, \|\cdot\|)$ is a real normed space, *if* τ and τ^* are triangle functions such that

$$\tau(\varepsilon_a, \varepsilon_b) \leqslant \varepsilon_{a+b} \leqslant \tau^*(\varepsilon_a, \varepsilon_b), \tag{11}$$

for all $a, b \geqslant 0$, and if we define $\mathcal{N}: S \to \Delta^+$ via $N_p = \varepsilon_{\|p\|}$, then $(S, \mathcal{N}, \tau, \tau^*)$ is a PN space. Since (5) shows that there are many pairs of triangle functions that satisfy (11), it follows that any real normed space may be viewed as a PN space.

If T is a continuous t-norm and T^* is the t-conorm of T, then it is known that $\tau_T \leqslant \tau_{T^*}$, i.e., that $\tau_T(F, G) \leqslant \tau_{T^*}(F, G)$ for all F, G in Δ^+ [2, Sec. 7.3]. It is therefore consistent with (10) to introduce the following definition.

DEFINITION 2. A *Menger PN space*, denoted by $(S, \mathcal{N}, T)$, is a PN space $(S, \mathcal{N}, \tau, \tau^*)$ in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its t-conorm T^*.

The name is appropriate, since the PM space derived from a Menger PN space by setting $F_{pq} = N_{p-q}$ is a Menger space [2, Sec. 8.1], and the right-hand inequality in (10) in a Menger PN space is the assertion that points on the linear segment joining θ and p are Menger-between θ and p [2, Sec. 14.3]. Similarly, if in a given PN space, $\tau^* = \tau$, then (10) reduces to the equality $N_p = \tau(N_{\alpha p}, N_{(1-\alpha)p})$, which means that points on the segment joining θ and p are Wald-between θ and p [2, Sec. 14.1].

It is known that $\tau_M = \tau_{M^*}$ [2, Cor. 7.5.8]. Thus, if $\tau = \tau_M$, then the notions of a PN space in the sense of Šerstnev and a Menger PN space are equivalent; and Menger-betweenness and Wald-betweenness also coincide. In general, since $\tau_M \leqslant \tau_{T^*}$ for any t-norm T, if $(S, \mathcal{N}, \tau_T)$ is a PN space in the sense of Šerstnev, then it is a Menger PN space. The converse is false, as the following example shows:

Let $\mathcal{N}: \mathbb{R} \to \Delta^+$ be defined by $N_0 = \varepsilon_0$ and

$$N_x(t) = \begin{cases} 0, & t \leqslant 0, \\ \exp(-\sqrt{|x|}), & 0 < t < \infty, \\ 1, & t = \infty. \end{cases}$$

A straightforward calculation shows that $(R, \mathcal{N}, \tau_\pi, \tau_{\pi^*})$, where $\pi(x, y) = xy$ and $\pi^*(x, y) = x + y - xy$, is a Menger PN space but that (Š) fails.

Acknowledgements

We thank Professor J. Rätz for several helpful suggestions concerning Theorem 1.

The second author's research was supported in part by ONR Contract N-00014-87-K-0379.

REFERENCES

[1] MENGER, K., *Statistical metrics.* Proc. Nat. Acad. Sci. U.S.A. *28* (1942), 535–537.
[2] SCHWEIZER, B. and SKLAR, A., *Probabilistic metric spaces.* Elsevier North-Holland, New York, 1983.
[3] ŠERSTNEV, A. N., *Random normed spaces: problems of completeness.* Kazan. Gos. Univ. Učen. Zap. *122* (1962), 3–20.
[4] ŠERSTNEV, A. N., *On the notion of a random normed space.* Dokl. Akad. Nauk SSSR *149*(2) (1963), 280–283.
[5] ŠERSTNEV, A. N., *Best approximation problems in random normed spaces.* Dokl. Akad. Nauk SSSR *149*(3) (1963), 539–542.
[6] ŠERSTNEV, A. N., *Some best approximation problems in random normed spaces.* Rev. Roumaine Math. Pures Appl. *9* (1963), 771–789.

Department of Mathematics and Statistics,
Universitat Politècnica de Catalunya,
E-08028 Barcelona,
Spain.

Department of Mathematics and Statistics,
University of Massachusetts,
Amherst, Massachusetts 01003,
U.S.A.

Department of Mathematics,
Illinois Institute of Technology,
Chicago, Illinois 60616,
U.S.A.